

Pacific Journal of Mathematics

HOMOLOGICAL DIMENSION OF ORE-EXTENSIONS

N. S. GOPALAKRISHNAN AND RAMAIYENGAR SRIDHARAN

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Let S be a ring with unit element and let $R = S\{x, d\}$ be the Ore-extension of S with respect to a derivation d of S . Our object in this paper is to show that $l. \text{ gl. dim } R = 1 + l. \text{ gl. dim } S$, if S is a commutative Noetherian ring and d is suitably restricted.

It was shown in [3] that $l. \text{ gl. dim } R \leq 1 + l. \text{ gl. dim } S$. While equality does not hold in general, we show that it does under suitable conditions (Theorem 2, § 5).

This is achieved in three steps. The first is to show that for any ring S , any R -module M and an S -projective resolution for M , there exists an R -projective resolution of M which "lifts" the given resolution (Theorem 1, § 3). The next step is to use this resolution to prove Theorem 2 in the special case in which S is a local ring (Proposition 1, § 4). The final step consists in deducing Theorem 2 by the method of localisation.

The authors would like to express their thanks to M. P. Murthy and A. Roy for their kind help during the preparation of this paper.

2. Preliminaries on Ore-extensions. Let S be a ring with unit element (denoted by 1), which is not necessarily commutative, and let d be a derivation of S into itself. Let $S\{x, d\}$ denote the Ore-extension of S with respect to d (see [5]). We recall that $R = S\{x, d\}$ is the ring generated by an indeterminate x over S with the relations $xs - sx = ds$ for every $s \in S$. We identify S with a subring of R . We collect here some properties of R which will be used in the later sections.

(2.1) For any ring S' , a ring homomorphism $\varphi: S \rightarrow S'$ and an element $\alpha \in S'$, with the property $\alpha\varphi(s) - \varphi(s)\alpha = \varphi(ds)$, there exists a unique ring homomorphism $\bar{\varphi}: R \rightarrow S'$ such that $\bar{\varphi}(x) = \alpha$ and $\bar{\varphi}|_S = \varphi$. (In fact R can be characterised by this property).

The proof is straightforward.

(2.2) Let S_1, S_2 be rings with derivations d_1, d_2 respectively and let $\varphi: S_1 \rightarrow S_2$ be a ring homomorphism such that $d_2 \circ \varphi = \varphi \circ d_1$. Then there exists a ring homomorphism $\bar{\varphi}: R_1 \rightarrow R_2$ such that $\bar{\varphi}|_{S_1} = \varphi$.

Proof. This follows from (2.1) by taking $S' = R_2$ and $\alpha = x \in R_2$.

(2.3) A left S -module M can be converted to a left- R -module if

and only if there exists an $f \in \text{Hom}_Z(M, M)$ such that $f(s.m) - s.f(m) = ds.m$, for every $s \in S, m \in M$.

Proof. If M is an R -module we may take $f \in \text{Hom}_Z(M, M)$ defined by $f(m) = x.m$. The converse follows from (2.1) by taking

$$S' = \text{Hom}_Z(M, M), \alpha = f \quad \text{and} \quad \varphi: S \rightarrow S'$$

to be the mapping which defines the S -module structure on M .

(2.4) If M is a projective left S -module, then M can be converted into a left R -module.

Proof. We first remark that S can be considered as a left R -module. In fact, with the notation of (2.3) we choose $f = d \in \text{Hom}_Z(S, S)$. By a direct sum argument, it is clear that any free left S -module can be regarded as an R -module. Now let M be any projective left S -module and let M be a direct summand of a free S -module F . Since F is a left R -module, there exists an $f \in \text{Hom}_Z(F, F)$ such that $f(s.m) - s.f(m) = ds.m; s \in S, m \in F$. Let $p: F \rightarrow M$ be an S -projection of F on M . It is easily seen that $g = f \circ p|_M$ satisfies $g(s.m) - s.g(m) = ds.m$. Hence M can be regarded as an R -module.

(2.5) R becomes a filtered ring by setting $F_p R = \sum_{0 \leq i \leq p} S.x^i$. The associated graded ring $E^\circ(R)$ of R is isomorphic to $S[x]$, the usual polynomial ring in one variable x over S .

Proof. See [3].

