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A theory of representations for compact semigroups has been lacking due in large part to the absence of a translation-invariant carrying measure that exists for compact groups. The object in this paper is to show that for a compact, group-extremal affine semigroup there is a sufficient system of representations by linear operators on finite-dimensional complex linear spaces; in the abelian case, a sufficient system of affine semicharacters is obtained. As a result, a compact group-extremal affine semigroup is the inverse limit of compact, finite-dimensional, group-extremal affine semigroups.

A subset S of a locally convex topological linear space X (over the reals or complexes) will be called an affine semigroup if:

(1) S is convex.

(2) There is an associative multiplication defined in S which is jointly continuous in the topology on S inherited from X.

(3) For fixed $x \in S$ the functions $y \rightarrow yx$ and $y \rightarrow xy$ are affine functions of S into S.

In this paper, S will always be compact. By a theorem due to Wendel [2], if S is a compact affine semigroup with identity u, then each point of S with inverse is an extreme point of S. If, conversely, each extreme point has an inverse then the set of extreme points of S is the maximal group of the idempotent u and is, therefore, compact [9]. In this case, we shall say S is *group-extremal*.

Following [2], we will say two affine semigroups S and T are *equivalent* if there exists a bicontinuous isomorphism of S onto T which is also an affine function.

DEFINITION 1. A representation of an affine semigroup S is a function P from S to $B(M)$ the set of bounded linear operator on some finite-dimensional complex linear space M satisfying:

(a) P is continuous (with any locally convex topology on $B(M)$, all of which are equivalent).

(b) P is a homomorphism.

(c) P is affine.

DEFINITION 2. An affine semicharacter on S is any complex-valued continuous affine homomorphism defined on S. We point out that if S is compact and f is any affine semicharacter on S then $|f(x)| \leqq 1$ for each $x \in S$.

In the remainder of this paper, S will be a compact, group-extremal affine semigroup with identity u, and whose extreme points form the compact topological group G.

1. Representations of S. In this section, we shall prove the following:

THEOREM 1. *For $x_0, y_0 \in S$, $x_0 \neq y_0$ there exists a representation P of S in $B(M)$, M a finite-dimensional complex linear space, satisfying*

(1) $P(x_0) \neq P(y_0)$.

(2) $P^*(\sigma) \in P(S)$ *for all* $\sigma \in S$ (*where* $P^*(\sigma)$ *is the adjoint of the operator* $P(\sigma)$).

Many of the details of the proof are quite similar to those in group representations (cf. [1], [6], [7]) but we shall include them for the sake of completeness. By $C(S)$ $(C(G))$ we mean the collection of all complex-valued continuous functions on $S(G)$. The supremum norm in $C(S)$ is denoted by $\|\cdot\|$ and in $C(G)$ by $\|\cdot\|_*$. $A(S)$ will denote the norm closed subspace of $C(S)$ consisting of all *affine* continuous complex-valued functions. $A(G)$ denotes the set of restrictions to G of elements of $A(S)$.

LEMMA 1.1. (a) *$A(G)$ is a closed subspace of $C(G)$.*

(b) *If $f, g \in A(S)$ and $f(x) = g(x)$ for $x \in G$ then $f(x) = g(x)$ for all $x \in S$.*

(c) *If $f_n \in A(G)$, $g_n \in A(S)$ for $n = 0, 1, 2, \cdots$ if $f_n(x) = g_n(x)$ for $x \in G$, $n = 0, 1, 2, 3, \cdots$ and if $\|f_n - f_0\|_* \to 0$ then $\|g_n - g_0\| \to 0$.*

Proof of (a). Let $f_n \to f$ where $f_n \in A(G)$, $n = 1, 2, 3, \cdots$ and $f \in C(G)$. There exist $g_n \in A(S)$ such that $g_n(x) = f_n(x)$ for $x \in G$. For $\varepsilon > 0$ there exists an N such that if $m, n \geqq N$ and $x \in G$ then $|f_n(x) - f_m(x)| < \varepsilon/2$. If $x_1, \cdots, x_r \in G$, $\lambda_i \geqq 0$, $\sum_{i=1}^r \lambda_i = 1$ and $x = \sum_{i=1}^r \lambda_i x_i$ then

$$\begin{aligned} |g_n(x) - g_m(x)| &= \left|\sum_{i=1}^r \lambda_i [g_n(x_i) - g_m(x_i)]\right| \\ &= \left|\sum_{i=1}^r \lambda_i [f_n(x_i) - f_m(x_i)]\right| < \frac{\varepsilon}{2} . \end{aligned}$$

