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MAXIMUM AND MONOTONICITY PROPERTIES OF INITIAL-BOUNDARY VALUE PROBLEMS FOR HYPERBOLIC EQUATIONS

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Various maximum and monotonicity properties of some initial boundary value problems for classes of linear second order hyperbolic partial differential operators in two independent variables are established. For example, let M be such an operator in Cartesian coordinates (x, y) and let T be a domain bounded by a characteristic curve of M with everywhere negative slope, and segments OA and OB of the positive x-axis and the positive y-axis, respectively; under certain restrictions on the coefficients of the operator M, if $Mu \leqq 0$ in T, $u = 0$ on $OA \cup OB$ and $\partial u/\partial y \leqq 0$ on OA then $u(x, y) \leqq 0$ in T.

Such maximum and monotonicity properties also have applications to ordinary differential equations; the above mentioned maximum property yields a comparison theorem on the distance between zeros of solutions to some ordinary differential equations.

The first maximum principles for a class of linear second order hyperbolic operators in two independent variables were formulated for problems in which conditions are imposed on the solution along characteristic curves [1; 3].

A maximum property of Cauchy's problem, in which the hypotheses on the solutions are imposed along noncharacteristic curves rather than characteristic curves, was first formulated by Weinberger [12] for a class of hyperbolic operators of the form

$$(1.1)\qquad Hu = \frac{\partial}{\partial x}\left(a\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(b\frac{\partial u}{\partial y}\right) + c\frac{\partial u}{\partial x} + d\frac{\partial u}{\partial y} \qquad a > 0\ ,\ b > 0\ .$$

Namely, under certain restrictions on the coefficients of the operator H, if $\partial u/\partial y \leqq 0$ on the initial line $y = 0$ and if $Hu \geqq 0$ for $y > 0$ then u attains its maximum on $y = 0$.

A generalized maximum property of Cauchy's problem was established by Protter [7] for essentially any smooth operator of the form (1.1). That is, the maximum of u divided by an appropriate function of the form $e^{\gamma x}(1 - \beta e^{-\alpha y})$, over a sufficiently small strip $0 \leqq y \leqq y_0$, is attained on $y = 0$.

Recently, additional maximum properties and even some monotonicity properties of Cauchy and characteristic initial value problems have been obtained by Gloistehn [4] for some classes of linear and nonlinear hyperbolic operators in two independent variables. For example, under certain restrictions on the coefficients of the operator H in (1.1), if $u \leqq 0$ and $\partial u/\partial y + \sqrt{a} \cdot \partial u/\partial x \leqq 0$ on $y = 0$, and if $Hu \geqq 0$ for $y > 0$ then $u \leqq 0$ and $\partial u/\partial y + \sqrt{a} \cdot \partial u/\partial x + \alpha u \leqq 0$ for $y \geqq 0$; here $\alpha(x, y)$ depends only on the coefficients of the operator H.

In the case of linear second order hyperbolic operators in more than two independent variables, Weinstein [14, 15], Weinberger [13] and the author [8; 9; 10] have established maximum properties of Cauchy's problem. A typical result for the wave operator

$$Wu = \frac{\partial^2 u}{\partial t^2} - \Delta u , \tag{1.2}$$

where Δ is the n-dimensional Laplace operator, is the following [10; 13; 14]. Let $N = ((n-2)/2)$ (n even), $N = ((n-3)/2)$ (n odd). If $\partial^k u/\partial t^k = 0$ ($k = 0, 1, \cdots, N$) and $\partial^{N+1} u/\partial t^{N+1} \leqq 0$ on the initial plane $t = 0$, and if $(\partial^N/\partial t^N) \cdot (Wu) \leqq 0$ for $t \geqq 0$ then $u \leqq 0$ for $t \geqq 0$. Here the t-derivatives of u on the initial plane $t = 0$ are to be determined from the Cauchy data.

In this paper, we derive various maximum and monotonicity properties of some initial-boundary value problems for linear second order hyperbolic equations in two independent variables. These initial-boundary value problems, first considered by Hadamard [5; 6], may be formulated in the following way.

Let L be a hyperbolic equation in characteristic coordinates (cf. [2]) of the form[1]

$$Lu = u_{\xi\eta} + a u_\xi + b u_\eta + cu = F . \tag{1.3}$$

Let C_l, C_0 and C_r be three curves with the following properties: (1) C_l, C_0 and C_r may be represented as $\eta = F_l(\xi)$, $\eta = f(\xi)$ and $\eta = F_r(\xi)$, respectively, where F_l, f and F_r are continuously differentiable and $F_l' > 0$, $f' < 0$ and $F_r' > 0$, (2) C_0 and C_l intersect at the point $O(0, 0)$, (3) C_0 and C_r intersect at $D(\bar{\xi}_0, \bar{\eta}_0)$, where $\bar{\xi}_0 > 0$ and $\bar{\eta}_0 < 0$, and (4) C_l and C_r do not intersect. Let C_l^+ and C_0^+ be the parts of C_l and C_0, respectively, where $\xi \geqq 0$. Let C_r' and C_0' be the parts of C_r and C_0, respectively, where $\eta \geqq \bar{\eta}_0$.

In the initial-boundary value problem I_l, we assume that the coefficients of the operator L are defined in the region "between" C_0^+ and C_l^+ and on the boundary $C_0^+ \cup C_l^+$, u and u_ξ (Cauchy data) are prescribed on C_0^+ and u is prescribed on C_l^+.

In the initial-boundary value problem I_r, the operator L is defined in the region "between" C_0' and C_r' and on the boundary $C_0' \cup C_r'$, u and u_η (Cauchy data) are prescribed on C_0' and u is prescribed on C_r'.