3. Lifting of resolutions. Let M be a left R -module and let

$$\cdots \longrightarrow X_i \xrightarrow{d_i} X_{i-1} \longrightarrow \cdots \longrightarrow X_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

be an S -projective resolution of M . Our aim in this section is to construct an R -projective resolution which ‘‘lifts’’ the above resolution.

We first prove the following

LEMMA. *There exist $f_i \in \text{Hom}_Z(X_i, X_i)$ such that*

(i) $f_i(s.\alpha) - s.f_i(\alpha) = ds.\alpha$ for $s \in S, \alpha \in X_i$;

(ii) $d_i \circ f_i = f_{i-1} \circ d_i, i \geq 1$, and $\varepsilon \circ f_0 = f \circ \varepsilon$,

where $f \in \text{Hom}_Z(M, M)$ is the mapping given by $f(m) = x.m$.

Proof. Since X_0 is S -projective, it follows from (2.4) and (2.3) that there exists an $f'_0 \in \text{Hom}_Z(X_0, X_0)$ such that $f'_0(s\alpha) - sf'_0(\alpha) = ds.\alpha$ for $s \in S, \alpha \in X_0$. The map $\varepsilon \circ f'_0 - f \circ \varepsilon: X_0 \rightarrow M$ is easily verified to be S -linear. Since X_0 is S -projective there exists an $f''_0 \in \text{Hom}_S(X_0, X_0)$

such that $\varepsilon \circ f'_0 - f_0 \circ \varepsilon = \varepsilon \circ f''_0$. We choose $f_0 = f'_0 - f''_0$. Then (i) and (ii) are verified for $i = 0$.

Assume inductively that f_j $0 \leq j \leq i - 1$ have already been defined satisfying (i) and (ii). Since X_i is S -projective, there exists $f'_i \in \text{Hom}_Z(X_i, X_i)$ such that $f'_i(s\alpha) - sf'_i(\alpha) = ds\alpha$ for $s \in S, \alpha \in X_i$. The map $d_i \circ f'_i - f_{i-1} \circ d_i : X_i \rightarrow X_{i-1}$ is easily verified to be S -linear. We have, (with the convention $f_1 = f$ and $d_0 = \varepsilon$),

$$\begin{aligned} d_{i-1}(d_i \circ f'_i - f_{i-1} \circ d_i) &= -d_{i-1} \circ f_{i-1} \circ d_i \\ &= -f_{i-2} \circ d_{i-1} \circ d_i \quad (\text{by induction}) \\ &= 0. \end{aligned}$$

Hence the image of X_i by $d_i \circ f'_i - f_{i-1} \circ d_i$ is contained in the kernel of $d_{i-1} = \text{Im. } d_i$. Since X_i is S -projective, there exists $f''_i \in \text{Hom}_S(X_i, X_i)$ such that $d_i \circ f'_i - f_{i-1} \circ d_i = d_i \circ f''_i$. We may choose $f_i = f'_i - f''_i$ and f_i satisfies (i) and (ii). This completes the proof of the lemma.

We set $X_{-1} = 0$ and define for $i \geq 0$

$$\bar{X} = R \otimes_S X_i + Ry \otimes_S X_{i-1},$$

where y is a dummy. We set $d_0 = 0$ and define for $i \geq 1$, the R -homomorphism $\bar{d}_i : \bar{X}_i \rightarrow \bar{X}_{i-1}$ by

$$\bar{d}_i(1 \otimes \alpha') = 1 \otimes d_i \alpha, \alpha \in X_i$$

and

$$\bar{d}_i(y \otimes \alpha') = y \otimes d_{i-1} \alpha' + (-1)^{i-1} x \otimes \alpha' + (-1)^i 1 \otimes f_{i-1}(\alpha'), \alpha' \in X_{i-1}.$$

We define the R -homomorphism $\bar{\varepsilon} : \bar{X}_0 = R \otimes_S X_0 \rightarrow M$ by

$$\bar{\varepsilon}(1 \otimes \alpha) = \varepsilon(\alpha), \alpha \in X_0.$$

THEOREM 1. *The sequence*

$$(*) \quad \dots \longrightarrow \bar{X}_i \xrightarrow{\bar{d}_i} \bar{X}_{i-1} \longrightarrow \dots \longrightarrow \bar{X}_0 \xrightarrow{\bar{\varepsilon}} M \longrightarrow 0$$

is an R -projective resolution of M .

