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HOMOLOGICAL DIMENSION OF ORE-EXTENSIONS

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Let S be a ring with unit element and let $R = S\{x, d\}$ be the Ore-extension of S with respect to a derivation d of S. Our object in this paper is to show that $l.$ gl. $\dim R = 1 + l.$ gl. $\dim S$, if S is a commutative Noetherian ring and d is suitably restricted.

It was shown in [3] that l. gl. $\dim R \leqq 1 +$ l. gl. $\dim S$. While equality does not hold in general, we show that it does under suitable conditions (Theorem 2, § 5).

This is achieved in three steps. The first is to show that for any ring S, any R-module M and an S-projective resolution for M, there exists an R-projective resolution of M which "lifts" the given resolution (Theorem 1, § 3). The next step is to use this resolution to prove Theorem 2 in the special case in which S is a local ring (Proposition 1, § 4). The final step consists in deducing Theorem 2 by the method of localisation.

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2. **Preliminaries on Ore-extensions.** Let S be a ring with unit element (denoted by 1), which is not necessarily commutative, and let d be a derivation of S into itself. Let $S\{x, d\}$ denote the Ore-extension of S with respect to d (see [5]). We recall that $R = S\{x, d\}$ is the ring generated by an indeterminate x over S with the relations $xs - sx = ds$ for every $s \in S$. We identify S with a subring of R. We collect here some properties of R which will be used in the later sections.

(2.1) For any ring S', a ring homomorphism $\varphi: S \rightarrow S'$ and an element $\alpha \in S'$, with the property $\alpha\varphi(s) - \varphi(s)\alpha = \varphi(ds)$, there exists a unique ring homomorphism $\bar{\varphi}: R \rightarrow S'$ such that $\bar{\varphi}(x) = \alpha$ and $\bar{\varphi} \mid S = \varphi$. (In fact R can be characterised by this property).

The proof is straightforward.

(2.2) Let S_1, S_2 be rings with derivations d_1, d_2 respectively and let $\varphi: S_1 \rightarrow S_2$ be a ring homomorphism such that $d_2 \circ \varphi = \varphi \circ d_1$. Then there exists a ring homomorphism $\bar{\varphi}: R_1 \rightarrow R_2$ such that $\bar{\varphi} \mid S_1 = \varphi$.

Proof. This follows from (2.1) by taking $S' = R_2$ and $\alpha = x \in R_2$.

(2.3) A left S-module M can be converted to a left-R-module if

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and only if there exists an $f \in \mathrm{Hom}_Z(M, M)$ such that $f(s.m) - s.f(m) = ds.m$, for every $s \in S$, $m \in M$.

Proof. If M is an R-module we may take $f \in \mathrm{Hom}_Z(M, M)$ defined by $f(m) = x.m$. The converse follows from (2.1) by taking

$$S' = \mathrm{Hom}_Z(M, M), \alpha = f \quad \text{and} \quad \varphi: S \to S'$$

to be the mapping which defines the S-module structure on M.

(2.4) If M is a projective left S-module, then M can be converted into a left R-module.

Proof. We first remark that S can be considered as a left R-module. In fact, with the notation of (2.3) we choose $f = d \in \mathrm{Hom}_Z(S, S)$. By a direct sum argument, it is clear that any free left S-module can be regarded as an R-module. Now let M be any projective left S-module and let M be a direct summand of a free S-module F. Since F is a left R-module, there exists an $f \in \mathrm{Hom}_Z(F, F)$ such that $f(s.m) - s.f(m) = ds.m$; $s \in S$, $m \in F$. Let $p: F \to M$ be an S-projection of F on M. It is easily seen that $g = f \circ p \mid M$ satisfies $g(s.m) - s.g(m) = ds.m$. Hence M can be regarded as an R-module.

(2.5) R becomes a filtered ring by setting $F_pR = \sum_{0 \leqq i \leqq p} S.x^i$. The associated graded ring $E^\circ(R)$ of R is isomorphic to $S[x]$, the usual polynomial ring in one variable x over S.

Proof. See [3].

3. **Lifting of resolutions.** Let M be a left R-module and let

$$\cdots \longrightarrow X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be an S-projective resolution of M. Our aim in this section is to construct an R-projective resolution which "lifts" the above resolution.