In the initial-boundary value problem II_{lr}, the operator L is defined

[1] A subscript $\xi(\eta)$ denotes partial differentiation with respect to $\xi(\eta)$.

in the region "between" C_l^+, C_r' and the segment OD of the curve C_0 and also on the boundary $C_l^+ \cup OD \cup C_r'$, u and either u_ξ or u_η are prescribed on OD and u is prescribed on $C_l^+ \cup C_r'$.

In § 2 and § 3, under certain conditions on the coefficients of the operator L, we establish some maximum properties of the initial-boundary value problems I_l, I_r and II_{lr}. In § 4, the results of § 2 and § 3 are extended to an operator that is not expressed in terms of characteristic coordinates; namely, we consider a hyperbolic operator of the form

$$(1.4) \quad Mu = u_{yy} - h^2(x, y)u_{xx} + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u, \quad h > 0 .$$

In § 5, we obtain a sort of a monotonicity property, as well as another maximum property, of an initial-boundary value problem for an operator of the form (1.4); in § 6, an application of this maximum property yields a comparison theorem on the distance between zeros of solutions to some ordinary differential equations.

2. Maximum properties of the initial-boundary value problems I_l and I_r. We consider a hyperbolic operator L in characteristic coordinates of the form

$$(2.1) \qquad Lu = u_{\xi\eta} + au_\xi + bu_\eta + cu .$$

Let $A(\bar{\xi}_1, \bar{\eta}_1)$ and $B(\bar{\xi}_1, \bar{\eta}_2)$ be points on C_0^+ and C_l^+, respectively. Let OA and OB be the indicated segments of C_0^+ and C_l^+; the points O and A are assumed to belong to OA. Let T_B denote the domain bounded by OA, OB and the line $\xi = \bar{\xi}_1 > 0$ and let $\bar{T}_B$ denote the closure of T_B. We assume that the coefficients of L are continuous in $\bar{T}_B$ and $b(\xi, \eta)$ has continuous first derivatives in $\bar{T}_B - OB$. We consider functions u that are twice continuously differentiable in $\bar{T}_B - OB$ and continuous, together with their first derivatives, in $\bar{T}_B$.

We consider problem I_l; that is, u and u_ξ are prescribed on C_0^+ and u is prescribed on C_l^+. In addition, suppose that

$$(2.2) \qquad u_\xi < 0 \quad \text{on} \quad OA - \{O\} .$$ [2]

We have the following maximum property of problem I_l.

THEOREM 1. *Let the coefficients of L satisfy the inequalities*

$$(2.3) \qquad b_\eta + ab - c \geqq 0$$

and

$$(2.4) \qquad c \geqq 0 ,$$

[2] The set $\{O\}$ contains only the point O.

in T_B, *and*

$$b \geqq 0 \quad on \quad OA . \tag{2.5}$$

Let u *satisfy the inequality* (2.2) *and*

$$Lu \leqq 0 \quad in \quad T_B . \tag{2.6}$$

Then if the maximum of u *in* $\bar{T}_B$ *is nonnegative it can only be attained on* $OA \cup OB$.

Proof. Let the maximum of u in $\bar{T}_B$ occur at the point Q and suppose that Q does not lie on $OA \cup OB$. Then

$$u_\xi(Q) \geqq 0 . \tag{2.7}$$

Let P denote the unique point of intersection of OA and the characteristic $\Gamma(\xi = \text{constant})$ through Q.

The following fundamental identity is also used in the discussion of maximum principles for mixed elliptic-hyperbolic operators [1, p. 456]:

$$vLu = (vu_\xi)_\eta + (bvu)_\eta + [cv - (bv)_\eta]u , \tag{2.8}$$

where v is a positive solution of the equation

$$v_\eta = av . \tag{2.9}$$

We integrate (2.8) along Γ from P to Q and obtain

$$\begin{aligned} vu_\xi \,|_Q &= vu_\xi \,|_P + \int_P^Q vLud\eta - bvu \,|_P^Q + \int_P^Q vu(b_\eta + ab - c)d\eta \\ &= vu_\xi \,|_P + \int_P^Q vLud\eta + (bv)\,|_P\,[u(P) - u(Q)] - u(Q)\int_P^Q cvd\eta \\ &\quad + \int_P^Q v[u - u(Q)](b_\eta + ab - c)d\eta . \end{aligned} \tag{2.10}$$

Since $u(Q) \geqq 0$ and $u \leqq u(Q)$ in $\bar{T}_B$, the equation (2.10) and (2.2) through (2.7) imply a contradiction. This completes the proof of Theorem 1.

The conditions (2.3), (2.4) and (2.5) are "best possible" in the sense that one can give examples where the maximum property in Theorem 1 does not hold when these conditions fail to be satisfied (see Examples 1, 3 and 2, respectively, in § 4).

COROLLARY 1. *If* $c = 0$ *then the result of Theorem* 1 *holds without the requirement that the maximum of* u *be nonnegative.*

COROLLARY 2. *If, in Corollary* 1, *we have* $u \leqq 0$ *on* $OA \cup OB$ *then* $u \leqq 0$ *in* $\bar{T}_B$ *holds without the requirement that the inequality* (2.2) *is strict.*

The proof of Corollary 2 consists of applying Corollary 1 to functions of the form $\omega = u - \varepsilon e^{\lambda(\xi+\eta)}$, with λ chosen so large that $L\omega \leqq 0$, and then letting $\varepsilon \to 0$.

If we impose further restrictions on the data along OA and OB we can eliminate the restrictions (2.4) and (2.5) on the operator L.