Proof. For $\alpha \in X_1, \bar{\varepsilon} \circ \bar{d}_1(1 \otimes \alpha) = \bar{\varepsilon}(1 \otimes d_1 \alpha) = \varepsilon d_1(\alpha) = 0$, and for

$$\begin{aligned} \alpha' \in X_0, \bar{\varepsilon} \circ \bar{d}_1(y \otimes \alpha') &= \bar{\varepsilon}(x \otimes \alpha' - 1 \otimes f_0(\alpha')) \\ &= f_0 \circ \varepsilon(\alpha') - \varepsilon \circ f_0(\alpha') = 0. \end{aligned}$$

For $i \geq 1$, we have

$$\bar{d}_{i-1} \circ \bar{d}_i(1 \otimes \alpha) = 1 \otimes d_{i-1} \circ d_i \alpha = 0, \alpha \in X_i,$$

and

$$\begin{aligned}
& \bar{d}_{i-1} \circ \bar{d}_i(y \otimes \alpha') \\
&= \bar{d}_{i-1}[y \otimes d_{i-1}\alpha' + (-1)^{i-1}x \otimes \alpha' + (-1)^i 1 \otimes f_{i-1}(\alpha')], \alpha' \in X_{i-1} \\
&= y \otimes d_{i-2} \circ d_{i-1} + (-1)^{i-2}x \otimes d_{i-1}\alpha' + (-1)^{i-1} 1 \otimes f_{i-2}d_{i-1}\alpha' \\
&\quad + (-1)^{i-1}x \otimes d_{i-1}\alpha' + (-1)^i 1 \otimes d_{i-1} \circ f_{i-1}\alpha' \\
&= (-1)^i 1 \otimes (d_{i-1} \circ f_{i-1} - f_{i-2} \circ d_{i-1}), \\
&\hspace{15em} \text{(with the convention that } f_{-1} = 0) \\
&= 0.
\end{aligned}$$

Thus (*) is a complex of left R -modules. To prove that the complex is acyclic, we define a suitable filtration on the complex whose associated graded is acyclic. By a well-known lemma on filtered complexes the acyclicity of (*) follows immediately. For $i \geq 0$, let

$$F_p \bar{X}_i = F_p R \otimes_S X_i + F_{p-1} R \cdot y \otimes_S X_{i-1},$$

where $\{F_p R\}$ is the filtration on R defined in (2.5). We define

$$F_p M = M \text{ for every } p.$$

[It is easily seen that $\{F_p \bar{X}_i\}$ defines a filtration on \bar{X}_i and that $\bar{d}_i(F_p \bar{X}_i) \subset F_p \bar{X}_{i-1}$ for $i \geq 1$ and $\varepsilon(F_p X_0) \subset F_p M$. We thus get for $p \geq 0$ the complex

$$\dots \longrightarrow E_p^0(\bar{X}_i) \xrightarrow{E_p^0(\bar{d}_i)} E_p^0(\bar{X}_{i-1}) \longrightarrow \dots \longrightarrow E_p^0(\bar{X}_0) \xrightarrow{E_p^0(\varepsilon)} E_p^0(M) \longrightarrow 0.$$

We note that $E_p^0(M) = 0$ for $p \neq 0$ and $E_0^0(M) = M$.

Let $S[x]$ denote the polynomial ring in one variable x over S . We regard M as an $S[x]$ -module by setting $xM = 0$. We set $X'_{-1} = 0$ and define X'_i for $i \geq 0$ by

$$X'_i = S_p[x] \otimes_S X_i + S_{p-1}[x] \cdot y \otimes_S X_{i-1}.$$

We set $d'_0 = 0$ and for $i \geq 1$ define the left $S[x]$ -homomorphism $d'_i: X'_i \rightarrow X'_{i-1}$ by

$$\begin{aligned}
d'_i(1 \otimes \alpha) &= 1 \otimes d_i \alpha, \alpha \in X_i, \\
d'_i(y \otimes \alpha') &= y \otimes d_{i-1} \alpha' + (-1)^{i-1} x \otimes \alpha', \alpha' \in X_{i-1}.
\end{aligned}$$

We define the $S[x]$ -homomorphism $\varepsilon': X'_0 \rightarrow M$ by setting

$$\varepsilon'(1 \otimes \alpha) = \varepsilon(\alpha).$$

It is easily verified [4, p. 210] that (X'_i, d'_i) is a left $S[x]$ -projective resolution for M .

