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**SOME LOWER BOUNDS FOR LEBESGUE AREA**

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It is well known in area theory that a continuous map  $f$  of the unit square  $Q^2$  into Euclidean space  $E^2$  can have zero Lebesgue area even though its range has a nonempty interior. This cannot happen if  $f$  is suitably well-behaved, for example, if  $f$  is light, Lipschitzian, or as we shall see below, if  $f$  satisfies a certain interiority condition. The purpose of this paper is to determine conditions under which an arbitrary measurable set  $A \subset Q^2$  will support the Lebesgue area of  $f$ . The results imply that if  $f|A$  is Lipschitz and if one of the coordinate functions of  $f$  is *BVT* (and continuous), then the Lebesgue area of  $f$  is no less than the integral of the multiplicity function  $N(f, A, y)$ , where  $N(f, A, y)$  is the number (possibly  $\infty$ ) of points in  $f^{-1}(y) \cap A$ . We show that the *BVT* condition cannot be omitted. The proofs of theorems involving Lebesgue area depend upon a new co-area formula for real valued *BVT* functions.

2. Preliminaries. Our proofs rely heavily upon the following topological theorem [3, p. 513] which was first proved by Federer in the 2-dimensional case [8, p. 358]. We believe that this result is yet to be fully exploited in area theory.

**THEOREM 2.1.** *If  $X$  is a  $k$ -dimensional finitely triangulable space and  $u: X \rightarrow E^1, v: X \rightarrow E^{k-1}, f: X \rightarrow E^1 \times E^{k-1}$  are continuous maps such that  $f(x) = (u(x), v(x))$  for  $x \in X$ , then there is a countable set  $D \subset E^1$  such that*

$$S[f, (s, t)] = S[v|u^{-1}(s), t] \quad \text{for } (s, t) \in (E^1 - D) \times E^{k-1}.$$

Here  $S[f, (s, t)]$  denotes the stable multiplicity of  $f$  at  $(s, t)$  [9, (3.10)].

In the case  $X = Q^2$ , the unit square, (and this will be the only case of interest to us throughout the remainder of this paper) this theorem provides a very simple criterion to determine the stability of  $f$  at a point  $(s, t)$ ; for  $t$  is a positive stable value of  $v|u^{-1}(s)$  if and only if there is a nondegenerate continuum  $C \subset u^{-1}(s)$  such that  $t \in \text{interior } v(C)$ . Thus, the stable multiplicity function is positive at almost all points in the range of a monotone map and in the case of a light map, it is positive on an open dense set. In view of the following proposition, we see that mappings which are similar to Whyburn's quasi-open maps [19, p. 110], [22, (3.9)] also have positive stable values.

**PROPOSITION 2.2.** Suppose  $f: Q^2 \rightarrow E^2$  is a continuous map such that

for each  $y \in f(Q^2)$ , there is a component  $K$  of  $f^{-1}(y)$  with the property that for each sufficiently small open connected set  $U$  containing  $y$ , there is a component  $V$  of  $f^{-1}(U)$  containing  $K$  which maps onto  $U$  by  $f$ . Then, for all but countably many  $y \in f(Q^2)$ ,  $S(f, y) > 0$ .

*Proof.* Select a point  $y \in f(Q^2)$  whose first coordinate is not contained in the set  $D$  of (2.1). Let  $U_i$  be a sequence of sufficiently small open connected sets such that  $U_i \supset \text{closure } U_{i+1}$  and whose intersection is a closed vertical line segment  $\lambda$  containing  $y$  in its interior. Then the intersection of the corresponding  $V_i$  will be a continuum  $C \supset K$  that will be mapped onto  $\lambda$ . By (2.1),  $S(f, y) > 0$ . Now by repeating this argument with horizontal line segments instead of vertical ones, the result follows.

It is easy to verify that if  $S(f, y) > 0$ , then the converse of (2.2) holds, c.f. [21, (2.4)].

The notion of stability is crucial in area theory since

$$(2.2.1) \quad \mathfrak{A}(f) = \int_{Q^2} S(f, y) dL_2(y) ,$$

where  $\mathfrak{A}(f)$  is the Lebesgue area of  $f$  and  $L_2$  is 2-dimensional Lebesgue measure. By a result of Cesari [1], (2.2.1) is a special case of a more general theorem due to Federer [9, (7.9)].

