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**DIAGONABILITY OF IDEMPOTENT MATRICES**

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A ring  $\mathcal{R}$  (commutative with identity) with the property that every idempotent matrix over  $\mathcal{R}$  is diagonalizable (i.e., similar to a diagonal matrix) will be called an *ID-ring*. We show that, in an *ID-ring*  $\mathcal{R}$ , if the elements  $a_1, a_2, \dots, a_n \in \mathcal{R}$  generate the unit ideal then the vector  $[a_1, a_2, \dots, a_n]$  can be completed to an invertible matrix over  $\mathcal{R}$ . We establish a canonical form (unique with respect to similarity) for the idempotent matrices over an *ID-ring*. We prove that if  $\mathcal{N}$  is the ideal of nilpotents in  $\mathcal{R}$  then  $\mathcal{R}$  is an *ID-ring* if and only if  $\mathcal{R}/\mathcal{N}$  is an *ID-ring*. The following are then shown to be *ID-rings*: elementary divisor rings, a restricted class of Hermite rings,  $\pi$ -regular rings, quasi-semi-local rings, polynomial rings in one variable over a principal ideal ring (zero divisors permitted), and polynomial rings in two variables over a  $\pi$ -regular ring with finitely many idempotents.

In this paper,  $\mathcal{R}$  will denote a commutative ring with identity, and  $\mathcal{R}_n$  will denote the set of  $n \times n$  matrices over  $\mathcal{R}$ . If  $A, B \in \mathcal{R}_n$ , then  $A \cong B$  will mean that  $A$  is similar to  $B$ . We remark that if  $\mathcal{R}$  is an *ID-ring* then every finitely generated projective  $\mathcal{R}$ -module is the finite direct sum of cyclic modules, and that  $\mathcal{R}$  is a directly indecomposable *ID-ring* if and only if every finitely generated projective  $\mathcal{R}$ -module is free. Most of the literature on this subject has been concerned with showing that a given ring  $\mathcal{R}$  has the property that every finitely generated projective  $\mathcal{R}$ -module is free. This necessarily imposes the condition that  $\mathcal{R}$  be indecomposable. In this paper, no such restriction is made.

## 2. Properties of *ID-rings*.

DEFINITION 1.  $\mathcal{R}$  is said to be an *ID-ring* provided that for every  $A = A^2 \in \mathcal{R}_n$ ,  $n = 1, 2, \dots$ , there exists an invertible matrix  $P \in \mathcal{R}_n$  such that  $PAP^{-1}$  is a diagonal matrix.

DEFINITION 2. The row vector  $[a_1, a_2, \dots, a_n]$  with components in  $\mathcal{R}$  is said to be a basal provided that it can be completed to an invertible matrix over  $\mathcal{R}$ .

DEFINITION 3. The row vector  $X$  is said to be a characteristic vector of  $A \in \mathcal{R}_n$  corresponding to  $r \in \mathcal{R}$  provided (1)  $X$  is a basal vector and (2)  $XA = rX$ .

The following lemma, due to A. L. Foster, is an important tool in our development.

**FOSTER'S LEMMA.**  *$\mathcal{R}$  is an ID-ring if and only if every idempotent matrix over  $\mathcal{R}$  has a characteristic vector.*

From this lemma, which appears essentially as Theorem 10 in [2], one can quickly deduce that quasi-local rings and principal ideal domains are ID-rings. Then, known structure theorems suffice to show that principal ideal rings (see [7], p. 66), rings with descending chain condition, and Boolean rings are ID. These results will be extended in the next section.

**THEOREM 1.** *Let  $A = A^2 \in \mathcal{R}_n$ . If there exist invertible matrices  $P, Q \in \mathcal{R}_n$  such that  $PAQ$  is a diagonal matrix then  $A$  is diagonal.*

*Proof.* Let  $PAQ = B = \text{diag}(b_1, b_2, \dots, b_n)$  and let  $U = Q^{-1}P^{-1} = (u_{ij})$ . Then  $(BU)^2 = BU$  and  $BUB = B$ . Hence  $b_i = b_i^2 u_{ii}$ ,  $b_i u_{ii}$  is idempotent, and by Lemma 2.1 of [9]  $b_i \sim b_i u_{ii}$  for each  $i$ . Thus, we may assume that  $Q$  has been adjusted so that  $b_i^2 = b_i$ ,  $i = 1, 2, \dots, n$ . The equation  $BUB = B$  now yields

- (1)  $b_i u_{ii} = b_i$ ,  $i = 1, 2, \dots, n$ , and
- (2)  $b_i b_j u_{ij} = 0$ ,  $i \neq j$ ,  $i, j = 1, 2, \dots, n$ .

From (1),

$$BU = \begin{bmatrix} b_1 & b_1 u_{12} & \cdots & b_1 u_{1n} \\ b_2 u_{21} & b_2 & \cdots & b_2 u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_n u_{n1} & b_n u_{n2} & \cdots & b_n \end{bmatrix}.$$

If  $X_k = [b_k u_{k1}, b_k u_{k2}, \dots, b_k u_{kk-1}, 1, b_k u_{kk+1}, \dots, b_k u_{kn}]$  then  $X_k BU = b_k X_k$ ,  $k = 1, 2, \dots, n$ . Now let

$$C = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

From (2), it follows that  $|C| = 1$ . Hence  $(CP)A(CP)^{-1} = CBUC^{-1} = \text{diag}(b_1, b_2, \dots, b_n)$ .

