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THE INVERSION OF A CLASS OF LINEAR OPERATORS

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THE INVERSION OF A CLASS OF LINER OPERATORS

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Let \bar{Q}_L denote the set of all quasi-continuous number valued functions on a number interval $[a, b]$ which vanish at a and are left continuous at each point of $(a, b]$. Every linear operator, \mathcal{L} , on \bar{Q}_L which is continuous relative to the sup norm topology for \bar{Q}_L has a unique representation of the form $\mathcal{L}f(s) = \int_a^b f(t)dL(t, s)$, $f \in \bar{Q}_L$, $a \leq s \leq b$, where all integrals are taken in the σ -mean Stieltjes sense, and L is a function on the square $a \leq t \leq b$, satisfying the conditions of Definition 1.2. This paper is concerned primarily with those linear operators, the P -operators, which are abstractions from that class of linear physical systems whose output signals at a given time do not depend on their input signals at a later time; and with a sub-family of the P -operators, the P_1 -operators which include all stationary linear operators. The P -operators are the Volterra operators on \bar{Q}_L . Necessary conditions and sufficient conditions for a P -operator to have an inverse which is a P -operator are found; and a necessary and sufficient condition for a P_1 -operator to have an inverse which is a P -operator is given in Theorem 3.1. In addition it is shown that if \mathcal{L} is a P_1 -operator and \mathcal{L}^{-1} is a P -operator then \mathcal{L}^{-1} may be written as the product of two operators whose generating functions may be found by successive approximation techniques. An analogue of Lane's inversion theorem for stationary operators on QC_{OL} is found as a special case of these results.

In [1] subspaces of the space of functions which are quasi-continuous on an interval $[a, b]$ for which every linear operator \mathcal{L} may be written as a σ -mean Stieltjes integral of the form $\mathcal{L}f(s) = \int_a^b f(t)dL(t, s)$ are investigated. In this paper we will be concerned with one such subspace, \bar{Q}_L , and with a class of linear operators on \bar{Q}_L , the P -operators, which are essentially the abstractions from that class of linear physical systems whose output signals at a given time do not depend on their input signals at a later time. In particular we shall be concerned with determining conditions which will guarantee that a P -operator has an inverse which is a P -operator.

In § 2 some of the basic properties of P -operators are developed and in § 3 a subfamily of these operators, the P_1 -operators, are introduced. The P_1 -operators have the property that if a P_1 -operator, \mathcal{H} , has an inverse which is a P -operator then the generating function for \mathcal{H}^{-1} may be determined by successive approximation techniques. In

Theorem 3.1 necessary and sufficient conditions for a P_1 -operator to have an inverse which is a P -operator are given. The inversion theorems obtained in § 2 and § 3 are related to some of Lane's results on stationary linear operators ([3] and [4]), and this relationship is also discussed in § 3. The author is grateful to the referee for his comments and suggestions in connection with this paper.

1. **Preliminary theorems.** In the main body of this paper it will be assumed that $[a, b]$ is a given number interval and that the statement " f is a left-continuous function on $[a, b]$ " means that f is a quasi-continuous function on $[a, b]$; i.e. f is the limit of a uniformly convergent sequence of step functions on $[a, b]$; and f is left-continuous at each point of (a, b) . All integrals referred to will be σ -mean Stieltjes integrals and the reader is referred to [2] or [5] for a definition. We will need the following lemma which is a trivial consequence of Corollaries 1.1 and 1.2 of [5] and the definition of the σ -mean Stieltjes integral.

LEMMA 1.1. *Suppose f is a bounded function on $[a, b]$ and g a function of bounded variation on $[a, b]$. If $\int_a^b f dg$ exists then*

$$\int_a^b |f(t)| d[V_{\xi=a}^t g(\xi)]$$

exists and

$$\left| \int_a^b f dg \right| \leq \int_a^b |f(t)| d[V_{\xi=a}^t g(\xi)].$$

The symbol \bar{Q}_L will denote the set of functions on $[a, b]$ to which a function f belongs if, and only if, f is left-continuous on $[a, b]$ and $f(a) = 0$. If f is in \bar{Q}_L then the norm of f is taken to be $\sup |f(s)|$ for s in $[a, b]$. It follows immediately from the properties of quasi-continuous functions that \bar{Q}_L is a Banach space. The following additional definitions will also be used.