Let $S_p[x]$ be the p^{th} homogeneous component of the usual gradation of $S[x]$ given by powers of x . We introduce a gradation on

X'_i by setting

$$X'_i{}^p = S_p[x] \otimes_S X_i + S_{p-1}[x]y \otimes_S X_{i-1}.$$

We take the trivial gradation on M i.e., $M^p = 0$ for $p > 0$ and $M^0 = M$. It is easily seen that $d'_i(X'_i{}^p) \subset X'_{i-1}{}^p$ and $\varepsilon'(X'_0{}^p) \subset M^p$ for every p . We thus get for every p an exact sequence

$$(**) \quad \dots \longrightarrow X'_i{}^p \xrightarrow{d'_i{}^p} X'_{i-1}{}^p \longrightarrow \dots \longrightarrow X'_0{}^p \xrightarrow{\varepsilon'{}^p} M^p \longrightarrow 0.$$

Clearly $E_p^0(\bar{X}_i) \approx X'_i{}^p$ and $E_p^0(M) \approx M^p$ for every p . Since for any $r \in F_{p-1}R$ and $\alpha' \in X_{i-1}$, we have $r \otimes f_{i-1}(\alpha') \in F_{p-1}\bar{X}_{i-1}$, it follows that $E_p^0(\bar{d}_i) = d'_i{}^p$. Since $(**)$ is exact, it follows that $(E_p^0(\bar{X}_i), E_p^0(\bar{d}_i))$ is exact and hence $(*)$ is exact. Since \bar{X}_i is clearly R -projective, the theorem is proved.

4. The case of local rings. Our aim in this section is to prove the following.

PROPOSITION 1. Let S be a (commutative, Noetherian) local ring and let \mathfrak{M} denote its unique maximal ideal. Let d be a derivation of S such that $d(S) \subset \mathfrak{M}$ and let $R = S\{x, d\}$. Then

$$\text{l.gl. dim } R = 1 + \text{gl. dim } S.$$

For proving this proposition, we need the following.

LEMMA. Let S be a commutative ring and let M be an R -module. Suppose

$$0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} \dots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

is an S -projective resolution of M . Assume that the following conditions hold.

(1) X_n is S -free of rank 1.

(2) There exists an S -module N with $xN = 0$ and $\text{Ext}_S^n(M, N) \neq$

(0).

Then $hd_R M = n + 1$.

Proof. Using the complex $(*)$ of Theorem 1, we find that $hd_R M \leq n + 1$. We now compute $\text{Ext}_R^{n+1}(M, N')$ for any R -module N' . We have

$$\text{Ext}_R^{n+1}(M, N') = \text{Hom}_S(X_n, N')/B^n$$

where B^n is the set of all $g \in \text{Hom}_S(X_n, N')$ such that there exist $g_1 \in \text{Hom}_S(X_n, N')$ and $g_2 \in \text{Hom}_S(X_{n-1}, N')$ with

$$g(\alpha) = g_2(d_n \alpha) + (-1)^{n-1} x g_1(\alpha) + (-1)^n g_1(f_n(\alpha))$$

for any $\alpha \in X_n$.

Let β be a free generator of X_n as an S -module and let $f_n(\beta) = s\beta$; $s \in S$. If $g \in B^n$, we have

$$g(\beta) = g_2(d_n\beta) + (-1)^{n-1}(x-s)g_1(\beta).$$

Let θ be the automorphism of R such that $\theta(x) = x + s$ and $\theta|_S =$ identity. (This exists in view of (2.1)). If we choose $N' = {}_\theta N$ (i.e., N considered as an R -module through θ), we find $g(\beta) = g_2(d_n\beta)$ and hence $g(\alpha) = g_2(d_n\alpha)$ for any $\alpha \in X_n$. Thus, $B^n = B_1^n = \{g \in \text{Hom}_S(X_n, N') \mid g(\alpha) = g_2(d_n\alpha) \text{ for some } g_2 \in \text{Hom}_S(X_{n-1}, N') \text{ for every } \alpha \in X_{n-1}\}$. However, using the resolution (X_i, d_i) for M to compute Ext , we find $\text{Ext}_S^2(M, N') \approx \text{Hom}_S(X_n, N')/B_1^n$. Hence

$$\begin{aligned} \text{Ext}_R^{n+1}(M, N') &\approx \text{Ext}_S^2(M, N') \\ &\approx \text{Ext}_S^2(M, N) \neq (0), \end{aligned}$$

since N and N' are isomorphic as S -modules. This proves the lemma.