**DEFINITIONS 2.3.**  $H_n^k$  will denote  $k$ -dimensional Hausdorff measure in  $E^n$ ,  $F_n^k$   $k$ -dimensional Favard measure [7, (2.18)],  $L_n$   $n$ -dimensional Lebesgue measure, and  $\text{dim}(A, x)$  will denote the topological dimension of a set  $A$  at a point  $x$ . A real valued map  $f$  on a topological space is called *almost light* if  $f^{-1}(y)$  is totally disconnected for  $L_1$  almost all  $y \in E^1$ . A map  $f: Q^2 \rightarrow E^1$  is said to satisfy *condition  $N_1$  on a set  $A$*  if it maps sets of  $H_2^1$  measure zero of  $A$  into sets of  $L_1$  measure zero.

We will use the following notion which was first introduced in [6, p. 48]. An  $L_n$  measurable set  $E \subset E^n$  has the unit vector  $n(x)$  as the *exterior normal* to  $E$  at  $x$  if, letting

$$(2.3.1) \quad \begin{aligned} S(x, r) &= \{y: |y - x| < r\} , \\ S_+(x, r) &= S(x, r) \cap \{y: (y - x) \cdot n(x) \geq 0\} , \\ S_-(x, r) &= S(x, r) \cap \{y: (y - x) \cdot n(x) \leq 0\} , \\ \alpha(n) &= L_n[S(x, 1)] , \end{aligned}$$

we have

$$2 \lim_{r \rightarrow 0^+} L_n[S_-(x, r) \cap E] / \alpha(n)r^n = 1 , \quad 2 \lim_{r \rightarrow 0^+} L_n[S_+(x, r) \cap E] / \alpha(n)r^n = 0 .$$

Let  $BV$  denote the class of all locally integrable functions  $u: Q^n \rightarrow E^1$

such that the  $i$ th partial derivative of  $u$  in the sense of distributions is a totally finite measure  $\mu_i$ . This class contains those functions which are *BVT*. For  $u \in BV$  and  $B$  any Borel subset of  $Q^n$  let  $I(u, E) = |\mu|(E)$  where  $|\mu|$  is the total variation of the vector-valued measure  $(\mu_1, \mu_2, \dots, \mu_n)$ . In the case that  $u$  is *ACT* observe that for any Borel set  $B \subset Q^n$ ,

$$(2.3.2) \quad I(u, B) = \int_B |\text{grad } u(x)| dL_n(x)$$

where  $\text{grad } u$  is the ordinary gradient of  $u$ . Thus, in this case  $I(u, \cdot)$  can be extended to all Lebesgue measurable sets.

If  $B \subset E^n$  is a Borel set then  $P(B)$  will denote the *perimeter* of  $B$ . If  $F$  is the set of  $x$  for which the exterior normal to  $B$  exists at  $x$  and if  $P(B) < \infty$ , then we see from [2] and [10] that

$$(2.3.3) \quad P(B) = H_n^{n-1}(F).$$

$F$  is called the *reduced boundary* of  $B$  and note that  $F \subset \text{bdry } B$ . For  $u: Q^n \rightarrow E^1$  in *BV* and  $E(s) = \{x: u(x) > s\}$ , Fleming and Rishel [14] proved that

$$(2.3.4) \quad I(u, Q^n) = \int_{E^1} P[E(s)] dL_1(s).$$

In the case that  $u$  is Lipschitzian, theorems obtained independently by Federer [11, (3.1)] and Young [20, Th. 4] imply that

$$(2.3.5) \quad I(u, A) = \int_{E^1} H_n^{n-1}[u^{-1}(s) \cap A] dL_1(s)$$

whenever  $A \subset Q^n$  is a Lebesgue measurable set.

**3. Metric theorems.** The following co-area formula is an extension of (2.3.5) and although the proof is only given for functions defined on  $E^2$ , it is clear that it will generalize to  $E^n$  without any essential change. The author is indebted to Casper Goffman for his suggestion that this co-area formula might be valid.