We first prove the following

LEMMA. *There exist $f_i \in \mathrm{Hom}_Z(X_i, X_i)$ such that*
(i) $f_i(s.\alpha) - s.f_i(\alpha) = ds.\alpha$ *for* $s \in S$, $\alpha \in X_i$;
(ii) $d_i \circ f_i = f_{i-1} \circ d_i$, $i \geqq 1$, *and* $\varepsilon \circ f_0 = f \circ \varepsilon$,
where $f \in \mathrm{Hom}_Z(M, M)$ *is the mapping given by* $f(m) = x.m$.

Proof. Since X_0 is S-projective, it follows from (2.4) and (2.3) that there exists an $f_0' \in \mathrm{Hom}_Z(X_0, X_0)$ such that $f_0'(s\alpha) - sf_0'(\alpha) = ds, \alpha$ for $s \in S$, $\alpha \in X_0$. The map $\varepsilon \circ f_0' - f \circ \varepsilon : X_0 \to M$ is easily verified to be S-linear. Since X_0 is S-projective there exists an $f_0'' \in \mathrm{Hom}_S(X_0, X_0)$

such that $\varepsilon\circ f_0' - f\circ\varepsilon = \varepsilon\circ f_0''$. We choose $f_0 = f_0' - f_0''$. Then (i) and (ii) are verified for $i = 0$.

Assume inductively that f_j $0 \leqq j \leqq i-1$ have already been defined satisfying (i) and (ii). Since X_i is S-projective, there exists $f_i' \in \mathrm{Hom}_Z(X_i, X_i)$ such that $f_i'(s\alpha) - sf_i'(\alpha) = ds\alpha$ for $s \in S$, $\alpha \in X_i$. The map $d_i\circ f_i' - f_{i-1}\circ d_i : X_i \to X_{i-1}$ is easily verified to be S-linear. We have, (with the convention $f_1 = f$ and $d_0 = \varepsilon$),

$$\begin{aligned} d_{i-1}(d_i\circ f_i' - f_{i-1}\circ d_i) &= -d_{i-1}\circ f_{i-1}\circ d_i \\ &= -f_{i-2}\circ d_{i-1}\circ d_i \quad \text{(by induction)} \\ &= 0\,. \end{aligned}$$

Hence the image of X_i by $d_i\circ f_i' - f_{i-1}\circ d_i$ is contained in the kernel of $d_{i-1} = \mathrm{Im}.\, d_i$. Since X_i is S-projective, there exists $f_i'' \in \mathrm{Hom}_S(X_i, X_i)$ such that $d_i\circ f_i' - f_{i-1}\circ d_i = d_i\circ f_i''$. We may choose $f_i = f_i' - f_i''$ and f_i satisfies (i) and (ii). This completes the proof of the lemma.

We set $X_{-1} = 0$ and define for $i \geqq 0$

$$\bar{X} = R\otimes_S X_i + Ry\otimes_S X_{i-1}\,,$$

where y is a dummy. We set $d_0 = 0$ and define for $i \geqq 1$, the R-homomorphism $\bar{d}_i\colon \bar{X}_i \to \bar{X}_{i-1}$ by

$$\bar{d}_i(1\otimes\alpha') = 1\otimes d_i\alpha,\ \alpha\in X_i$$

and

$$\bar{d}_i(y\otimes\alpha') = y\otimes d_{i-1}\alpha' + (-1)^{i-1}x\otimes\alpha' + (-1)^i 1\otimes f_{i-1}(\alpha'),\ \alpha'\in X_{i-1}\,.$$

We define the R-homomorphism $\bar{\varepsilon}\colon \bar{X}_0 = R\otimes_S X_0 \to M$ by

$$\bar{\varepsilon}(1\otimes\alpha) = \varepsilon(\alpha),\ \alpha\in X_0\,.$$

THEOREM 1. *The sequence*

$$(*) \qquad \cdots \longrightarrow \bar{X}_i \xrightarrow{\bar{d}_i} \bar{X}_{i-1} \longrightarrow \cdots \longrightarrow \bar{X}_0 \xrightarrow{\bar{\varepsilon}} M \longrightarrow 0$$

is an R-projective resolution of M.

