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CONTRACTION SEMI-GROUPS IN A FUNCTION SPACE

JAMES ROBERT DORROH

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J. R. DORROH

Using the concepts of a semi inner-product and a dissipative operator, it is proven that if X is a complex Banach space (under the supremum norm) of bounded complex valued functions on a set S , p is a bounded positive function on S which is bounded away from zero, $pX \subset X$, and A is the infinitesimal generator of a strongly continuous (class (C_0)) semi-group of contraction operators in X , then pA is also the infinitesimal generator of such a semi-group.

The notion of a *semi inner-product* was introduced by G. Lumer in [3].

DEFINITION 1. A *semi inner-product* for a complex (real) Banach space X is a function $[\cdot, \cdot]$ from $X \times X$ into the complex (real) numbers which satisfies

$$\begin{aligned} [\alpha x + \beta y, z] &= \alpha[x, z] + \beta[y, z], \\ |[x, z]| &\leq \|x\| \cdot \|z\|, \end{aligned}$$

and

$$[x, x] = \|x\|^2.$$

There is at least one semi inner-product for every Banach space X , because we can define $[x, y] = f(x)y$, where f is a bounded linear functional on X such that $\|f\| = \|y\|$, and $|f(y)| = \|y\|^2$ (see [4]).

By an operator in a Banach space X , we mean a linear transformation (not necessarily bounded) from a subspace of X to a subspace of X . The notion of a *dissipative operator* in a Banach space is treated by G. Lumer and R. S. Phillips in [4].

DEFINITION 2. An operator A in a Banach space X is said to be *dissipative* (with respect to a given semi inner-product for X) if

$$\operatorname{re}[Ax, x] \leq 0$$

for all x in the domain of A .

By a contraction semi-group in a Banach space X we mean a strongly continuous semi-group of contraction operators in X which is of class (C_0) (see [2]). A contraction operator in X is a bounded linear transformation T from X into X with $\|T\| \leq 1$. Lumer and Phillips have given the following characterization [4, Theorem 3.1] of the infinitesimal generator of a contraction semi-group.

THEOREM (Lumer and Phillips). *Suppose A is an operator in a Banach space X , the domain of A is dense in X , and $[\cdot, \cdot]$ is a semi inner-product for X . Then A is the infinitesimal generator of a contraction semi-group in X if and only if A is dissipative with respect to $[\cdot, \cdot]$, and the range of $I - A$ is all of X , where I denotes the identity transformation on X .*

THEOREM. *Suppose S is a set, X is a complex Banach space (under the supremum norm) of bounded complex valued functions on S , p is a bounded positive function on S which is bounded away from zero, $pX \subset X$, and A is the infinitesimal generator of a contraction semi-group in X . Then pA is also the infinitesimal generator of a contraction semi-group in X .*

Proof. Let U denote the Banach algebra of all bounded complex valued functions on S , and let S_1 denote the set of all nonzero multiplicative linear functionals on S . It follows from [1, pp. 272-277], especially [1, Corollary 19, p. 276], that

(i) if m is in S_1 , and q is a nonnegative function in U , then $m(q) \geq 0$, and

(ii) if x is in U , then there is an m in S_1 such that $|m(x)| = \|x\|$. For each x in X , let m_x denote an element m of S_1 such that $|m(x)| = \|x\|$, and for each x, y in X , let

$$[x, y] = m_y(x)[m_y(y)]^* ,$$

where the $*$ denotes complex conjugation. Then $[\cdot, \cdot]$ is a semi inner-product for X ; it is the only one to be used from this point on. A dissipative operator in X will mean one which is dissipative with respect to this semi inner-product.

If q is a bounded nonnegative function on S , and $qX \subset X$, then

$$\operatorname{re} [qAx, x] = m_x(q) \operatorname{re} [Ax, x] \leq 0 ,$$

for all x in $\mathfrak{D}(A)$, the domain of A , since A is dissipative by [4, Theorem 3.1]. Therefore, qA is dissipative. Also, the domain of qA is $\mathfrak{D}(A)$, which is dense in X by [2, Theorem 12.3.1, p. 360]. If

$$\sup_{s \in S} |1 - q(s)| < 1/2 ,$$

then $\|I - q\|$, the operator norm of $I - q$, is less than $1/2$, so that $I - qA$ is invertible, since

$$I - qA = I - A + (I - q)A = \{I + (I - q)AR(1, A)\}(I - A) ,$$

and

$$\|AR(1, A)\| = \|R(1, A) - I\| \leq 2$$

by [2, Theorem 12.3.1, p. 360]. Thus the range of $I - qA$ is all of X , and qA generates a contraction semi-group in X by [4, Theorem 3.1].

Since $F(p)X \subset X$ for every polynomial F , and p is bounded and nonnegative, it follows from the classical Weierstrass theorem that $p^{(1/n)}X \subset X$ for every positive integer n . Choose n so that

$$\sup_{s \in S} |1 - [p(s)]^{(1/n)}| < 1/2,$$

and let $r = p^{(1/n)}$. This is possible because the range of p is contained in a closed and bounded interval of positive numbers. By what was shown in the previous paragraph, rA generates a contraction semi-group in X . If $1 \leq j < n$, and r^jA generates a contraction semi-group in X , then $r^{j+1}A$ does also, for

$$r^{j+1}A = r(r^jA),$$

and we can substitute r for q and r^jA for A in the argument given in the previous paragraph.

REMARK. An argument similar to the one given will establish the theorem if X is taken to be a real Banach space (under the supremum norm) of bounded real valued functions on S , and the rest of the hypothesis remains the same. Also, we could take A to be the generator of a class (C_0) semi-group $[T(t); 0 \leq t < \infty]$ of operators in X such that for some $\omega > 0$,

$$\|T(t)\| \leq e^{\omega t} \quad \text{for } t \geq 0.$$

If

$$\tilde{T}(t) = e^{-\omega t}T(t) \quad \text{for } t \geq 0,$$

then $[\tilde{T}(t)]$ is a contraction semi-group in X and has the generator $\tilde{A} = A - \omega$.

If

$$V(t) = e^{\omega t p} \tilde{V}(t) \quad \text{for } t \geq 0,$$

where $[\tilde{V}(t); 0 \leq t < \infty]$ is the contraction semi-group generated by $p\tilde{A}$, then $[V(t)]$ is a class (C_0) semi-group of operators in X ,

$$\|V(t)\| \leq e^{\omega t \|p\|} \quad \text{for } t \geq 0,$$

and $[V(t)]$ is generated by pA . The author wishes to express his thanks to the referee for his suggestions.

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