Proof of proposition. By [2, p. 74, Prop. 2], it follows that $\text{gl. dim } R \geq \text{gl. dim } S$. Thus, if $\text{gl. dim } S = \infty$, we have $\text{gl. dim } R = \infty$ and the proposition is proved. We therefore assume that $\text{gl. dim } S = n < \infty$. If $M = S/\mathfrak{M}$, we have $hd_S M = n$. Let

$$0 \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \cdots \longrightarrow X_0 \longrightarrow M \longrightarrow 0$$

be the ‘‘Koszul resolution’’ for M [1, p. 151]. Since $X_n = E_n^s(y_1, \dots, y_n)$, where $E_n^s(y_1, \dots, y_n)$ is the n th component of the exterior algebra on y_1, \dots, y_n over S , condition (i) of the above lemma is satisfied. Since $d(S) \subset \mathfrak{M}$, it is clear that M can be regarded as an R -module satisfying $xM = 0$ (See (2.3)). Since $\text{Ext}_S^2(M, M) \neq (0)$, [1, p. 153], condition (2) of the lemma is satisfied with $N = M$. Thus, by the above lemma, we have $hd_R M = n + 1$. Hence $\text{gl. dim } R \geq n + 1$. Since $\text{gl. dim } R \leq n + 1$ [6, Th. 1 or 3], the proposition is proved.

5. The case of Noetherian rings. In this section, we prove the following

THEOREM 2. *Let S be a commutative Noetherian ring and let d be a derivation of S such that any one of the following two conditions is satisfied:*

(1) $d(S) \subset \text{Radical of } S$,

(2) $d(S)$ generates a proper ideal of S and $\text{Krull dim } S_{\mathfrak{M}}$ is the same for all the maximal ideals \mathfrak{M} of S .

If $R = S\{x, d\}$, we have

$$l. \text{ gl. dim } R = 1 + \text{gl. dim } S.$$

Proof. As in the proof of Proposition 1, we need only prove that $\text{l. gl. dim } R \geq 1 + \text{gl. dim } S$ assuming $\text{gl. dim } S < \infty$. Since $\text{gl. dim } S = \sup_{\mathfrak{M}} \text{gl. dim } S_{\mathfrak{M}}$ where \mathfrak{M} runs over all the maximal ideals of S , it is clear that under either of the conditions of the theorem, there exists a maximal ideal \mathfrak{M} such that $\text{gl. dim } S = \text{gl. dim } S_{\mathfrak{M}}$ and $d(S) \subset \mathfrak{M}$. The derivation d of S induces a derivation \bar{d} of $S_{\mathfrak{M}}$ if we set

$$\bar{d}\left(\frac{s}{s'}\right) = \frac{ds \cdot s' - s \cdot ds'}{s'^2}; s, s' \in S, s' \in \mathfrak{M}.$$

It is clear that $\bar{d}(S_{\mathfrak{M}}) \subset \mathfrak{M}S_{\mathfrak{M}}$. Hence by Proposition 1, § 4, we have

$$\begin{aligned} \text{l. gl. dim } S_{\mathfrak{M}}\{x, \bar{d}\} &= 1 + \text{gl. dim } S_{\mathfrak{M}} \\ &= 1 + \text{gl. dim } S. \end{aligned}$$

Thus, the theorem will be proved if we prove the following

LEMMA. *If \mathfrak{M} is any maximal ideal of S , we have*

$$\text{l. gl. dim } S\{x, d\} \geq \text{l. gl. dim } S_{\mathfrak{M}}\{x, \bar{d}\}.$$

Proof of the lemma. Let us set $R = S\{x, d\}$ and $\bar{R} = S_{\mathfrak{M}}\{x, \bar{d}\}$. Let $\eta: S \rightarrow S_{\mathfrak{M}}$ denote the ring homomorphism defined by $\eta(s) = \text{class of } s/1$. Since $\bar{d} \circ \eta = \eta \circ d$, η induces (see (2.2)) a ring homomorphism $\bar{\eta}: R \rightarrow \bar{R}$ such that $\bar{\eta}|_S = \eta$.