Proof. For $\alpha\in X_1$, $\bar{\varepsilon}\circ\bar{d}_1(1\otimes\alpha) = \bar{\varepsilon}(1\otimes d_1\alpha) = \varepsilon d_1(\alpha) = 0$, and for

$$\begin{aligned} \alpha'\in X_0,\ \bar{\varepsilon}\circ\bar{d}_1(y\otimes\alpha') &= \bar{\varepsilon}(x\otimes\alpha' - 1\otimes f_0(\alpha')) \\ &= f\circ\varepsilon(\alpha') - \varepsilon\circ f_0(\alpha') = 0\,. \end{aligned}$$

For $i \geqq 1$, we have

$$\bar{d}_{i-1}\circ\bar{d}_i(1\otimes\alpha) = 1\otimes d_{i-1}\circ d_i\alpha = 0,\ \alpha\in X_i\,,$$

and

$$\begin{aligned}
&\bar{d}_{i-1} \circ \bar{d}_i(y \otimes \alpha') \\
&\quad = \bar{d}_{i-1}[y \otimes d_{i-1}\alpha' + (-1)^{i-1}x \otimes \alpha' + (-1)^i 1 \otimes f_{i-1}(\alpha')],\ \alpha' \in X_{i-1} \\
&\quad = y \otimes d_{i-2} \circ d_{i-1} + (-1)^{i-2}x \otimes d_{i-1}\alpha' + (-1)^{i-1} 1 \otimes f_{i-2}d_{i-1}\alpha' \\
&\qquad + (-1)^{i-1}x \otimes d_{i-1}\alpha' + (-1)^i 1 \otimes d_{i-1} \circ f_{i-1}\alpha' \\
&\quad = (-1)^i 1 \otimes (d_{i-1} \circ f_{i-1} - f_{i-2} \circ d_{i-1}), \\
&\qquad\qquad\qquad\qquad \text{(with the convention that } f_{-1} = 0) \\
&\quad = 0\,.
\end{aligned}$$

Thus $(*)$ is a complex of left R-modules. To prove that the complex is acyclic, we define a suitable filtration on the complex whose associated graded is acyclic. By a well-known lemma on filtered complexes the acyclicity of $(*)$ follows immediately. For $i \geqq 0$, let

$$F_p\bar{X}_i = F_pR \underset{S}{\otimes} X_i + F_{p-1}R.\, y \underset{S}{\otimes} X_{i-1}\,,$$

where $\{F_pR\}$ is the filtration on R defined in (2.5). We define

$$F_pM = M \quad \text{for every } p\,.$$

It is easily seen that $\{F_p\bar{X}_i\}$ defines a filtration on $\bar{X}_i$ and that $\bar{d}_i(F_p\bar{X}_i) \subset F_p\bar{X}_{i-1}$ for $i \geqq 1$ and $\varepsilon(F_pX_0) \subset F_pM$. We thus get for $p \geqq 0$ the complex

$$\cdots \longrightarrow E_p^0(\bar{X}_i) \xrightarrow{E_p^0(\bar{d}_i)} E_p^0(\bar{X}_{i-1}) \longrightarrow \cdots \longrightarrow E_p^0(\bar{X}_0) \xrightarrow{E_p^0(\bar{\varepsilon})} E_p^0(M) \longrightarrow 0\,.$$

We note that $E_p^0(M) = 0$ for $p \neq 0$ and $E_0^0(M) = M$.

Let $S[x]$ denote the polynomial ring in one variable x over S. We regard M as an $S[x]$-module by setting $xM = 0$. We set $X'_{-1} = 0$ and define X'_i for $i \geqq 0$ by

$$X'_i = S_p[x] \underset{S}{\otimes} X_i + S_{p-1}[x]\cdot y \underset{S}{\otimes} X_{i-1}\,.$$

We set $d'_0 = 0$ and for $i \geqq 1$ define the left $S[x]$-homomorphism $d'_i \colon X'_i \to X'_{i-1}$ by

$$\begin{aligned}
&d'_i(1 \otimes \alpha) = 1 \otimes d_i\alpha,\ \alpha \in X_i\,, \\
&d'_i(y \otimes \alpha') = y \otimes d_{i-1}\alpha' + (-1)^{i-1}x \otimes \alpha',\ \alpha' \in X_{i-1}\,.
\end{aligned}$$

We define the $S[x]$-homomorphism $\varepsilon' \colon X'_0 \to M$ by setting

$$\varepsilon'(1 \otimes \alpha) = \varepsilon(\alpha)\,.$$

It is easily verified [4, p. 210] that (X'_i, d'_i) is a left $S[x]$-projective resolution for M.