DEFINITION 1.1. Suppose \bar{t} is a number, $a \leq \bar{t} < b$. The statement that $\tau_{\bar{t}}$ is a test function means that $\tau_{\bar{t}}$ is that function in \bar{Q}_L defined by $\tau_{\bar{t}}(s) = -J_L(s - \bar{t})$, $a \leq s \leq b$, where J_L denotes the function defined by $J_L(s) = 0$, $s \leq 0$, $J_L(s) = 1$, $s > 0$.

It is clear that the span of the set of all test functions is dense in \bar{Q}_L .

DEFINITION 1.2. The statement that A is a generating function means that A is a function on the square $a \leq \frac{t}{s} \leq b$ and that

- (i) $A(b, s) = 0, a \leq s \leq b,$
- (ii) for each number $\bar{s}, a \leq \bar{s} \leq b, A(t, \bar{s})$ is of bounded variation on $[a, b],$
- (iii) for each number $\bar{t}, a \leq \bar{t} < b, 1/2[A(\bar{t}, s) + A(\bar{t}+, s)]$ is in $\bar{Q}_L,$ and
- (iv) there exists a positive number $M,$ such that if \bar{s} is in $[a, b], V_{t=a}^b A(t, \bar{s}) \leq M.$ The smallest such number M will be denoted by $V_A.$ It is clear from this definition that any finite linear combination of generating functions is also a generating function.

If A is a generating function, then \bar{A} will denote the function defined by

$$\bar{A}(t, s) = \begin{cases} A(b, s) & t = b, \quad a \leq s \leq b \\ \frac{1}{2}[A(t, s) + A(t+, s)], & a \leq t < b, \quad a \leq s \leq b. \end{cases}$$

While the space \bar{Q}_L introduced here differs from the space Q_L studied in [1], it is easy to show that the basic results of [1] can also be developed for the space $\bar{Q}_L.$ In particular, Theorem 1.1, stated here without proof, can be established by the same techniques used for the analogous theorem in [1].

For the purposes of this paper, it will be assumed that operator means a continuous mapping whose domain is \bar{Q}_L and whose range is a subset of $\bar{Q}_L.$ The statement that an operator, $\mathcal{A},$ has an inverse will mean that the mapping inverse to \mathcal{A} is an operator.

THEOREM 1.1. *If A is a generating function then there exists a linear operator, $\mathcal{A},$ on \bar{Q}_L such that if s is in $[a, b]$ and f is in \bar{Q}_L then $\mathcal{A}f(s) = \int_a^b f(t)dA(t, s),$ with $\|\mathcal{A}\| \leq V_A.$ Conversely if \mathcal{B} is a linear operator on \bar{Q}_L then \mathcal{B} admits a representation of this type for some generating function, $B,$ with $V_B \leq 3 \|\mathcal{B}\|.$ Furthermore B is unique.*

COROLLARY 1.11. *Suppose \mathcal{A} is a linear operator on $\bar{Q}_L,$ and that A is the generating function for $\mathcal{A}.$ If \bar{t} is a number such that $a \leq \bar{t} < b,$ and $\tau_{\bar{t}}$ is a test function then $\mathcal{A}\tau_{\bar{t}}(s) = \bar{A}(\bar{t}, s), a \leq s \leq b.$*

Proof. By Theorem 1.1, $\mathcal{A}\tau_{\bar{t}}(s) = \int_a^b \tau_{\bar{t}}(\xi)dA(\xi, s) = \bar{A}(\bar{t}, s).$

It follows from this corollary that the generating function, $I,$ for the identity operator, $\mathcal{I},$ on \bar{Q}_L is given by $I(t, s) = -J_L(s - t), a \leq \frac{t}{s} \leq b.$ Also, if each of $\mathcal{K}, \mathcal{L},$ and \mathcal{M} is a linear operator on $\bar{Q}_L,$ with generating functions $K, L,$ and M respectively, and $\mathcal{K}\mathcal{L} = \mathcal{M}$

then $\bar{M}(t, s) = \int_a^b \bar{L}(t, \xi) dK(\xi, s)$, $a \leq \frac{t}{s} \leq b$.