Since $g_n - g_m$ is continuous on S, and the elements x of the above form are dense in S [4], we have $|g_n(x) - g_m(x)| < \varepsilon$ for $x \in S$. Thus, $\{g_n\}_{n=1}^\infty$ is a Cauchy sequence in $C(S)$ and, hence, converges to $g \in C(S)$. Since $A(S)$ is clearly closed, $g \in A(S)$. Now for $x \in G$, $f_n(x) \to f(x)$ but $f_n(x) = g_n(x) \to g(x)$ so that $f(x) = g(x)$ and $f \in A(G)$.

Proof of (b). An application of the Krein-Milman Theorem.

Proof of (c). By an argument similar to the proof of (a) $\|g_n - h\| \to 0$ for some $h \in A(S)$. But $f_n(x) = g_n(x)$ for all $x \in G$ so that $h(x) = f_0(x) = g_0(x)$ for $x \in G$. By (b), $h(x) = g_0(x)$ for all $x \in S$.

Proof of theorem. By $L^2(G)$, we mean the Hilbert space of all functions on G which are square-integrable with respect to Haar measure on G, where the inner product is defined as usual. $\left(\text{i.e. } (f, g) = \int f\bar{g}dx\right)$. We denote the norm of an element $f \in L^2(G)$ by $\|f\|_2 = \left(\int |f|^2 dx\right)^{1/2}$.

We now fix $x_0, y_0 \in S$ where $x_0 \neq y_0$. There exists a set U which is open in G, $u \in U$, and $\langle U\rangle x_0 \cap \langle U\rangle y_0 = \varnothing$. ($\langle U\rangle$ denotes the closed convex hull of U). This follows from $ux_0 \neq uy_0$, the continuity of multiplication in S, and the local convexity of the containing space X.

There exists a real-valued function $f_0 \in A(S)$ satisfying:

$$\min_{z \in \langle U\rangle x_0} \{f_0(z)\} > \max_{z \in \langle U\rangle y_0} \{f_0(z)\}$$

[3]. Choose $h \in C(G)$, $h(u) = 1$, $h = 0$ in $G\backslash U$, and $0 \leqq h \leqq 1$. For $z \in G$, let

$$k(z) = \frac{h(z) + h(z^{-1})}{2}$$

then $k \in C(G)$, $0 \leqq k \leqq 1$, $k(u) = 1$, $k = 0$ in $G\backslash U$ and $k(z) = k(z^{-1})$. We then have

$$\begin{aligned} \int k(z^{-1})f_0(zx_0)dz &= \int_U k(z^{-1})f_0(zx_0)dz > \int_U k(z^{-1})f_0(zy_0)dz \\ &= \int k(z^{-1})f_0(zy_0)dz \, . \end{aligned}$$

Hence,

(1) $\int k(z^{-1})f_0(zx_0)dz \neq \int k(z^{-1})f_0(zy_0)dz$.

The operator in $L^2(G)$ defined by

(2) $Tf(x) = \int k(z^{-1})f(zx)dz$ for $f \in L^2(G), x \in G$ takes $L^2(G)$ into $C(G)$ and is a completely continuous, symmetric bounded linear operator in $L^2(G)$ [8; p. 242]. Further, $\|Tf\|_* \leqq \|k\|_2 \cdot \|f\|_2$ so that $f \to Tf$ is continuous in the norm topology on $C(G)$. If $f \in A(G)$ then there is a $g \in A(S)$ such that $g(x) = f(x)$ for $x \in G$. If we define:

(3) $g'(x) = \int k(z^{-1})g(zx)dz$ then $g' \in A(S)$ and for

$$x \in G,\ g'(x) = \int k(z^{-1})g(zx)dz = \int k(z^{-1})f(zx)dz = Tf(x) \, .$$

Thus, if $f \in A(G)$, then $Tf \in A(G)$. If we let H denote the closure of

$A(G)$ in $L^2(G)$, then H is a closed invariant subspace of T. In fact, if $f \in H$, there exists a sequence $f_n \in A(G)$ such that $\|f_n - f\|_2 \to 0$. But then $\|Tf_n - Tf\|_* \to 0$ and since $Tf_n \in A(G)$, which is norm closed in $C(G)$, we have $Tf \in A(G)$. Hence, T takes H into $A(G)$. Using T again to denote the restriction of T to H, we have again that T is a completely-continuous, symmetric bounded linear operator in H. By a well-known theorem (cf. [8; p. 233]) there exists a sequence $\{\psi_i\}_{i=1}^{\infty}$ where