Let $S_p[x]$ be the p^{th} homogeneous component of the usual gradation of $S[x]$ given by powers of x. We introduce a gradation on

X_i' by setting

$$X_i'^p = S_p[x] \underset{S}{\otimes} X_i + S_{p-1}[x]y \underset{S}{\otimes} X_{i-1}\,.$$

We take the trivial gradation on M i.e., $M^p = 0$ for $p > 0$ and $M^0 = M$. It is easily seen that $d_i'(X_i'^p) \subset X_{i-1}'^p$ and $\varepsilon'(X_0'^p) \subset M^p$ for every p. We thus get for every p an exact sequence

$$(**) \qquad \cdots \longrightarrow X_i'^p \xrightarrow{d_i'^p} X_{i-1}'^p \longrightarrow \cdots \longrightarrow X_0'^p \xrightarrow{\varepsilon'^p} M^p \longrightarrow 0\,.$$

Clearly $E_p^0(\bar{X}_i) \approx X_i'^p$ and $E_p^0(M) \approx M^p$ for every p. Since for any $r \in F_{p-1}R$ and $\alpha' \in X_{i-1}$, we have $r \otimes f_{i-1}(\alpha') \in F_{p-1}\bar{X}_{i-1}$, it follows that $E_p^0(\bar{d}_i) = d_i'^p$. Since $(**)$ is exact, it follows that $(E_p^0(\bar{X}_i), E_p^0(\bar{d}_i))$ is exact and hence $(*)$ is exact. Since $\bar{X}_i$ is clearly R-projective, the theorem is proved.

4. **The case of local rings.** Our aim in this section is to prove the following.

PROPOSITION 1. Let S be a (commutative, Noetherian) local ring and let $\mathfrak{M}$ denote its unique maximal ideal. Let d be a derivation of S such that $d(S) \subset \mathfrak{M}$ and let $R = S\{x, d\}$. Then

$$\text{l.gl. dim}\, R = 1 + \text{gl. dim}\, S\,.$$

For proving this proposition, we need the following.

LEMMA. *Let S be a commutative ring and let M be an R-module. Suppose*

$$0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

is an S-projective resolution of M. Assume that the following conditions hold.

(1) *X_n is S-free of rank* 1.

(2) *There exists an S-module N with $xN = 0$ and* $\text{Ext}_S^n(M, N) \neq (0)$.

Then $hd_R M = n + 1$.

Proof. Using the complex $(*)$ of Theorem 1, we find that $hd_R M \leqq n + 1$. We now compute $\text{Ext}_R^{n+1}(M, N')$ for any R-module N'. We have

$$\text{Ext}_R^{n+1}(M, N') = \text{Hom}_S(X_n, N')/B^n$$

where B^n is the set of all $g \in \text{Hom}_S(X_n, N')$ such that there exist $g_1 \in \text{Hom}_S(X_n, N')$ and $g_2 \in \text{Hom}_S(X_{n-1}, N')$ with

$$g(\alpha) = g_2(d_n\alpha) + (-1)^{n-1}xg_1(\alpha) + (-1)^n g_1(f_n(\alpha))$$

for any $\alpha \in X_n$.

Let β be a free generator of X_n as an S-module and let $f_n(\beta) = s\beta$; $s \in S$. If $g \in B^n$, we have

$$g(\beta) = g_2(d_n\beta) + (-1)^{n-1}(x - s)g_1(\beta) .$$

Let θ be the automorphism of R such that $\theta(x) = x + s$ and $\theta \mid S =$ identity. (This exists in view of (2.1)). If we choose $N' = {}_\theta N$ (i.e., N considered as an R-module through θ), we find $g(\beta) = g_2(d_n\beta)$ and hence $g(\alpha) = g_2(d_n\alpha)$ for any $\alpha \in X_n$. Thus, $B^n = B_1^n = \{g \in \mathrm{Hom}_S(X_n, N') \mid g(\alpha) = g_2(d_n\alpha)$ for some $g_2 \in \mathrm{Hom}_S(X_{n-1}N')$ for every $\alpha \in X_{n-1}\}$. However, using the resolution (X_i, d_i) for M to compute Ext, we find $\mathrm{Ext}_S^n(M, N') \approx \mathrm{Hom}_S(X_n, N')/B_1^n$. Hence

$$\begin{aligned} \mathrm{Ext}_R^{n+1}(M, N') &\approx \mathrm{Ext}_S^n(M, N') \\ &\approx \mathrm{Ext}_S^n(M, N) \neq (0) , \end{aligned}$$

since N and N' are isomorphic as S-modules. This proves the lemma.