The following notation will be used throughout the proof. Let  $(q, r, s)$  be coordinates in  $E^3$  and define  $\delta: E^3 \rightarrow E^1$ ,  $\Pi_2: E^3 \rightarrow E^2$ ,  $\Pi_1: E^2 \rightarrow E^1$  by  $\delta(q, r, s) = s$ ,  $\Pi_2(q, r, s) = (r, s)$  and  $\Pi_1(q, r) = r$ . If  $u: Q^2 \rightarrow E^1$  then  $u': Q^2 \rightarrow E^3$  is defined by  $u'(q, r) = (q, r, u(q, r))$ .  $G^2$  will denote the group of orthogonal transformations on  $E^2$  and  $\varphi$  the unique Haar measure on  $G^2$  for which  $\varphi(G^2) = 1$ . For  $R \in G^2$  let  $R^*: E^3 \rightarrow E^3$  be defined by  $R^*(q, r, s) = (q', r', s)$  where  $R(q, r) = (q', r')$ .

**THEOREM 3.1.** *If  $u: Q^2 \rightarrow E^1$  is *BVT*(*ACT*), then*

$$I(u, D) = \int_{E^1} H_2^1[u^{-1}(s) \cap D] dL_1(s)$$

whenever  $D \subset \mathbb{Q}^2$  is a Borel ( $L_2$  measurable) set.

*Proof.* Let

$$g(s) = H_2^1[u^{-1}(s) \cap D] = H_3^1[\delta^{-1}(s) \cap u'(D)] .$$

If  $u$  is *BVT* and  $D$  a Borel set, then  $A = u'(D)$  is an analytic set and therefore it is the union of an increasing sequence of compact sets and a set  $N$  of  $H_3^2$  measure zero. Using the Eilenberg inequality [4] we see that

$$H_3^1[\delta^{-1}(s) \cap N] = 0$$

for  $L_1$  almost all  $s \in E^1$ . Thus, in order to show that  $g$  is  $L_1$  measurable it is sufficient to consider the case when  $A$  is compact; but then, it can be shown as in [11, (3.1)] that  $g$  is the limit of upper semi-continuous functions.

If  $u: \mathbb{Q}^2 \rightarrow E^1$  is *ACT* and  $N \subset \mathbb{Q}^2$  a set for which  $L_2(N) = 0$ , then [18, (3.17)] and [12] imply that  $H_3^2[u'(N)] = 0$ . Thus,  $u'(D)$  is  $H_3^2$  measurable whenever  $D \subset \mathbb{Q}^2$  is  $L^2$  measurable and the measurability of  $g$  follows as it did above.

Let

$$\alpha(D) = \int_{E^1} H_3^1[u^{-1}(s) \cap D] dL_1(s) .$$

It is now clear that  $\alpha$  is a measure on Borel ( $L_2$  measurable) sets if  $u$  is *BVT*(*ACT*). Moreover, from [18, (3.17)], [12], and [4] we see that  $\alpha$  is absolutely continuous with respect to  $L_2$  if  $u$  is *ACT*. Hence, it is only necessary to prove the theorem in case  $u$  is *BVT*. For this purpose we only need to show that  $I(u, W) = \alpha(W)$  for rectangles  $W \subset \mathbb{Q}^2$  because both  $I(u, \cdot)$  and  $\alpha$  are measures over the Borel sets. We may as well assume that  $W = \mathbb{Q}^2$ .

In view of (2.3.4) and (2.3.3) it is obvious that  $I(u, \mathbb{Q}^2) \leq \alpha(\mathbb{Q}^2)$ . The opposite inequality will follow from the last of four parts into which the remainder of the proof is divided.

**PART 1.** For  $L_1$  almost all  $s \in E^1$ ,  $u^{-1}(s)$  is  $(H_2^1, 1)$  rectifiable.

*Proof.* Since  $u$  is *BVT*,  $\mathcal{L}(u') < \infty$  [16, p. 516]. If  $A = u'(Q)$  then it follows from [12] that  $H_3^2(A) < \infty$  and that  $A$  is  $(H_3^2, 2)$  rectifiable. Now apply [13, (8.16)] to obtain a countable number of 2-dimensional proper regular submanifolds  $M_i$  of class  $C^1$  for which

$$H_3^2 \left[ A - \bigcup_{i=1}^{\infty} M_i \right] = 0 .$$

Letting  $M = \bigcup_{i=1}^{\infty} M_i$  the Eilenberg inequality [4] implies

$$H_3^1[\delta^{-1}(s) \cap (A - M)] = 0$$

and

$$H_3^1[\delta^{-1}(s) \cap A] < \infty$$

for  $L_1$  almost all  $s$ . In view of (2.3.5) one can easily verify that for each  $i$ ,  $\delta^{-1}(s) \cap M_i$  is  $(H_2^1, 1)$  rectifiable and therefore that  $\delta^{-1}(s) \cap M_i \cap A$  is  $(H_2^1, 1)$  rectifiable for  $L_1$  almost all  $s \in E^1$ . But the union of  $\delta^{-1}(s) \cap M_i \cap A$  occupies  $H_2^1$  almost all of  $\delta^{-1}(s) \cap A$  and thus the result follows.