2. P-operators.

DEFINITION 2.1. Suppose \mathcal{A} is a linear operator on \bar{Q}_L . The statement that \mathcal{A} is a P -operator means that if f is in \bar{Q}_L , and has the property that for some number c , $a < c < b$, $f(t) = 0$, $t \leq c$ then $\mathcal{A}f(s) = 0$, $s \leq c$.

It follows immediately from the definition that the identity operator, \mathcal{I} , is a P -operator; and that sums and products of P -operators are P -operators. A more interesting result however is:

THEOREM 2.1. Suppose \mathcal{K} is a linear operator with generating function K . A necessary and sufficient condition that \mathcal{K} be a P -operator is that $K(t, s) = 0$, $a \leq t \leq b$, $a \leq s \leq t$.

Proof. Since $K(t, s) = 2\bar{K}(t, s) - \bar{K}(t+, s)$, $a \leq t < b$, $a \leq s \leq b$, the necessity of this condition follows Corollary 1.11 and the definition of a test function.

Conversely, if $K(t, s) = 0$, $a \leq t \leq b$, $a \leq s \leq t$ then $\bar{K}(t, s) = 0$, $a \leq t \leq b$, $a \leq s \leq t$, and by Corollary 1.11 $\mathcal{K}\tau_i(s) = 0$, $a \leq s \leq \bar{t}$. If g is in \bar{Q}_L , and for some number k , $a < k < b$, $g(t) = 0$, $t \leq k$, then any sequence of linear combinations of test functions which converges to g need contain only test functions τ_i for which $\bar{t} \geq k$, therefore $\mathcal{K}g(s) = 0$, $s \leq k$, and \mathcal{K} is a P -operator.

From Theorem 2.1 and the properties of the mean integral, it is clear that if \mathcal{K} is a P -operator with generating function K and f is in \bar{Q}_L then $\mathcal{K}f(s) = \int_a^s f(t) dK(t, s)$, $a \leq s \leq b$; or in other words, a P -operator is an operator of Volterra type.

Throughout the remainder of this section it will be assumed that \mathcal{K} denotes a P -operator with generating function K . If \mathcal{K} has an inverse then the generating function for \mathcal{K}^{-1} will be denoted by $K^{(-1)}$.

The remaining theorems of this section are concerned with necessary conditions and sufficient conditions for \mathcal{K} to have an inverse which is a P -operator.

THEOREM 2.2. Suppose there exists a number \bar{t} , $a \leq \bar{t} < b$, such that $\mathcal{K}\tau_i(\bar{t}+) = \bar{K}(\bar{t}, \bar{t}+) = 0$. Then \mathcal{K} does not have an inverse which is a P -operator.

Proof. If \mathcal{K} has an inverse then $\mathcal{K}^{-1}\mathcal{K}\tau_i(\bar{t}+) = -1$. If \mathcal{L} is a P -operator then $\mathcal{L}\mathcal{K}\tau_i(s) = \int_i^s \mathcal{K}\tau_i(\xi) dL(\xi, s)$. Hence from

Lemma 1.1, $|\mathcal{L}\mathcal{K}\tau_i(s)| \leq \sup_{\xi \in [\bar{t}, s]} |\mathcal{K}\tau_i(\xi)| V_L$, and $\mathcal{L}\mathcal{K}\tau_i(\bar{t}+) = 0$.