THEOREM 2. *Let the coefficients of L satisfy the inequality*

$$b_\eta + ab - c \geqq 0 \ \ in\ \ T_B. \tag{2.3}$$

Let u satisfy the conditions

$$u = 0 \quad and \quad u_\xi \leqq 0\,, \quad on \quad OA\,, \tag{2.11}$$

$$u \leqq 0 \quad on \quad OB \tag{2.12}$$

and the differential inequality

$$Lu \leqq 0 \quad in \quad T_B\,. \tag{2.13}$$

Then

$$u \leqq 0 \quad in \quad \bar{T}_B\,. \tag{2.14}$$

Moreover, if the strict inequality in (2.11) *holds on $OA - \{O\}$ then $u < 0$ in $T_B \cup AB$.*

Proof. We define the functions

$$u^\delta = e^{-\delta(\xi+\eta)}u\,, \qquad \delta > 0\,.$$

Each function u^δ satisfies a differential inequality

$$L^\delta u^\delta \equiv u^\delta_{\xi\eta} + a^\delta u^\delta_\xi + b^\delta u^\delta_\eta + c^\delta u^\delta \leqq 0 \quad \text{in} \quad T_B\,, \tag{2.15}$$

where the coefficients of the hyperbolic operator L^δ are given by

$$a^\delta = a + \delta\,, \tag{2.16}$$

$$b^\delta = b + \delta\,, \tag{2.17}$$

$$c^\delta = c + \delta(a + b) + \delta^2\,. \tag{2.18}$$

We note that for δ sufficiently large we have $b^\delta \geqq 0$ on OA and $c^\delta \geqq 0$ in $\bar{T}_B$. Since the expression $b_\eta + ab - c$ is one of the two Laplace Invariants[3] under transformations of the dependent variable u of the form $u = gU$, where g is any positive function (cf. [1, p. 460]), we have

$$b^\delta_\eta + a^\delta b^\delta - c^\delta = b_\eta + ab - c\,. \tag{2.19}$$

[3] The other invariant is $a_\xi + ab - c$.

Suppose that the strict inequality in (2.11) holds on $OA - \{O\}$. Since

$$u_\xi^\delta = e^{-\delta(\xi+\eta)} u_\xi \quad \text{on} \quad OA , \tag{2.20}$$

Theorem 1 implies that $u^\delta < 0$ in $T_B \cup AB$. Therefore $u < 0$ in $T_B \cup AB$. This establishes the part of Theorem 2 when u_ξ is negative on $OA - \{0\}$.

In order to complete the proof of Theorem 2, we introduce the class of functions

$$\omega = u - \varepsilon \phi e^{\lambda(\xi+\eta)} ,$$

where ϕ is given by

$$\phi(\xi, \eta) = \eta - f(\xi) \qquad (\xi, \eta) \quad \text{in} \quad \bar{T}_B$$

and $\eta = f(\xi)$ is the equation of the curve C_0. We note that

$$\omega_\xi |_{OA} = u_\xi |_{OA} + \varepsilon f' e^{\lambda(\xi+\eta)} |_{OA} , \tag{2.21}$$

$$L\omega = Lu - \varepsilon e^{\lambda(\xi+\eta)}[\lambda(1 - f') - af' + b + \phi(\lambda^2 + \lambda(a + b) + c)] . \tag{2.22}$$

Since $f' < 0$ on OA and $\phi \geqq 0$, we may choose λ independently of ε and so large that $L\omega \leqq Lu$ in T_B. It follows from (2.11) through (2.13) that ω satisfies the conditions of the first part of this proof and hence

$$u < \varepsilon \phi e^{\lambda(\xi+\eta)} \quad \text{in} \quad T_B \cup AB . \tag{2.23}$$

Finally, if we let $\varepsilon \to 0$ in (2.23), we obtain the desired result (2.14).

We remark that the condition (2.3) in Theorem 2 is "best possible" (see Example 1 in § 4). In addition we wish to emphasize that the condition (2.3) is invariant under a wide class of transformations of the dependent variable u of the form $u = gU$ and also under transformations of the independent variables ξ and η which leave the form of the operator L unchanged [1, p. 461].

Let $C(\xi_2, 0)$ be a point on C'_r. Take A to be the point D and let DC be the indicated segment of C'_r. Let T_G denote the domain bounded by OD, DC and the line $\eta = 0$ and let $\bar{T}_G$ denote the closure of T_G. If we interchange ξ and η, together with a and b, in the above discussion we can establish, for example, the following maximum property of problem I_r (see Theorem 2).

THEOREM 3. *Let the coefficients of L satisfy the inequality*

$$a_\xi + ab - c \geqq 0 \quad in \quad T_G . \tag{2.24}$$

Let u satisfy the conditions

$$u = 0 \quad and \quad u_\eta \leqq 0 , \quad on \quad OD , \tag{2.25}$$

(2.26) $$u \leqq 0 \quad on \quad DC$$

and the differential inequality

(2.27) $$Lu \leqq 0 \quad in \quad T_G .$$

Then

(2.28) $$u \leqq 0 \quad in \quad \bar{T}_G .$$

Moreover, if the strict inequality in (2.25) *holds on* $OD - \{D\}$ *then* $u < 0$ *in* $T_G \cup OC$.

The condition (2.24) is also "best possible" (see Example 1 in §4).