PART 2. For  $L_1$  almost all  $s \in E^1$ ,  $F_2^1[u^{-1}(s)] = H_2^1[u^{-1}(s)]$ .

*Proof.* This follows from Part 1 and [7, (5.14)].

PART 3.

$$\int_{E^1} H_2^1[u^{-1}(s)] dL_1(s) = \Pi 2^{-1} \int_{G^2} \int_{E^1} N[\Pi_2 R^* u', Q^2, y] dL_2(y) d\varphi(R) .$$

*Proof.* For each  $s \in E^1$  apply [7, (5.11)] to obtain

$$\begin{aligned} F_2^1[u^{-1}(s)] &= \Pi 2^{-1} \int_{G^2} \int_{E^1} N[\Pi_1 R, u^{-1}(s), r] dL_1(r) d\varphi(R) \\ &= \Pi 2^{-1} \int_{G^2} \int_{E^1} N[\Pi_2 R^* u', Q^2, (r, s)] dL_1(r) d\varphi(R) . \end{aligned}$$

By integrating with respect to  $s$ , the result follows from Part 2 and Fubini's theorem.

PART 4.

$$I(u, Q^2) \geq \int_{E^1} H_2^1[u^{-1}(s)] dL_1(s) .$$

*Proof.* Select a sequence of Lipschitz functions  $u_k: Q^2 \rightarrow E^1$  which converge uniformly to  $u$  and for which  $I(u_k, Q^2) \rightarrow I(u, Q^2)$  as  $k \rightarrow \infty$ . A result of [18, (3.5)] states that for each  $R \in G^2$  and continuous  $v: Q^2 \rightarrow E^1$ ,

$$(1) \quad N[\Pi_2 R^* v', Q^2, y] = S[\Pi_2 R^* v', y]$$

for  $L_2$  almost all  $y \in E^2$ . Recall that the stable multiplicity function

is lower semi-continuous with respect to uniform convergence. Thus, from Part 3, (1), Fatou's lemma, and (2.3.5)

$$\begin{aligned}
 \int_{E^1} H_2^1[u^{-1}(s)]dL_1(s) &= \Pi 2^{-1} \int_{Q^2} \int_{E^2} N[\Pi_2 R^* u', Q^2, y] dL_2(y) d\varphi(R) \\
 &= \Pi 2^{-1} \int_{Q^2} \int_{E^2} S[\Pi_2 R^* u', y] dL_2(y) d\varphi(R) \\
 &\leq \liminf_{k \rightarrow \infty} \Pi 2^{-1} \int_{Q^2} \int_{E^2} S[\Pi_2 R^* u'_k, y] dL_2(y) d\varphi(R) \\
 &= \liminf_{k \rightarrow \infty} \Pi 2^{-1} \int_{Q^2} \int_{E^3} N[\Pi_2 R^* u'_k, Q^2, y] dL_2(y) d\varphi(R) \\
 &= \lim_{k \rightarrow \infty} \int_{E^1} H_2^1[u_k^{-1}(s)]dL_1(s) \\
 &= \lim_{k \rightarrow \infty} I(u_k, Q^2) = I(u, Q^2) .
 \end{aligned}$$

**COROLLARY 3.2.** *If  $u: Q^2 \rightarrow E^1$  is BVT, then the following hold for  $L_1$  almost all  $s \in E^1$ :*

- (i)  $H_2^1[u^{-1}(s)] < \infty$  and  $u^{-1}(s)$  is  $(H_2^1, 1)$  rectifiable,
- (ii) the exterior normal to  $E(s)$  exists at  $H_2^1$  almost all  $x \in u^{-1}(s)$ .

*Proof.* The first statement follows from the proof of Part 1 in (3.1) and the second from (3.1), (2.3.4), and (2.3.3).