(4) $\psi_i \in H$ for $i = 1, 2, \cdots$

(5) $T\psi_i = \lambda_i \psi_i$ for some real number $\lambda_i \neq 0$

(6) $(\psi_i, \psi_j) = \delta_{ij}$ (δ_{ij} is the Kronecker delta function)

(7) $Tf = \sum_{i=1}^{\infty} (Tf, \psi_i)\, \psi_i$ for each $f \in H$ and where the series converges in $L^2(G)$ norm.

(8) For each $\lambda \neq 0$, $M_\lambda = \{f \in H\colon Tf = \lambda \cdot f\}$ is finite-dimensional.

Note that $\psi_i = T((1/\lambda_i)\psi_i)$ and since $(1/\lambda_i)\psi_i \in H$, it follows that $\psi_i \in A(G)$. Also, using a computation that can be found in [1; p. 209] the series in (7) converges to Tf in the supremum norm on $C(G)$.

Now since $\psi_i \in A(G)$ for each i, there exists $\hat{\psi}_i \in A(S)$ such that $\hat{\psi}_i(x) = \psi_i(x)$ for $x \in G$. Further, if $g \in A(S)$ and f denotes the restriction of g to G then $f \in A(G)$ so that $Tf = \sum_{i=1}^{\infty} (Tf, \psi_i)\psi_i$ where the series converges in supremum norm on $C(G)$. As in (3), if $g'(x) = \int k(z^{-1})g(zx)dz$ for $x \in S$ then $g' \in A(S)$ and for $x \in G$, $g'(x) = Tf(x)$. Also for $x \in G$, $n \geqq 1$, $\sum_{i=1}^{n} (Tf, \psi_i)\hat{\psi}_i(x) = \sum_{i=1}^{n} (Tf, \psi_i)\psi_i(x)$ and, hence, Lemma 1.1(c) implies that $g' = \sum_{i=1}^{\infty} (Tf, \psi_i)\hat{\psi}_i$ where the series converges in $A(S)$. In particular, if f_0 is our original function (1) and g_0 is the restriction to G of f_0 then $f_0' = \sum_{i=1}^{\infty} (Tg_0, \psi_i)\hat{\psi}_i$. But by (1), $f_0'(x_0) \neq f_0'(y_0)$ so that for some i, $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$.

For $\lambda = \lambda_i$, $M_\lambda = \{f \in H, Tf = \lambda \cdot f\}$ is a finite-dimensional subspace of H; hence, by Lemma 1.1(b) $N_\lambda = \{f \in A(S)\colon f' = \lambda f\}$ is a finite-dimensional subspace of $A(S)$, and there exists $\hat{\psi}_i \in N_\lambda$ for which $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$. N_λ is easily seen to be a finite-dimensional Hilbert space with inner product again $(f, g) = \int f\bar{g}dx$. In fact, if $f \in A(S)$ and $(f, f) = 0$ then $\int |f|^2 dx = 0$ so that $f(x) = 0$ for $x \in G$. By Lemma 1.1(b), $f(x) = 0$ for all $x \in S$. For $f \in N_\lambda$, it is easily seen $(|\lambda|/\|k\|_2)\, \|f\| \leqq (f, f)^{1/2} \leqq \|f\|$ so that N_λ is complete with respect to this inner product. For $\sigma \in S$, we define the linear operator $P(\sigma)$ in N_λ by:

(9) $[P(\sigma)f](x) = f(x\sigma)$ where $f \in N_\lambda$, $x \in S$. We have

$$[P(\sigma)f]'(x) = \int k(z^{-1})P(\sigma)f(zx)dz = \int k(z^{-1})f(zx\sigma)dz$$
$$= f'(x\sigma) = \lambda f(x\sigma) = \lambda[P(\sigma)f](x)\ .$$