3. A maximum property of the initial-boundary value problem II_{lr}. Let $B(\bar{\xi}_0, \bar{\eta}_2)$ be the point of intersection of C_l^+ and the line $\xi = \bar{\xi}_0$ and let T_B and T_G be defined as in §2. Let u satisfy the conditions

(3.1) $$u = 0 \quad \text{and either} \quad u_\xi < 0 \quad \text{or} \quad u_\eta < 0 , \quad \text{on} \quad OD ,$$

(3.2) $$u \leqq 0 \quad \text{on} \quad OB \cup DC$$

and the differential inequality

(3.3) $$Lu \leqq 0 \quad \text{in} \quad T_B \cup T_G .$$

Since $f' < 0$ on OD, $u = 0$ and $u_\xi < 0 (u_\eta < 0)$, on OD, imply $u_\eta < 0 (u_\xi < 0)$ on OD. Hence, if the coefficients of L satisfy the inequalities (2.3) and (2.24) then Theorem 2 and Theorem 3 imply

(3.4) $$u < 0 \quad \text{in} \quad T_B \cup T_G \cup DB \cup OC .$$

In this section, we determine a domain Σ such that (1) $T_B \cup T_G \cup DB \cup OC \subset \Sigma$ and (2) under certain "invariant" conditions[4] on the coefficients of L, if (3.1) through (3.3) are satisfied then $u < 0$ in Σ.

Let $P(\xi_1, \eta_1)$ be any point such that $\bar{\xi}_0 < \xi_1 < \bar{\xi}_2$ and $0 < \eta_1 < \bar{\eta}_2$. Let $Q(\xi_1, \eta_0)$ denote the unique point of intersection of DC and $\xi = \xi_1$ and let $R(\xi_0, \eta_0)$ denote the unique point of intersection of OD and $\eta = \eta_0$. Hence, to each point $P(\xi_1, \eta_1)$ we may associate a unique point $S_P(\xi_0, \eta_1)$ and a characteristic rectangle with corners P, Q, R and S_P such that Q and R lie on DC and OD, respectively; let T denote the set of all points $P(\xi_1, \eta_1)$ such that S_P is contained in T_B.[5] The set T is a domain.

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[4] The conditions are stated in terms of Laplace Invariants (see footnote 3).

[5] In the definition of the set T we may also use OB instead of DC so that S_P lies on OB and T consists of all points P such that Q is contained in T_G.

Let $P(\xi_1, \eta_1)$ be any point in T and let Q, R and S_P have coordinates as in the definition of the domain T. We integrate (2.8) along the characteristic from $P_1(\xi, \eta_0)$ to $P_2(\xi, \eta_1)$ and obtain[6]

$$(3.5)\quad \int_{P_1}^{P_2} vLud\eta = (vu)_\xi \,|_{P_1}^{P_2} + (bv - v_\xi)u \,|_{P_1}^{P_2} + \int_{P_1}^{P_2} uv(c - ab - b_\eta)d\eta \,.$$

We integrate (3.5) with respect to $\xi(\xi_0 \leqq \xi \leqq \xi_1)$ and obtain

$$(3.6)\quad \begin{aligned}(vu)(P) = (vu)(Q) + (vu)(S_P) - (vu)(R) + \int_R^Q (bv - v_\xi)ud\xi \\ - \int_{S_P}^P (bv - v_\xi)ud\xi + \iint v[Lu + u(b_\eta + ab - c)]d\xi d\eta \,,\end{aligned}$$

where the double integral denotes integration over $\xi_0 \leqq \xi \leqq \xi_1$ and $\eta_0 \leqq \eta \leqq \eta_1$. Let v^0 be the particular solution of (2.9) given by

$$(3.7)\quad v^0 = \exp\left[\int_{\xi_0}^{\xi} b(\tau, \eta_0)d\tau + \int_{\eta_0}^{\eta} a(\xi, \rho)d\rho\right].$$

Then

$$(3.8)\quad (v^0)^{-1}(bv^0 - v^0_\xi) = 0 \quad \text{on} \quad \eta = \eta_0 \,,$$

$$(3.9)\quad \begin{aligned}(v^0)^{-1}(bv^0 - v^0_\xi) &= b(\xi, \eta_1) - b(\xi, \eta_0) - \int_{\eta_0}^{\eta_1} a_\xi(\xi, \rho)d\rho \\ &= \int_{\eta_0}^{\eta_1} [b_\eta(\xi, \rho) - a_\xi(\xi, \rho)]d\rho \quad \text{on} \quad \eta = \eta_1 \,.\end{aligned}$$

It follows from (3.1) and (3.6) through (3.9) that

$$(3.10)\quad \begin{aligned}(v^0u)(P) = (v^0u)(Q) + (v^0u)(S_P) + \int_{\xi_0}^{\xi_1}\left[\int_{\eta_0}^{\eta_1}(a_\xi - b_\eta)d\rho\right](v^0u)(\xi, \eta_1)d\xi \\ + \iint v^0[Lu + u(b_\eta + ab - c)]d\xi d\eta \,.\end{aligned}$$

Let $\Sigma = T \cup T_B \cup T_C \cup DB \cup OC$. Suppose that there is a point P in Σ such that $u(P) = 0$. The inequality (3.4) implies that (1) P is in T and (2) we may assume without loss of generality that $u(P) = 0$ and $u \leqq 0$ in the characteristic rectangle with corners P, Q, R and S_P. Let Σ_B and Σ_C denote the parts of Σ where $\eta > 0$ and $\xi > \bar{\xi}_0$, respectively. Under the assumptions (2.24) and

$$(3.11)\quad b_\eta + ab - c \geqq 0 \quad \text{in} \quad \Sigma \,,$$

$$(3.12)\quad a_\xi \geqq b_\eta \quad \text{in} \quad \Sigma_B \,,$$

it follows from (3.2), (3.3) and (3.10) that $(v^0u)(S_P) \geqq 0$. Since S_P is

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[6] In this section, u and the coefficients of L are assumed to be sufficiently smooth in T (see § 2).

in T_B, this is a contradiction. Hence $u < 0$ in Σ.

If we interchange ξ and η, together with a and b, in the above discussion, the conditions (compare (3.11) and (3.12))

$$a_\xi + ab - c \geqq 0 \quad \text{in} \quad \Sigma , \tag{3.13}$$

$$b_\eta \geqq a_\xi \quad \text{in} \quad \Sigma_C , \tag{3.14}$$

also imply that $u < 0$ in Σ. We have established the following maximum property of problem II_{lr}.