We first prove the following two statements:

(1) \bar{R} is R -flat as a right R -module (through $\bar{\eta}$).

(2) If M is any left \bar{R} -module, there exists a left R -module M' and a left \bar{R} -isomorphism $M \approx \bar{R} \otimes_R M'$.

The left $S_{\mathfrak{M}}$ -isomorphism $\varphi: S_{\mathfrak{M}} \otimes_S R \rightarrow \bar{R}$ given by $\varphi(1 \otimes x^i) = x^i \in \bar{R}$ satisfies $\varphi(1 \otimes f) = \bar{\eta}(f)$ for any $f \in R$. We have

$$\varphi(1 \otimes fg) = \bar{\eta}(fg) = \bar{\eta}(f)\bar{\eta}(g) = \varphi(1 \otimes f)\bar{\eta}(g).$$

Thus, φ is an isomorphism of right R -modules. Since $S_{\mathfrak{M}} \otimes_S R$ is right R -flat, (1) is proved. Let

$$\bar{F}_1 \xrightarrow{\lambda} \bar{F} \xrightarrow{\mu} M \longrightarrow 0$$

be an exact sequence where \bar{F}_1 and \bar{F} are \bar{R} -free with bases $\{e_\alpha\}$ and $\{f_\beta\}$ respectively. We then have

$$\lambda(e_\alpha) = \eta\left(\frac{1}{s_\alpha}\right) \sum_{\beta} \frac{1}{\eta} (a_{\alpha\beta}) f_\beta; a_{\alpha\beta} \in R, s_\alpha \in S - \mathfrak{M}.$$

Let θ be the \bar{R} -automorphism of \bar{F}_1 defined by $\theta(e_\alpha) = \eta(s_\alpha)e_\alpha$. Let

$\lambda' = \lambda \circ \theta$. We then have

$$\lambda'(e_\alpha) = \sum_{\beta} \frac{1}{\eta} (\alpha_{\alpha\beta}) f_\beta,$$

and the sequence

$$\bar{F}_1 \xrightarrow{\lambda'} \bar{F} \xrightarrow{\mu} M \longrightarrow 0$$

is exact. Let F_1 (resp. F) be the free R -module generated by $\{e_\alpha\}$ (resp. $\{f_\beta\}$) and let $\lambda'': F_1 \rightarrow F$ be the R -homomorphism defined by

$$\lambda''(e_\alpha) = \sum_{\beta} \alpha_{\alpha\beta} f_\beta.$$

It is easily seen that if we take $M' = c \circ \ker \lambda''$, we have $M \approx \bar{R} \otimes_R M'$. This proves (2). We now complete the proof of the lemma.

Let M be any left \bar{R} -module and let M' be a left R -module such that (2) is satisfied. Let

$$\dots \longrightarrow X_n \xrightarrow{d_n} X_{n-1} \longrightarrow \dots \longrightarrow X_0 \longrightarrow M' \longrightarrow 0$$

be a resolution of M' as a left R -module. Then

$$\bar{R} \otimes_R X_n \xrightarrow{I \otimes d_n} \bar{R} \otimes_R X_{n-1} \longrightarrow \dots \longrightarrow \bar{R} \otimes_R X_0 \longrightarrow M \longrightarrow 0$$

is exact in view of (1). Since $\bar{R} \otimes_R X_i$ is \bar{R} -projective, it follows that $(\bar{R} \otimes_R X_i, 1 \otimes d_i)$ is an \bar{R} -projective resolution of M . In particular, we have $hd_{\bar{R}} M \leq hd_R M' \leq \text{gl. dim } R$. Since M is arbitrary, it follows that $\text{gl. dim } \bar{R} \leq \text{gl. dim } R$. This proves the lemma and hence the theorem.

REMARK. Let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field K . It is well-known [7, Chap. III Cor. 4 to Th. 5] that $\text{Krull dim } S_{\mathfrak{M}}$ is the same for all maximal ideals \mathfrak{M} of S . Let d be a K -derivation of S given by $d(x_i) = f_i$. Then the derivation d satisfies condition (2) of Theorem 2 if and only if $f_i, 1 \leq i \leq n$ are not coprime and in this case we may apply the theorem and we have $\text{gl. dim } R = n + 1$. This includes the special case of Theorem 1 of [6] in which K is a field.

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