If \bar{t} in Theorem 2.2 is a , then \mathcal{K} not only does not have an inverse which is a P -operator, \mathcal{K} does not have an inverse. This follows because $\mathcal{K}\tau_a(a+) = 0$ implies that for any $f \in Q_L$, $\mathcal{K}f(a+) = 0$, since for any P -operator \mathcal{K} , $\mathcal{K}\tau_i(a+) = 0$, $a < \bar{t} < b$. If \bar{t} is not a then \mathcal{K} may have an inverse. As an example, suppose that $[a, b]$ is $[0, 1]$ and let \mathcal{L} be the P -operator defined by:

$$\mathcal{L}\tau_t = \tau_{(3/2)t}, \quad 0 \leq t \leq 1/2; \quad \mathcal{L}\tau_t = \tau_{(1/2)(t+1)}, \quad 1/2 < t < 1.$$

Here, $\mathcal{L}\tau_i(t+) = 0$, $0 < t < 1$. However, \mathcal{L} has an inverse, \mathcal{L}^{-1} being the linear operator defined by:

$$\mathcal{L}^{-1}\tau_t = \tau_{(2/3)t}, \quad 0 \leq t \leq \frac{3}{4}; \quad \mathcal{L}^{-1}\tau_t = \tau_{2t-1}, \quad \frac{3}{4} < t < 1.$$

\mathcal{L}^{-1} is clearly not a P -operator.

If \mathcal{K} is a P -operator whose generating function has the property that for some number \bar{t} , $a \leq \bar{t} < b$ and every positive number c , there exists a positive number d such that if s is in $(\bar{t}, \bar{t} + d)$, $V_{\xi=\bar{t}}^s K(\xi, s) < c$, then $\mathcal{K}\tau_i(\bar{t}+) = 0$. This follows from Lemma 1.1 since

$$|\mathcal{K}\tau_i(s)| = \left| \int_{\bar{t}}^s \tau_i(\xi) dK(\xi, s) \right|, \quad \bar{t} \leq s \leq b.$$

This condition will be needed in § 3.

THEOREM 2.3. *Suppose that \mathcal{U} is an operator on \bar{Q}_L whose generating function has the property that for some number h ,*

$$0 < h < b - a, \quad U(t, s) = 0, \quad a \leq s \leq b, \quad s - h \leq t \leq b.$$

Then $\mathcal{L} = \mathcal{F} - \mathcal{U}$ has an inverse and $\mathcal{L}^{-1} = \sum_{n=0}^{p-1} \mathcal{U}^n$, where p is the smallest integer such that $a + ph \geq b$.

Proof. Suppose that g is in \bar{Q}_L . Since $U(t, s) = 0$, $a \leq s \leq b$, $s - h \leq t \leq b$, it follows that

$$(1) \quad \mathcal{U}g(s) = \begin{cases} 0, & a \leq s \leq a + h \\ \int_a^{s-h} g(t) dU(t, s), & a + h < s \leq b. \end{cases}$$

By successive applications of equation (1) it can be shown that if q is an integer, $q \geq p$, then $\mathcal{U}^q g(s) = 0$, $a \leq s \leq b$. The theorem then follows.

It should be noted that the hypothesis on \mathcal{U} implies that \mathcal{U} is a P -operator. Hence \mathcal{L} and \mathcal{L}^{-1} are P -operators.

THEOREM 2.4. *Suppose that K has the property that there exists a number k , $0 < k < b - a$, such that*

- (i) $K(s - k, s)$ is a left-continuous function on $[a + k, b]$, and
- (ii) the P -operator, \mathcal{A} , generated by

$$A(t, s) = \begin{cases} K(s - k, s), & a + k < s \leq b, a \leq t \leq s - k \\ K(t, s), & a \leq s \leq a + k, a \leq t \leq b \text{ or} \\ & a + k < s \leq b, s - k \leq t \leq b \end{cases}$$

has an inverse.

Then \mathcal{K} has an inverse, and \mathcal{K}^{-1} is a P -operator if, and only if, \mathcal{A}^{-1} is a P -operator.