**LEMMA 3.3.** *If  $u: Q^2 \rightarrow E^1$  is BVT, then for  $L_1$  almost all  $s \in E^1$ ,  $\dim [u^{-1}(s), x] > 0$  for  $H_2^1$  almost all  $x \in u^{-1}(s)$ .*

*Proof.* If  $B \subset E^2, x \in E^2$ , denote by  $W(x)$  the set of all straight lines passing through  $x$  and by  $U(B, x)$  those  $\lambda \in W(x)$  for which  $x$  is not a cluster point of  $\lambda \cap B$ . Since we may identify  $W(x)$  with the unit semi-circle  $S_+^1$ , we can regard the restriction of  $H_2^1$  to  $S_+^1$  as defining a measure  $\mu$  on  $W(x)$ . In the same manner, we can define a measure  $\nu$  on the homogeneous space of all orthogonal projections  $p: E^2 \rightarrow E^1$ .

Suppose, for some  $s \in E^1$ , that  $H_2^1[u^{-1}(s)] < \infty$  and that  $u^{-1}(s)$  is  $(H_2^1, 1)$  rectifiable. Letting

$$D_s = u^{-1}(s) \cap \{x: \mu[U(u^{-1}(s), x)] = 0\} ,$$

it follows from [7, (8.3)] that  $L_1[p(D_s)] = 0$  for  $\nu$  almost all  $p$ . But  $D_s$  is also  $(H_2^1, 1)$  rectifiable and therefore, from [7, (5.14)] it follows that  $H_2^1(D_s) = 0$ . Thus, in view of (3.2), for  $L_1$  almost all  $s \in E^1$  the following two conditions hold at  $H_2^1$  almost all  $x \in u^{-1}(s)$ :

- (i) the exterior normal to  $E(s)$  exists at  $x$ ,
- (ii)  $\mu[U^{-1}(s), x] > 0$ .

We will conclude the proof by showing that for all such  $s$  and  $x$ ,  $\dim[u^{-1}(s), x] > 0$ . For if we assume that  $\dim[u^{-1}(s), x] = 0$ , this means that there exist arbitrarily small open sets  $G$  containing  $x$  whose boundaries do not intersect  $u^{-1}(s)$ . By the Phragmen-Brouwer theorem, it can be assumed that  $\text{bdry } G$  is connected. For every  $r > 0$ , let

$$U_r[u^{-1}(s), x] = W(x) \cap \{\lambda: S(x, r) \cap u^{-1}(s) \cap (\lambda - \{x\}) = 0\}.$$

From (ii) we know that there exists  $\alpha > 0$  and  $r_0 > 0$  such that  $\mu[U_{r_0}(u^{-1}(s), x)] = \alpha$ . Choose  $G \subset S(x, r_0/2)$ . Since  $\text{bdry } G$  is connected and  $\text{bdry } G \cap u^{-1}(s) = 0$ , either  $\text{bdry } G \subset E(s)$  or  $\text{bdry } G \subset F(s) = \{x: u(x) < s\}$ . Suppose  $\text{bdry } G \subset E(s)$  and because of (i),  $r_0$  may be assumed to have been chosen so small that (see (2.3.1)),

$$(3) \quad 2L_2[S_+(r_0, x) \cap E(s)]/\Pi r_0^2 < \alpha/\Pi.$$

Now, for each  $\lambda \in U_{r_0}(u^{-1}(s), x)$ ,  $S(x, r_0) \cap u^{-1}(s) \cap (\lambda - \{x\}) = 0$  and  $\lambda \cap \text{bdry } G \neq 0$ . Therefore, since  $\text{bdry } G \subset E(s)$ , the union of all such  $\lambda$  in  $S(x, r_0) - \{x\}$  is contained in  $E(s)$  and its  $L_2$  measure is no less than  $\alpha r_0^2$ , which contradicts (3). The case of  $\text{bdry } G \subset F(s)$  is treated in a similar way and thus the proof is concluded.