Proof of proposition. By [2, p. 74, Prop. 2], it follows that $\mathrm{gl.\,dim}\, R \geqq \mathrm{gl.\,dim}\, S$. Thus, if $\mathrm{gl.\,dim}\, S = \infty$, we have $\mathrm{gl.\,dim}\, R = \infty$ and the proposition is proved. We therefore assume that $\mathrm{gl.\,dim}\, S = n < \infty$. If $M = S/\mathfrak{M}$, we have $hd_S M = n$. Let

$$0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

be the "Koszul resolution" for M [1, p. 151]. Since $X_n = E_n^s(y_1, \cdots, y_n)$, where $E_n^S(y_1, \cdots, y_n)$ is the nth component of the exterior algebra on $y_1, \cdots, y_n$ over S, condition (i) of the above lemma is satisfied. Since $d(S) \subset \mathfrak{M}$, it is clear that M can be regarded as an R-module satisfying $xM = 0$ (See (2.3)). Since $\mathrm{Ext}_S^n(M, M) \neq (0)$, [1, p. 153], condition (2) of the lemma is satisfied with $N = M$. Thus, by the above lemma, we have $hd_R M = n + 1$. Hence $\mathrm{gl.\,dim}\, R \geqq n + 1$. Since $\mathrm{gl.\,dim}\, R \leqq n + 1$ [6, Th. 1 or 3], the proposition is proved.

5. The case of Noetherian rings. In this section, we prove the following

THEOREM 2. *Let S be a commutative Noetherian ring and let d be a derivation of S such that any one of the following two conditions is satisfied:*

(1) *$d(S) \subset$ Radical of S,*

(2) *$d(S)$ generates a proper ideal of S and Krull dim $S_{\mathfrak{M}}$ is the same for all the maximal ideals $\mathfrak{M}$ of S.*

If $R = S\{x, d\}$, we have

$$l.\,gl.\,dim\, R = 1 + gl.\,dim\, S .$$

Proof. As in the proof of Proposition 1, we need only prove that l. gl. dim $R \geqq 1 +$ gl. dim S assuming gl. dim $S < \infty$. Since gl. dim $S = \sup_{\mathfrak{M}}$ gl. dim $S_{\mathfrak{M}}$ where $\mathfrak{M}$ runs over all the maximal ideals of S, it is clear that under either of the conditions of the theorem, there exists a maximal ideal $\mathfrak{M}$ such that gl. dim $S =$ gl. dim $S_{\mathfrak{M}}$ and $d(S) \subset \mathfrak{M}$. The derivation d of S induces a derivation $\bar{d}$ of $S_{\mathfrak{M}}$ if we set

$$\bar{d}\left(\frac{s}{s'}\right) = \frac{ds.\, s' - s.\, ds'}{s'^2}\,;\ s,\, s' \in S,\ s' \in \mathfrak{M}\ .$$

It is clear that $\bar{d}(S_{\mathfrak{M}}) \subset \mathfrak{M}S_{\mathfrak{M}}$. Hence by Proposition 1, § 4, we have

$$\begin{aligned} \text{l. gl. dim}\, S_{\mathfrak{M}}\{x, \bar{d}\} &= 1 + \text{gl. dim}\, S_{\mathfrak{M}} \\ &= 1 + \text{gl. dim}\, S\ . \end{aligned}$$

Thus, the theorem will be proved if we prove the following

LEMMA. *If $\mathfrak{M}$ is any maximal ideal of S, we have*

$$\text{l. gl. dim}\, S\{x, d\} \geqq \text{l. gl. dim}\, S_{\mathfrak{M}}\{x, \bar{d}\}\ .$$

Proof of the lemma. Let us set $R = S\{x, d\}$ and $\bar{R} = S_m\{x, \bar{d}\}$. Let $\eta\colon S \to S_{\mathfrak{M}}$ denote the ring homomorphism defined by $\eta(s) =$ class of $s/1$. Since $\bar{d} \circ \eta = \eta \circ d$, η induces (see (2.2)) a ring homomorphism $\bar{\eta}\colon R \to \bar{R}$ such that $\bar{\eta} \mid S = \eta$.