Proof. There exists a unique operator \mathcal{L} such that $\mathcal{A}\mathcal{L} = \mathcal{K}$. From Theorem 1.1 and its corollary it then follows that if \bar{t} is a number in $[a, b]$ then $\bar{L}(\bar{t}, s)$ is the unique solution in \bar{Q}_L to the integral equation

$$\bar{K}(\bar{t}, s) = \int_a^s f(\bar{t}, \xi) dA(\xi, s), \quad a \leq s \leq b.$$

By direct computation it can be seen that

$$\bar{L}(\bar{t}, s) = -J_L(s - \bar{t}) = L(\bar{t}, s), \quad \text{if } a \leq s \leq a + k, a \leq \bar{t} \leq b,$$

or $a + k < s \leq b, s - k \leq \bar{t} \leq b$. Hence, $\mathcal{J} - \mathcal{L}$ satisfies the hypotheses of Theorem 2.3 and \mathcal{L} has an inverse. Therefore $\mathcal{K}^{-1} = \mathcal{L}^{-1}\mathcal{A}^{-1}$. The remaining assertions follow immediately since \mathcal{L} and \mathcal{L}^{-1} are P -operators. This completes the proof.

If there exists an integer q such that $\|\mathcal{J} - \mathcal{A}\|^q < 1$, then K satisfies the hypotheses of Theorem 2.4, and in this case, for each number t in $[a, b]$ $\bar{A}^{(-1)}$ is the successive approximations solution to the integral equation

$$-J_L(s - t) = g(t, s) - \int_a^s g(t, \xi) d[-J_L(s - \xi) - A(\xi, s)], \quad a \leq t \leq b,$$

and \bar{L} is the successive approximations solution to

$$\bar{K}(t, s) = f(t, s) - \int_a^s f(t, \xi) d[-J_L(s - \xi) - A(\xi, s)], \quad a \leq t \leq b;$$

the approximating sequences converging in both cases uniformly in s on $[a, b]$ for each number t in $[a, b]$. Consideration of the approximating sequence for $\bar{A}^{(-1)}$ shows that in this case \mathcal{A}^{-1} , and hence \mathcal{K}^{-1} , is a P -operator. In particular, if it is true that

$$(2) \quad V_{t=a}^b[-J_L(s-t) - A(t,s)] \leq M < 1, \quad a \leq s \leq b,$$

it can be shown that these two approximating sequences are in fact uniformly convergent on the square $a \leq \frac{t}{s} \leq b$.

If K is right continuous in t for each number s in $[a, b]$ and A satisfies equation (2) one would have usable computational techniques for the determination of $\bar{K}^{(-1)}$, since if K has this right continuity property then so does A . It can then be shown from the approximating sequences for $\bar{A}^{(-1)}$ and \bar{L} that they are also right continuous in t for each number s in $[a, b]$. Hence, in this case, $\bar{A}^{(-1)} = A^{(-1)}$ and $\bar{L} = L$. One would still have the problem of obtaining $K^{(-1)}$ from $\bar{K}^{(-1)}$, but for the solution of many operator problems knowledge of $\bar{K}^{(-1)}$ would suffice. The P_1 -operators to be considered in § 3 will have this right continuity property.

3. P_1 -operators. In this section we will consider a class of P -operators that are of interest in the study of electrical networks.

DEFINITION 3.1. The statement that a linear operator, \mathcal{S} , on \bar{Q}_L is a stationary operator means that if each of f and g is in \bar{Q}_L and for some number k , $0 < k < b - a$,

$$g(t) = \begin{cases} 0, & a \leq t \leq a + k \\ f(t - k), & a + k < t \leq b, \end{cases}$$

then

$$\mathcal{S}g(s) = \begin{cases} 0 & a \leq s \leq a + k \\ \mathcal{S}f(s - k), & a + k < s \leq b. \end{cases}$$

It follows from this definition, Definition 1.1, and Theorem 1.1, that a linear operator, \mathcal{S} , is stationary if, and only if, \mathcal{S} has a representation of the form, $\mathcal{S}f(s) = \int_a^s f(\xi)d[u(s - \xi)]$, where

$$u(t) = \begin{cases} 0, & a - b \leq t \leq 0 \\ \mathcal{S}\tau_a(t + a), & 0 < t \leq b - a. \end{cases}$$

Hence, every stationary operator is a P -operator. It is also a trivial consequence of Definition 3.1 and this representation that \mathcal{S} is a stationary operator and that sums and products of stationary operators are stationary.