THEOREM 4. *Let the coefficients of L satisfy the inequalities*

$$\begin{aligned} a_\xi + ab - c &\geqq 0 \quad in \quad \Sigma , \\ b_\eta + ab - c &\geqq 0 \quad in \quad \Sigma \end{aligned} \tag{3.15}$$

and either

$$a_\xi + ab - c \geqq b_\eta + ab - c \quad in \quad \Sigma_B \tag{3.16}$$

or

$$b_\eta + ab - c \geqq a_\xi + ab - c \quad in \quad \Sigma_C . \tag{3.17}$$

Let u satisfy the conditions[7]

$$u = 0 \quad and\ either \quad u_\xi < 0 \quad or \quad u_\eta < 0 , \quad on \quad OD , \tag{3.18}$$

$$u \leqq 0 \quad on \quad OB \cup DC \tag{3.19}$$

and the differential inequality

$$Lu \leqq 0 \quad in \quad \Sigma . \tag{3.20}$$

Then

$$u < 0 \quad in \quad \Sigma . \tag{3.21}$$

We remark that the domain Σ is the "largest possible" in the sense that if we relax the strict inequalities in (3.18)—and hence also the strict inequality in (3.21)—then one can give examples where the maximum property $u \leqq 0$ holds only in the closure of Σ (see Example 4 in § 4).

4. Maximum properties of the initial-boundary value problems I_l' and II_{lr}'. In this section we extend the results of § 2 and § 3 to a hyperbolic operator of the form

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[7] We may replace the condition "either $u_\xi < 0$ or $u_\eta < 0$ on OD" by a condition involving the normal derivative of u on OD (cf. (4.15) in § 4).

(4.1) $Mu = u_{yy} - h^2(x, y)u_{xx} + \alpha(x, y)u_x + \beta(x, y)u_y + \gamma(x, y)u, \;\; h > 0 \, .$

For the sake of simplicity we consider only initial-boundary value problems for M where u and u_y are prescribed on a portion of the x-axis and u is prescribed on either the line $x = 0$ (problem I_l') or the lines $x = 0$ and $x = d_0 > 0$ (problem II_{lr}').

We recall that the characteristic curves of M are the solutions of the ordinary differential equations

$$\frac{dx}{dy} = h \, , \tag{4.2}$$

$$\frac{dx}{dy} = -h \, . \tag{4.3}$$

Let $A'(d, 0)$ and $D'(d_0, 0)$ be points on the positive x-axis. Let $B'(0, y_1)$ [respectively $C'(d_0, y_2)$] be the unique point of intersection of the line $x = 0[x = d_0]$ and the characteristic curve $\Gamma_-[\Gamma_+]$ with slope (4.3) [(4.2)] that passes through $A'(d, 0)$ [$O(0, 0)$]. Let OA', OD', OB' and $D'C'$ be the indicated straight line segments. Let $T_{B'}$ and $T_{C'}$ be the domains bounded by OB', OA', Γ_- and $D'C'$, OD', Γ_+, respectively.[8]

We consider functions u that are twice continuously differentiable in $\bar{T}_{B'} - OB'$ and continuous, together with their first derivatives, in $\bar{T}_{B'}$. We assume that the coefficients of M are continuous in $\bar{T}_{B'}$, α and β are continuously differentiable in $\bar{T}_{B'} - OB'$ and h has continuous second derivatives in $\bar{T}_{B'} - OB'$. (We assume that analogous conditions hold when we consider the domain $T_{C'}$).

We define the operators

$$\delta = \frac{\partial}{\partial y} + h\frac{\partial}{\partial x} \, , \tag{4.4}$$

$$D = \frac{\partial}{\partial y} - h\frac{\partial}{\partial x} \, . \tag{4.5}$$

The operators δ and D are essentially the directional derivatives along the characteristic curves defined by (4.2) and (4.3), respectively.

In this section we assume also that h is continuously differentiable and positive in $\bar{T}_{B'}$ (and $\bar{T}_{C'}$). If we introduce characteristic coordinates $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ as new independent variables (cf. [2]) then we can apply the results of §2 to the transformed operator—an operator that is of the form (2.1). In terms of the operators δ and D the conditions (2.3), (2.5) and (2.24) become

[8] The points A' and O do not belong to either Γ_- or Γ_+.

$$(4.6)\quad 2E \equiv D\left(\frac{D(h)-\alpha+\beta h}{h}\right)$$
$$-\frac{1}{2h^2}(D(h)-\alpha+\beta h)(D(h)-\alpha-\beta h)-2\gamma \geqq 0 \quad \text{in} \quad T_{B'},$$

$$(4.7)\qquad D(h)-\alpha+\beta h \geqq 0 \quad \text{on} \quad OA'$$

and (compare [1, p. 464, (5″)])

$$(4.8)\quad 2F \equiv \delta\left(\frac{\delta(h)+\alpha+\beta h}{h}\right)$$
$$-\frac{1}{2h^2}(\delta(h)+\alpha+\beta h)(\delta(h)+\alpha-\beta h)-2\gamma \geqq 0 \quad \text{in} \quad T_{C'},$$

respectively. We have, for example, the following result.[9]

THEOREM 1′. *Let the coefficients of M satisfy the inequalities* (4.6), (4.7) *and*

$$(4.9)\qquad \gamma \geqq 0 \quad in \quad T_{B'}.$$

Let u satisfy the condition

$$(4.10)\qquad \delta(u) < 0 \quad on \quad OA' - \{O\}$$

and the differential inequality

$$(4.11)\qquad Mu \leqq 0 \quad in \quad T_{B'}.$$

Then if the maximum of u in $\bar{T}_{B'}$ is nonnegative it can only be attained on $OA' \cup OB'$.