LEMMA 3.4. *Suppose  $f: Q^2 \rightarrow E^2$  is continuous and  $f = (u, v)$  where  $u$  is BVT. Then  $f^{-1}(y)$  is totally disconnected for  $L_2$  almost all  $y \in E^2$ .*

*Proof.* Let  $\lambda$  be a horizontal (or vertical) line segment in  $Q^2$  on which  $u$  as a function of one variable is of bounded variation. Thus, if  $\lambda$  is the line  $r = r_0$ , the function  $u(\cdot, r_0)$  is of bounded variation and consequently,  $N[u(\cdot, r_0), \lambda, s] < \infty$  for  $L_1$  almost all  $s \in E^1$ . This implies that  $f(\lambda)$  intersects almost all vertical lines in a finite number of points and therefore, by Fubini's theorem,  $L_2[f(\lambda)] = 0$ . Since  $u$  is BVT, there exist a countable dense set of vertical lines and a countable dense set of horizontal lines such that the image of each line is a set of  $L_2$  measure zero. If  $A$  denotes the union of these vertical and horizontal lines, then  $L_2[f(A)] = 0$ . Now if  $C$  is a nondegenerate continuum of  $f^{-1}(y)$ , for some  $y \in E^2$ , then clearly  $C$  must intersect  $A$ . Thus  $y \in f(A)$  and the result follows.

COROLLARY 3.5. *With the same hypotheses as in 3.4, for  $L_1$  almost all  $s \in E^1$ ,  $v|u^{-1}(s)$  is almost light.*

THEOREM 3.6. *Suppose  $f: Q^2 \rightarrow E^2$  is continuous,  $f = (u, v)$ ,  $u$  is BVT and  $v$  satisfies condition  $N_1$  on an analytic set  $A \subset Q^2$ . Then*

$$\mathfrak{L}(f) \geq \int_{E^2} N(f, A, y) dL_2(y).$$



*Proof.* Let  $W_s = u^{-1}(s) \cap \{x: \dim[u^{-1}(s), x] > 0\}$ . It follows from (2.1), (3.3), (3.5) and [9, (3.3), (3.5), (3.12)] that for  $L_1$  almost all  $s \in E^1$

$$\begin{aligned} \int_{E^1} S[f, (s, t)] dL_1(t) &= \int_{E^1} S[v | u^{-1}(s), t] dL_1(t) \\ &\cong \int_{E^1} N[v, W_s, t] dL_1(t) \\ &\cong \int_{E^1} N[v, u^{-1}(s) \cap A, t] dL_1(t) \\ &= \int_{E^1} N[f, A, (s, t)] dL_1(t). \end{aligned}$$

Now by integrating with respect to  $s$  the result follows from Fubini's theorem and (2.2.1). The analyticity of  $A$  is needed only to assure the  $L_2$  measurability of the last integrand.

**COROLLARY 3.7.** *If  $f: Q^2 \rightarrow E^2$  is continuous, if  $f$  is Lipschitzian on an  $L_2$  measurable set  $A \subset Q^2$ , and  $f = (u, v)$  where  $u$  is BVT, then*

$$\mathfrak{L}(f) \cong \int_{E^2} N(f, A, y) dL_2(y).$$

**REMARK 3.8.** It is easy to see that if neither of the coordinate functions of  $f$  is BVT, then the conclusion of (3.7) may not hold. For this purpose let  $A \subset Q^2$  be a dendrite for which  $L_2(A) > 0$ . Then a result from [15, p. 290] implies that  $A$  is a retract of  $Q^2$ . If  $r: Q^2 \rightarrow A$  is the retraction and  $i: A \rightarrow A$  the identity map, then  $f = ir$  is clearly Lipschitzian on  $A$  and  $\mathfrak{L}(f) = 0$  since the range of  $f$  has no interior.

**THEOREM 3.9.** *Suppose  $f: Q^2 \rightarrow E^2$  is continuous,  $f = (u, v)$ ,  $u$  is ACT,  $v$  satisfies condition  $N_1$  on  $Q^2$ , the approximate partial derivatives of  $v$  exist  $L_2$  almost everywhere on  $Q^2$ , and  $Jf$ , the approximate Jacobian of  $f$ , is integrable. Then*

$$\mathfrak{L}(f) = \int_{Q^2} |Jf(x)| dL_2(x) = \int_{E^2} N(f, Q_2, y) dL_2(y).$$

*Proof.* Referring to [5, (5.4)] and (3.6) we see that we only need to prove that  $f$  carries sets of  $L_2$  measure zero into sets of  $L_2$  measure zero. If this were not the case, then there would exist an  $L_2$  null set  $N \subset Q^2$  for which  $L_2[f(N)] > 0$ . We may assume that  $f(N)$  is measurable since  $N$  can be taken as a  $G_\delta$  set. Thus,  $L_1[v(u^{-1}(s) \cap N)] > 0$  and therefore  $H_2^1[u^{-1}(s) \cap N] > 0$  for all  $s$  in some set of positive  $L_1$  measure. But, from (2.3.2) and (3.1)

$$0 = \int_N |\text{grad } u(x)| dL_2(x) = \int_{E^1} H_2^1[u^{-1}(s) \cap N] dL_1(s) > 0$$

a contradiction.