It may also be concluded from this representation that the generating function for a stationary operator is right continuous in t for each s in $[a, b]$, and that u is of bounded variation on $[a - b, b - a]$. Furthermore if u^* denotes the function defined by $u^*(t) = V_{\xi=a-b}^t u(t)$,

$a - b \leq t \leq b - a$, then the mapping, \mathcal{S}^* , given by

$$\mathcal{S}^* f(s) = \int_a^s f(\xi) du^*(s - \xi), \quad a \leq s \leq b$$

is a stationary operator on \bar{Q}_L .

In [3] and [4], Lane has developed the theory of a class of linear operators, T_{ol} , on the set, QC_{ol} , of functions on the real line, which are quasi-continuous on each closed bounded interval, are everywhere left continuous, and vanish for negative values of their argument. A linear operator, \mathcal{L} , is in T_{ol} if, and only if, there exists a function, u , in QC_{ol} , of bounded variation on each closed bounded interval, such that if f is in QC_{ol} and s is a number then $\mathcal{L}f(s) = \int_0^\infty f(s - t) du(t)$. Using the properties of the integral this condition can be rewritten

$$\mathcal{L}f(s) = \begin{cases} 0, & s \leq 0 \\ \int_0^s f(\xi) d[-u(s - \xi)], & s > 0. \end{cases}$$

Thus, the stationary operators on \bar{Q}_L are analogous to Lane's T_{ol} operators.

DEFINITION 3.2. The statement that a bounded linear operator, \mathcal{K} , on \bar{Q}_L is a P_1 -operator means that there exists a stationary operator, \mathcal{S} , such that if each of τ_p and τ_q is a test function then

$$|\mathcal{K}(\tau_p - \tau_q)(s)| \leq |\mathcal{S}(\tau_p - \tau_q)(s)|, \quad a \leq s \leq b.$$

From Corollary 1.11, an equivalent form of Definition 3.2 is

$$|\bar{K}(p, s) - \bar{K}(q, s)| \leq |u(s - p) - u(s - q)| \leq |u^*(s - p) - u^*(s - q)|.$$

Therefore if \mathcal{K} is a P_1 -operator, $\bar{K} = K$. Also if \mathcal{S} dominates \mathcal{K} then so does \mathcal{S}^* . From this it may be shown by direct computation that if each of \mathcal{K}_1 and \mathcal{K}_2 is a P_1 -operator with dominating stationary operators \mathcal{S}_1 and \mathcal{S}_2 respectively then $\mathcal{S}_1^* + \mathcal{S}_2^*$ dominates $\mathcal{K}_1 + \mathcal{K}_2$ and $\mathcal{S}_1^* \mathcal{S}_2^*$ dominates $\mathcal{K}_1 \mathcal{K}_2$.

In the remainder of this section it will be assumed that \mathcal{K} denotes a P_1 -operator, with generating function K , and that \mathcal{S} is a dominating stationary operator for \mathcal{K} .

THEOREM 3.1. \mathcal{K} has an inverse which is a P -operator if, and only if, $\inf |K(s-, s)|$ for s in $(a, b]$ is not zero. Furthermore if $\lim_{s \rightarrow a+} K(s-, s) = 0$ then \mathcal{K} has no inverse.

Proof. It can be shown from Definition 3.2 that if h , $0 < h < b - a$, is a point of continuity of u , then $K(s - h, s)$ is a left continuous

function on $[a + h, b]$. It then follows that if f is a function on $[a, b]$ such that $f(s) = K(s-, s)$, $a < s \leq b$ then f is a left continuous function on $[a, b]$. Suppose that $f(a) \neq 0$ and $\inf_{s \in (a, b]} |K(s-, s)| = L > 0$. If M is a number, $0 < M < 1$, and h is a point of continuity of u such that $u^*(h) - u^*(0+) \leq ML$, then it follows from Definition 3.2 that for every number s in $[a, b]$,