The following examples illustrate which conditions in the above theorems are "best possible".

EXAMPLE 1. We consider an operator M of the form $Mu = u_{yy} - u_{xx} + 3u$. Let OA' and OB' be the segments of the x-axis and the y-axis where $0 \leqq x \leqq 3\pi/4$ and $0 \leqq y \leqq 3\pi/4$, respectively. The domain $T_{B'}$ is given by $x + y < 3\pi/4$, $x > 0$ and $y > 0$. Since $h = 1$, $\gamma = 3$ and $\alpha = \beta = 0$, the conditions (4.7) and (4.9) are satisfied. However, the condition (4.6) becomes $\gamma \leqq 0$ which is not satisfied. Let $u(x, y) = -\sin 2y \cos(x - \pi/2)$. Then $Mu = 0$ in $T_{B'}$ and $\delta(u) = -2\cos(x - \pi/2) < 0$ when $y = 0$ and $0 < x \leqq 3\pi/4$. Since $u(r, (\pi + r)/2) = \sin^2 r > 0$ $(0 < r \leqq \pi/6)$ and $u = 0$ on $OA' \cup OB'$, the function u does not attain its maximum on $OA' \cup OB'$. Therefore, the condition (4.6) in Theorem 1′ is "best possible". Moreover, if we set $\xi = y + x$ and $\eta = y - x$, this example shows that the

[9] The desired extension of Theorem 2 is contained in Theorem 5.

condition (2.3) in Theorem 1 and Theorem 2 is also "best possible".

EXAMPLE 2. Let $Mu = u_{yy} - u_{xx} - 2u_y$. Let OA' and OB' be the segments of the x-axis and the y-axis where $0 \leqq x \leqq \pi/3$ and $0 \leqq y \leqq \pi/3$, respectively. Then domain $T_{B'}$ is given by $x + y < \pi/3$, $x > 0$ and $y > 0$. Since $h = 1$, $\beta = -2$ and $\alpha = \gamma = 0$, the conditions (4.6) and (4.9) are satisfied but the condition (4.7) becomes $\beta \geqq 0$ which is not satisfied. Let $u(x, y) = (y - 1)e^y \cos(x - \pi/2)$. Then $Mu = 0$ in $T_{B'}$, $u \leqq 0$ on $OA' \cup OB'$ and $\delta(u) = \sin(x - \pi/2) < 0$ when $y = 0$ and $0 \leqq x \leqq \pi/3$. Since $u(r, 1 + r) = re^{1+r} \sin r > 0$ $(0 < r < 1/2(\pi/3 - 1))$, the condition (4.7) in Theorem 1′ is also "best possible".

EXAMPLE 3. Let $Mu = u_{yy} - u_{xx} - \gamma_0^2 u$, where γ_0 is a positive constant. Let β_1 be the first positive zero of $J_1(\rho)$, the Bessel function of order 1. Let OA' and OB' be the segments of the x-axis and the y-axis where $0 \leqq x \leqq d$ and $0 \leqq y \leqq d$ $(0 < d < \beta_1/\gamma_0)$, respectively. We note that condition (4.9) is not satisfied. Let $u(x, y) = J_0(\gamma_0\sqrt{x^2 - y^2})$, where $J_0(\rho)$ denotes the Bessel function of order 0. It is well known that u has the properties (1) $Mu = 0$, (2) $u = 1$ on $y = x$ (and $y = -x$) and (3) $|u(x, y)| \leqq 1$ (cf. [2, p. 120] and [11]). Moreover, $\delta(u) = \gamma_0 J_0'(\gamma_0 x) = -\gamma_0 J_1(\gamma_0 x) < 0$ when $y = 0$ and $0 < x \leqq d$. Since u attains its maximum on $y = x$, the condition (4.9) is also "best possible".

In order to extend Theorem 4 to the operator M we first determine a domain T' that plays the role of the domain T in § 3. In the definition of the point B', we take A' to be the point $D'(d_0, 0)$. Let $\Gamma_{B'}$ and $\Gamma_{C'}$ be the characteristic curves given by (4.2) and (4.3), respectively, that pass through B' and C'. Let E be the characteristic quadrilateral bounded by $\Gamma_{B'}, \Gamma_{C'}, \Gamma_+$ and Γ_-. As in § 3, to each point $P'(x, y)$ in E, we may associate a unique point $S_{P'}$ and a characteristic quadrilateral with corners P', Q', R' and $S_{P'}$ such that Q' and R' lie on $D'C'$ and OD', respectively. Let T' denote the domain that consists of all points P' such that $S_{P'}$ is contained in $T_{B'}$. Moreover, as in § 3, let $\Sigma' = T' \cup T_{B'} \cup T_{C'} \cup \Gamma_- \cup \Gamma_+$ and let $\Sigma_{B'}$ and $\Sigma_{C'}$ be the parts of Σ' "above Γ_+" and "above Γ_-", respectively.

We can now formulate the desired extension of Theorem 4. Since the Laplace Invariants $b_\eta + ab - c$ and $a_\xi + ab - c$ are given essentially by (4.6) and (4.8), respectively, we need only restate the conditions (3.15) through (3.17) in terms of the operators δ and D.