We first prove the following two statements:

(1) $\bar{R}$ is R-flat as a right R-module (through $\bar{\eta}$).

(2) If M is any left $\bar{R}$-module, there exists a left R-module M' and a left $\bar{R}$-isomorphism $M \approx \bar{R} \otimes_R M'$.

The left $S_{\mathfrak{M}}$-isomorphism $\varphi : S_{\mathfrak{M}} \otimes_S R \to \bar{R}$ given by $\varphi(1 \otimes x^i) = x^i \in \bar{R}$ satisfies $\varphi(1 \otimes f) = \bar{\eta}(f)$ for any $f \in R$. We have

$$\varphi(1 \otimes fg) = \bar{\eta}(fg) = \bar{\eta}(f)\bar{\eta}(g) = \varphi(1 \otimes f)\bar{\eta}(g)\ .$$

Thus, φ is an isomorphism of right R-modules. Since $S_{\mathfrak{M}} \otimes_S R$ is right R-flat, (1) is proved. Let

$$\bar{F}_1 \xrightarrow{\ \lambda\ } \bar{F} \xrightarrow{\ \mu\ } M \longrightarrow 0$$

be an exact sequence where $\bar{F}_1$ and $\bar{F}$ are $\bar{R}$-free with bases $\{e_\alpha\}$ and $\{f_\beta\}$ respectively. We then have

$$\lambda(e_\alpha) = \eta\left(\frac{1}{s_\alpha}\right) \sum_\beta \frac{1}{\eta}\, (a_{\alpha\beta}) f_\beta;\ a_{\alpha\beta} \in R,\ s_\alpha \in S - \mathfrak{M}\ .$$

Let θ be the $\bar{R}$-automorphism of $\bar{F}_1$ defined by $\theta(e_\alpha) = \eta(s_\alpha)e_\alpha$. Let

$\lambda' = \lambda \circ \theta$. We then have

$$\lambda'(e_\alpha) = \sum_\beta \frac{1}{\eta}(a_{\alpha\beta})f_\beta ,$$

and the sequence

$$\bar{F}_1 \xrightarrow{\lambda'} \bar{F} \xrightarrow{\mu} M \longrightarrow 0$$

is exact. Let F_1 (resp. F) be the free R-module generated by $\{e_\alpha\}$ (resp. $\{f_\beta\}$) and let $\lambda'': F_1 \to F$ be the R-homomorphism defined by

$$\lambda''(e_\alpha) = \sum_\beta a_{\alpha\beta} f_\beta .$$

It is easily seen that if we take $M' = c \circ \ker \lambda''$, we have $M \approx \bar{R} \otimes_R M'$. This proves(2). We now complete the proof of the lemma.

Let M be any left $\bar{R}$-module and let M' be a left R-module such that (2) is satisfied. Let

$$\cdots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M' \longrightarrow 0$$

be a resolution of M' as a left R-module. Then

$$\bar{R} \otimes_R X_n \xrightarrow{I \otimes d_n} \bar{R} \otimes_R X_{n-1} \longrightarrow \cdots \longrightarrow \bar{R} \otimes_R X_0 \longrightarrow M \longrightarrow 0$$

is exact in view of (1). Since $\bar{R} \otimes_R X_i$ is $\bar{R}$-projective, it follows that $(\bar{R} \otimes_R X_i, 1 \otimes d_i)$ is an $\bar{R}$-projective resolution of M. In particular, we have $hd_{\bar{R}} M \leqq hd_R M' \leqq \mathrm{gl.\,dim}\, R$. Since M is arbitrary, it follows that $\mathrm{gl.\,dim}\, \bar{R} \leqq \mathrm{gl.\,dim}\, R$. This proves the lemma and hence the theorem.

REMARK. Let $S = K[x_1, \cdots, x_n]$ be the polynomial ring in n variables over a field K. It is well-known [7, Chap. III Cor. 4 to Th. 5] that Krull $\dim S_{\mathfrak{M}}$ is the same for all maximal ideals $\mathfrak{M}$ of S. Let d be a K-derivation of S given by $d(x_i) = f_i$. Then the derivation d satisfies condition (2) of Theorem 2 if and only if f_i, $1 \leqq i \leqq n$ are not coprime and in this case we may apply the theorem and we have $\mathrm{gl.\,dim}\, R = n + 1$. This includes the special case of Theorem 1 of [6] in which K is a field.

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