$$V_{t=\varepsilon}^b \{ -J_L(s-t) - [-f(s)]^{-1}K(t, s) \} \leq M,$$

where ε is the larger of a and $s - h$. Let \mathcal{K}^* denote the P -operator defined by $\mathcal{K}^*g(s) = [-f(s)]^{-1}\mathcal{K}g(s)$, $a \leq s \leq b$, $g \in \bar{Q}_L$. Then \mathcal{K}^* satisfies the hypotheses of Theorem 2.4, since $\mathcal{F} - \mathcal{A}^*$ is a contraction mapping. Hence, \mathcal{K}^{*-1} exists and is a P -operator. It then follows immediately from Theorems 1.1 and 2.1 that \mathcal{K}^{-1} exists, and $K^{(-1)}(t, s) = [-f(s)]^{-1}K^{*(-1)}(t, s)$ so that \mathcal{K}^{-1} is also a P -operator.

If $\inf_{s \in (a, b]} |K(s-, s)| = 0$ then either there exists a number q in $[a, b)$ such that $\lim_{s \rightarrow q+} K(s-, s) = 0$ or a number p in $(a, b]$ such that $K(p-, p) = 0$. In the first case it can be shown from Definition 3.2 that if c is a positive number there exists a positive number d such that $V_{\xi=q}^s K(\xi, s) < c$, $q < s < q + d$. Hence $K(q, q+) = 0$, \mathcal{K} does not have an inverse which is a P -operator, and if $q = a$, \mathcal{K} has no inverse.

In the second case, it can be shown in a similar manner that if c is a positive number, there exists a positive number d such that if s and t are in $(p - d, p]$, $t < s$, then $V_{\xi=t}^s K(\xi, s) < c$. Hence $|\mathcal{K}\tau_t(s)| < c$ for t in $(p - d, p]$ and $t < s \leq p$ by Lemma 1.1. From this it follows that there exists a strictly increasing sequence of numbers, $\{t_n\}_{n=1}^\infty$, $t_n < p$, $n = 1, 2, 3, \dots$, such that if for each positive integer n , g_n is defined by

$$g_n(s) = \begin{cases} \mathcal{K}\tau_{t_n}(s), & a \leq s \leq p \\ 0, & p < s \leq b, \end{cases}$$

then the sequence $\{g_n\}_{n=1}^\infty$ converges uniformly to zero on $[a, b]$. Suppose now that \mathcal{K} has an inverse and \mathcal{K}^{-1} is a P -operator. Since \mathcal{K}^{-1} is bounded, $\{\mathcal{K}^{-1}g_n\}_{n=1}^\infty$ is also uniformly convergent on $[a, b]$. But $\mathcal{K}^{-1}(g_n - \mathcal{K}\tau_{t_n})(s) = 0$, $a \leq s \leq p$, if \mathcal{K}^{-1} is a P -operator. Or, $\mathcal{K}^{-1}g_n(s) = \tau_{t_n}(s)$, $a \leq s \leq p$, and $\{\tau_{t_n}\}_{n=1}^\infty$ is not uniformly convergent on $[a, p]$. This completes the proof.

If \mathcal{K} is a stationary operator then Theorem 3.1 yields a stronger result, because in this case $K(s-, s) = \mathcal{K}\tau_a(a+)$, $a < s \leq b$. Therefore either $K(s-, s) \equiv 0$ or $\inf |K(s-, s)| > 0$ on $(a, b]$. Consequently a necessary and sufficient condition that a stationary operator, \mathcal{K} , have an inverse is that $\mathcal{K}\tau_a(a+) \neq 0$. Furthermore the inverse of a stationary operator must be a P -operator, and it can be shown by

applying the construction used in the proof of Theorem 2.4, and the remarks following Theorem 2.4, to a stationary generating function that the inverse must be stationary also. This is analogous to Lane's result for the operators in T_{ol} [3].

From Theorem 3.1 and its special form for a stationary operator, it may be concluded that if \mathcal{K} is a P_1 -operator and there exists a stationary operator, \mathcal{S} , which has no inverse and dominates \mathcal{K} , then \mathcal{K} has no inverse, since $|\mathcal{K}\tau_a(s)| \leq |\mathcal{S}\tau_a(s)|$, $a \leq s \leq b$, from Definition 3.2.

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