THEOREM 4′. *Let the coefficients of M satisfy the inequalities*

$$\begin{aligned} &E \geqq 0 \quad \textit{in} \quad \Sigma' \\ &F \geqq 0 \quad \textit{in} \quad \Sigma' \end{aligned} \tag{4.12}$$

and either

$$F \geqq E \quad in \quad \Sigma_{B'} \tag{4.13}$$

or

$$E \geqq F \quad in \quad \Sigma_{C'} . \tag{4.14}$$

Let u satisfy the conditions

$$u = 0 \quad and \quad u_y \leqq 0 , \quad on \quad OD' , \tag{4.15}$$

$$u \leqq 0 \quad on \quad OB' \cup D'C' \tag{4.16}$$

and the differential inequality

$$Mu \leqq 0 \quad in \quad \Sigma' . \tag{4.17}$$

Then

$$u \leqq 0 \quad in \quad \Sigma' . \tag{4.18}$$

Moreover, if the strict inequality holds in (4.15) *then the strict inequality holds also in* (4.18).

Proof. If the strict inequality holds in (4.15), Theorem 4 implies the desired result $u < 0$ in Σ'.

In order to complete the proof of Theorem 4′, we consider the functions

$$w = u - \varepsilon y e^{\lambda y} \qquad \varepsilon > 0 ,$$

where λ is chosen independently of ε and so large that $Mw \leqq Mu$ in Σ'. Since (4.15) through (4.17) imply that w satisfies the conditions of the first part of this proof, it follows that

$$u < \varepsilon y e^{\lambda y} \quad \text{in} \quad \Sigma' . \tag{4.19}$$

Hence, letting $\varepsilon \to 0$, we obtain (4.18).

The following example shows that the domain Σ' in Theorem 4′ is the "largest possible".

EXAMPLE 4. Let $Mu = u_{yy} - u_{xx}$. Let OD' and OB' be the segments of the x-axis and the y-axis where $0 \leqq x \leqq \pi$ and $0 \leqq y \leqq \pi$, respectively, and let $D'C'$ be the segment of the line $x = \pi$ where $0 \leqq y \leqq \pi$. Then the domain Σ' is given by $0 < x < \pi$ and $0 < y < \pi$. Let $u(x, y) = -\sin y \cos (x - \pi/2)$. Since $u \leqq 0$ in the closure of Σ' but $u > 0$ when $0 < x < \pi$ and $y = \pi + \varepsilon$ $(0 < \varepsilon < \pi)$, the set Σ' in Theorem 4′ is the "largest possible".

5. A monotonicity property of the initial-boundary value problem I_l'. In this section (the notation and the various smoothness assumptions are the same as in § 4) we consider the operator M without introducing characteristic coordinates. In addition to an extension of Theorem 2 this more direct approach also yields a sort of a monotonicity property for M.

Our discussion is based upon the fundamental identity (see (2.8) and [1, p. 465]; compare also [4, p. 385, (1.2)])

$$D[v\delta(u)] = vMu + [D(v) - \beta v]D(u) - \gamma vu , \tag{5.1}$$

where δ and D are the operators defined in (4.4) and (4.5) and v is a positive solution of the equation

$$2hD(v) + v[D(h) - \alpha - \beta h] = 0 .[10] \tag{5.2}$$

We rewrite (5.1) as

$$D[v(\delta(u) + \theta u)] = vMu + uvE , \tag{5.3}$$

where E is defined in (4.6) and

$$\begin{aligned} \theta &= v^{-1}[\beta v - D(v)] \\ &= \frac{D(h) - \alpha + \beta h}{2h} . \end{aligned} \tag{5.4}$$

The following theorem is a consequence of (5.1) and (5.3).

THEOREM 5. *Let the coefficients of M satisfy the inequality* (4.6). *Let u satisfy the conditions*

$$u = 0 \quad \textit{and} \quad u_y \leqq 0 , \quad \textit{on} \quad OA' , \tag{5.5}$$

$$u \leqq 0 \quad \textit{on} \quad OB' \tag{5.6}$$

and the differential inequality

$$Mu \leqq 0 \quad \textit{in} \quad T_{B'} . \tag{5.7}$$

Then

$$u \leqq 0 \tag{5.8}$$

and

$$\delta(u) + \theta u \leqq 0 , \tag{5.9}$$

in $T_{B'} \cup \Gamma_-$. Moreover, if the strict inequality in (5.5) *holds on*

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[10] On any characteristic curve given by $dx/dy = -h$, we see that $D(v) = dv/dy$ and, hence, the equation (5.2) becomes an ordinary differential equation.

$OA' - \{O\}$ then the strict inequality holds also in (5.8).

Proof. Suppose that the strict inequality in (5.5) holds on $OA' - \{O\}$. Since $D = d/dy$ on any characteristic curve $dx/dy = -h$, if we proceed as in the proof of Theorem 1 and Theorem 2—with the identity (5.1) playing the role of (2.8) and $u^\delta = e^{-\delta y}u$—we obtain $u < 0$ in $T_{B'} \cup \Gamma_-$. The remainder of the proof is a variation of a method used by Gloistehn [4] for the Cauchy problem. Assume that there is a point Q' in $T_{B'} \cup \Gamma_-$ such that $[\delta(u) + \theta u]\,|_{Q'} = 0$. Let $\Gamma_{Q'}$ be the characteristic curve given by (4.3) that passes through Q' and let P denote the point of intersection of $\Gamma_{Q'}$ and OA'. Since $[\delta(u) + \theta u]\,|_P < 0$ by our hypotheses there is a point Q on $\Gamma_{Q'}$ such that $[\delta(u) + \theta u]\,|_Q = 0$ and $\delta(u) + \theta u < 0$ on the arc of $\Gamma_{Q'}$ between P and Q. Therefore, since $v > 0$ and D is essentially differentiation along $\Gamma_{Q'}$, it follows that

$$D[v(\delta(u) + \theta u)]\,|_Q \geqq 0 \,. \tag{5.10}$$

The basic equation (5.3), together with $u(Q) < 0$, $Mu < 0$, (4.6) and (5.10), yields a contradiction. Thus $\delta(u) + \theta u$ is negative in $T_{B'} \cup \Gamma_-$ under the additional assumptions $u_y < 0$ on $OA' - \{O\}$ and $Mu < 0$ in $T_{B'} \cup \Gamma_-$.