COROLLARY 3.10. *If  $u$  is ACT and  $v$  Lipschitzian on  $Q^2$ , then*

$$\mathfrak{L}(f) = \int_{Q^2} |Jf(x)| dL_2(x) = \int_{E^2} N(f, Q^2, y) dL_2(y).$$

REMARK 3.11. The above corollary is an extension of a theorem proved in [17, p. 437], where only the first part of the equality is established. Both (3.8) and (3.9) are related to the following unsolved problem c.f. [16, p. 380], [17, p. 433]: Let  $f: Q^2 \rightarrow E^2$  where both coordinate functions of  $f$  are ACT and  $Jf$  is  $L_2$  integrable. Then, is

$$\mathfrak{L}(f) = \int_{Q^2} |Jf(x)| dL_2(x) ?$$

By using techniques employed in this paper, one can show that if the additional hypothesis is made that  $v$  satisfies condition  $N_1$  on  $W_s = u^{-1}(s) \cap \{x: \dim[u^{-1}(s), x] > 0\}$  for  $L_1$  almost all  $s \in E^1$ , then the question can be settled in the affirmative.

#### BIBLIOGRAPHY

1. L. Cesari, *Sui punti di diramazione delle trasformazioni continue e sull' area della superficie in forma parametrica*, Rend. Sem. Mat. Univ. Roma (5) **3** (1942), 37-62.
2. E. DeGiorgi, *Nuovi teoremi relativi alle misure  $(r-1)$ -dimensionali in uno spazio ad  $r$ -dimensioni*, Ricerche Mat. **4**(1955), 95-113.
3. M. R. Demers and H. Federer, *On Lebesgue area II*, Trans. Amer. Math. Soc. **90** (1959), 499-522.
4. S. Eilenberg, *On  $\varphi$  measure*, Annales de la Société Polonaise de Mathématique **17** (1938), 252-253.
5. H. Federer, *Surface area II*, Trans. Amer. Math. Soc. **55** (1944), 438-456.
6. ———, *The Gauss-Green theorem*, Trans. Amer. Math. Soc. **58** (1945), 44-76.
7. ———, *The  $(\varphi, k)$  rectifiable subsets of  $n$  space*, Trans. Amer. Math. Soc. **62** (1947), 114-192.
8. ———, *Measure and area*, Bull. Amer. Math. Soc. **58** (1952), 306-378.
9. ———, *On Lebesgue area*, Ann. of Math. **61** (1955), 289-353.
10. ———, *A note on the Gauss-Green theorem*, Proc. Amer. Math. Soc. **9** (1958), 447-451.
11. ———, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418-491.
12. ———, *The area of a nonparametric surface*, Proc. Amer. Math. Soc. **11** (1960), 436-439.
13. H. Federer and W. H. Fleming, *Normal and integral currents*, Ann. of Math. **72** (1960), 458-520.
14. W. H. Fleming and R. Rishel, *An integral formula for total gradient variation*, Arch. Math. **11** (1960), 218-222.
15. C. Kuratowski, *Topologie*, Monografie Matematyczne **21** (1950), Warsaw.
16. T. Rado, *Length and area*, Amer. Math. Soc. Colloquium Pub. **30** (1948).
17. T. Rado and P. V. Reichelderfer, *Continuous transformations in Analysis*, Grundlehren der mathematischen Wissenschaften, Springer, 1955.

18. R. N. Thompson, *Areas of  $k$  dimensional nonparametric surfaces in  $k + 1$  space*, Trans. Amer. Math. Soc. **77** (1954), 374-407.
19. G. T. Whyburn, *Topological Analysis*, Princeton, 1958.
20. L. C. Young, *Partial Area I*, Riv. Mat. Univ. Parma **10** (1959), 103-113.
21. W. Ziemer, *On a sufficient condition for onto-ness*, J. Math. Mech. **13** (1964), 503-509.
22. ———, *The structure of quasi-open maps*, Duke Math. J. **32** (1965), 661-672.

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