Hence, $P(\sigma)$ clearly takes N_λ to N_λ. It is clear that the map $\sigma \to P(\sigma)$ is continuous in the strong operator topology. Further, $[P(\sigma\tau)f](x) = f(x\sigma\tau) = P(\sigma)[P(\tau)f](x)$ so that $P(\sigma\tau) = P(\sigma)P(\tau)$ and $\sigma \to P(\delta)$ is a homomorphism. For $\sigma, \tau \in S$ $0 \leqq \lambda \leqq 1$ and $x \in S$ we have

$$\begin{aligned}[P(\lambda\sigma + (1-\lambda)\tau)f](x) &= f(x[\lambda\sigma + (1-\lambda)\tau]) \\ &= \lambda f(x\sigma) + (1-\lambda)f(x\tau) \\ &= [\lambda P(\sigma) + (1-\lambda)P(\tau)f](x)\end{aligned}$$

and $\sigma \to P(\sigma)$ is now an affine continuous homomorphism of S into the bounded linear operators on the finite-dimensional space N_λ.

Note further that there exists $\hat{\psi}_i \in N_\lambda$ where $\hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0)$. Then $[P(x_0)\hat{\psi}_i](u) = \hat{\psi}_i(x_0) \neq \hat{\psi}_i(y_0) = [P(y_0)\hat{\psi}_i](u)$ and $P(x_0) \neq P(y_0)$. Finally, for $x \in G$, $f, g \in N_\lambda$

$$\begin{aligned}(P(x)f, g) &= \int [P(x)f](y)\overline{g(y)}dy = \int f(yx)\overline{g(y)}dy \\ &= \int f(y)\overline{g(yx^{-1})}dy = \int f(y)\overline{[P(x^{-1})g](y)}dy = (f, P(x^{-1})g) .\end{aligned}$$

Hence, we have for $x \in G$, $P^*(x) = P(x^{-1})$. If $x_1, x_2, \cdots x_n \in G$, $\lambda_i \geqq 0$, $\sum_{i=1}^n \lambda_i = 1$ and $x = \sum_{i=1}^n \lambda_i x_i$ then

$$P^*(x) = \sum_{i=1}^n \lambda_i P^*(x_i) = \sum_{i=1}^n \lambda_i P(x_i^{-1}) = P\Big(\sum_{i=1}^n \lambda_i x_i^{-1}\Big) \in P(S) .$$

Since $P(S)$ is compact and convex, it follows by continuity of P and the Krein-Milman Theorem that $P^*(\sigma) \in P(S)$ for each $\sigma \in S$ and the proof is complete.

COROLLARY 1.1. *If G is metrizable, there is a countable number of representations which separate points.*

Proof. In Theorem 1, to separate two points we obtained a neighborhood of the identity, and then constructed a countable number of representations using this neighborhood. It is clear this neighborhood may be taken from a countable basis at the identity, giving rise to a countable number of representations which separate the points of S.

2. Affine semicharacters.

In this section, we assume the additional condition that S is abelian; then we have:

THEOREM 2. *If $x_0, y_0 \in S$, $x_0 \neq y_0$ there exists an affine semicharacter p such that $p(x_0) \neq p(y_0)$.*

Proof. By Theorem 1, there exists a representation P of S in the bounded linear operators $B(M)$ on the n-dimensional complex vector space M for which $P(x_0) \neq P(y_0)$ and $P^*(\sigma) \in P(S)$ for each $\sigma \in S$. The space M is then a finite-dimensional space invariant under the abelian family of operators $\{P(\sigma) : \sigma \in S\}$ satisfying $P^*(\sigma) \in P(S)$ for $\sigma \in S$ and, hence, is spanned by one dimensional invariant subspaces. We thus obtain a basis $e_1, \cdots, e_n$ for M where $P(\sigma)e_i = P_i(\sigma)e_i$ for each $i = 1, 2, \cdots, n$ and $p_i(\sigma)$ is a complex number. The functions $p_1, \cdots, p_n$ are easily seen to be affine semicharacters of S. Since $P(x) \neq P(y)$, $p_i(x) \neq p_i(y)$ for some integer i and we are finished. Using the representations of S and the fact that they are affine maps we have:

THEOREM 3. *A group-extremal affine semigroup is equivalent to the inverse limit of finite-dimensional group-extremal affine semigroups.*

The proof of this theorem is completely analogous to the proof of the well-known theorem that a compact group is the inverse limit of compact Lie groups, so we shall omit it.

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