In order to complete the proof of Theorem 5, we consider again the functions

$$w = u - \varepsilon y e^{\lambda y} \qquad \varepsilon > 0 \,,$$

where λ is chosen independently of ε and so large that $Mw < Mu$ in $T_{B'}$. It follows from (5.5) through (5.7) and the first part of this proof that

$$u < \varepsilon y e^{\lambda y} \tag{5.11}$$

and

$$\delta(u) + \theta u < \varepsilon e^{\lambda y}(1 + \lambda y + \theta y) \,, \tag{5.12}$$

in $T_{B'} \cup \Gamma_-$. Therefore, letting $\varepsilon \to 0$, we obtain (5.8) and (5.9).

COROLLARY 3. *Let $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ be two points in $T_{B'}$ that are joined by a characteristic curve Γ of the family* (4.2) *and suppose that $y_1 \leqq y_2$. If* (4.6) *and* (5.5) *through* (5.7) *are satisfied then*

$$u(Q_2) \leqq u(Q_1) \exp\left[\int_{\Gamma\, Q_1}^{Q_2} \theta dy\right]. \tag{5.13}$$

The proof consists of multiplying (5.9) by $\exp\left[\int_{\Gamma\, y_1}^{y} \theta dy\right]$ and integrating along Γ from Q_1 to Q_2.

6. An application to ordinary differential equations. In this section we establish a comparison theorem on the distance between zeros of solutions to some ordinary differential equations. Comparison theorems of this type have already been obtained by Weinberger [12] and Protter [7] as applications of some maximum properties of "pure" initial value problems. However, we show that in some cases a "stronger" result can be obtained by the use of a maximum property of an initial-boundary value problem.

We consider the ordinary differential equations[11]

$$(6.1)\quad (f_1(x)\phi'(x))' + g_1(x)\phi(x) = 0\,, \qquad f_1(x) > 0 \qquad c \leqq x \leqq d\,,$$

$$(6.2)\quad (f_2(y)\psi'(y))' + g_2(y)\psi(y) = 0\,, \qquad f_2(y) > 0 \qquad a \leqq y \leqq b\,.$$

Suppose that $\phi(x_1) = 0$ and $\phi(x) > 0$, $c \leqq x_1 < x \leqq x_2 \leqq d$. In addition, suppose that $\psi(y_1) = 0$ and $\psi'(y_1) < 0$, $a \leqq y_1 < b$. Let M be the hyperbolic operator given by

$$(6.3)\qquad Mu = u_{yy} - u_{xx} - f_1^{-1}f_1'u_x + f_2^{-1}f_2'u_y + (f_2^{-1}g_2 - f_1^{-1}g_1)u\,.$$

Then the function $u(x, y) = \phi(x)\psi(y)$ is such that

$$(6.4)\qquad u = 0 \quad \text{and} \quad u_y < 0\,, \quad \text{on} \quad y = y_1 \quad \text{and} \quad x_1 < x \leqq x_2\,,$$

$$(6.5)\qquad u = 0 \quad \text{on} \quad x = x_1 \quad \text{and} \quad y_1 \leqq y \leqq b\,,$$

$$(6.6)\qquad Mu = 0\,, \quad a \leqq y \leqq b \quad \text{and} \quad c \leqq x \leqq d\,.$$

Hence, if the functions $\alpha = -f_1^{-1}f_1'$, $\beta = f_2^{-1}f_2'$ and $\gamma = f_2^{-1}g_2 - f_1^{-1}g_1$ are such that the operator M satisfies the condition (4.6), Theorem 5 implies that $u < 0$ in the domain bounded by the lines $x = x_1$, $y = y_1$ and $x + y = x_2 + y_1$. Thus $\psi(y) < 0$ when $y_1 < y < y_1 + (x_2 - x_1)$. Since ψ and ψ' cannot vanish simultaneously and x_1, x_2 and y_1 were arbitrary, we have established the following comparison theorem (see [12, p. 512] and [7, pp. 123–125]).

THEOREM 6. *Let m be the greatest lower bound of the distance between zeros of ψ on the interval $a \leqq y \leqq b$ and let m^* be the least upper bound of the distances between zeros of ϕ on the interval $c \leqq x \leqq d$. If*

$$(6.7)\qquad 2f_2^{-1}f_2'' - (f_2^{-1}f_2')^2 - 4f_2^{-1}g_2 \geqq 2f_1^{-1}f_1'' - (f_1^{-1}f_1')^2 - 4f_1^{-1}g_1$$

for $a \leqq y \leqq b$ and $c \leqq x \leqq d$, then

$$(6.8)\qquad m \geqq m^*\,.$$

[11] In this section, v' denotes the derivative of the function v.

COROLLARY 4. *If, in Theorem* 6, *we have* $f_1(x) \equiv 1$, $g_1(x) \equiv \lambda^2$ *and*

$$2f_2f_2'' - (f_2')^2 + 4f_2(\lambda^2 f_2 - g_2) \geqq 0 \qquad a \leqq y \leqq b\,, \tag{6.9}$$

then

$$m \geqq \pi\lambda^{-1}\,. \tag{6.10}$$

We remark that, even under the conditions $\lambda^2 f_2(y) \geqq g_2(y)$ and $f_2(y)f_2''(y) \geqq (f_2'(y))^2$, the direct application of a maximum property for a "pure" initial value problem would yield only the "weaker" result $m \geqq \pi\lambda^{-1}/2$ [7, p. 124 Corollary 3].

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