**THEOREM 2.** *Let  $\mathcal{R}$  be an ID-ring. If  $a_1, a_2, \dots, a_n \in \mathcal{R}$  generate the unit ideal in  $\mathcal{R}$  then the vector  $[a_1, a_2, \dots, a_n]$  is basal.*

*Proof.* Let  $\sum_{i=1}^n x_i a_i = 1$  and let  $B = (x_i a_j) \in \mathcal{R}_n$ . Then  $B^2 = B$  and  $\text{tr } B = 1$ . Since  $\mathcal{R}$  is ID,  $B \cong C = \text{diag}(c_1, c_2, \dots, c_n)$ . If  $X = [c_1, c_2, \dots, c_n]$  then  $XC = X$  and, since  $\sum_{i=1}^n c_i = 1$ ,

$$\begin{pmatrix} c_1 & c_2 & c_3 & \dots & c_n \\ -1 & 1 & 0 & \dots & 0 \\ -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & 1 \end{pmatrix} = 1.$$

Hence,  $B$  has a characteristic vector  $Y = [y_1, y_2, \dots, y_n]$  corresponding to 1. From  $YB = Y$ , we have  $\sum_{i=1}^n y_i x_i a_j = y_j, j = 1, 2, \dots, n$ . Thus  $(\sum_{i=1}^n y_i x_i) [a_1, a_2, \dots, a_n] = [y_1, y_2, \dots, y_n]$ . Since  $Y$  is basal, so also is  $[a_1, a_2, \dots, a_n]$ .

**THEOREM 3.** *If  $\mathcal{R}$  is an ID-ring then every invertible ideal in  $\mathcal{R}$  is principal.*

*Proof.* Let  $\mathcal{K}$  be an invertible ideal in  $\mathcal{R}$ . Then there exist elements  $a_1, a_2, \dots, a_n \in \mathcal{K}$  and elements  $x_1, x_2, \dots, x_n$  in the full ring of quotients of  $\mathcal{R}$  such that  $x_i \mathcal{K} \subseteq \mathcal{R}, i = 1, 2, \dots, n$ , and  $\sum_{i=1}^n x_i a_i = 1$ . It follows that  $\mathcal{K} = (a_1, a_2, \dots, a_n)$ . Let  $B = (x_i a_j) \in \mathcal{R}_n$ . Then, as in Theorem 2, there exists a basal vector  $Y = [y_1, y_2, \dots, y_n]$  such that  $y_j = \sum_{i=1}^n y_i x_i a_j, j = 1, 2, \dots, n$ . Now let  $x_i = c_i/d, c_i, d \in \mathcal{R}$  and  $d$  not a zero divisor. If  $p = \sum_{i=1}^n y_i c_i$  then  $[pa_1, pa_2, \dots, pa_n] = [dy_1, dy_2, \dots, dy_n]$ . Since  $Y$  is basal,  $p\mathcal{K} = (d)$ . Hence there is an  $a \in \mathcal{K}$  such that  $pa = d$ . Thus,  $p$  is not a zero divisor. If  $b \in \mathcal{K}$ , then for some  $r \in \mathcal{R}, pb = rd = pra$ . Hence,  $b = ra$  and  $\mathcal{K} = (a)$ .

Recall that if  $\mathcal{S}$  is the set of idempotents of  $\mathcal{R}$  then  $\langle \mathcal{S}, \cap, \cup, * \rangle$  where  $a \cap b = ab, a \cup b = a + b - ab$ , and  $a^* = 1 - a$ , is a Boolean algebra (see [1]). It follows that if  $a_1, a_2, \dots, a_n \in \mathcal{S}$  and  $a = \mathbf{U}_{i=1}^n a_i$  then  $a_1, a_2, \dots, a_n$  generate the principal ideal  $(a)$  in  $\mathcal{R}$ .

**THEOREM 4.** (Canonical Form) *Let  $\mathcal{R}$  be an ID-ring and let  $A = A^2 \in \mathcal{R}_n$ . Then  $A \cong \text{diag}(a_1, a_2, \dots, a_n)$  where  $a_i | a_{i+1}, i = 1, 2, \dots, n - 1$ . Moreover, if  $A \cong \text{diag}(b_1, b_2, \dots, b_n)$  with  $b_i | b_{i+1}, i = 1, 2, \dots, n - 1$ , then  $a_i = b_i, i = 1, 2, \dots, n$ .*

*Proof.* Since  $\mathcal{R}$  is ID, let  $A \cong C = \text{diag}(c_1, c_2, \dots, c_n)$  and let  $a_1 = \mathbf{U}_{i=1}^n c_i$ . Then there exist idempotents  $x_1, x_2, \dots, x_n$  such that  $x_i a_1 = c_i$  for each  $i$  and  $\mathbf{U}_{i=1}^n x_i = 1$ . Thus,  $(x_1, x_2, \dots, x_n) = 1$  and, by Theorem 2,  $X = [x_1, x_2, \dots, x_n]$  is basal. Since  $x_i$  is idempotent,  $i = 1, 2, \dots, n, XC = a_1 X$  and, as in the proof of Foster's Lemma,

$A \cong \text{diag}(a_1, d_2, \dots, d_n)$ . By induction,  $A \cong \text{diag}(a_1, a_2, \dots, a_n)$  where  $a_i | a_{i+1}$ ,  $i = 2, 3, \dots, n - 1$ . Since  $a_1$  divides each entry of  $C$ ,  $a_1 | a_2$ . If also,  $A \cong \text{diag}(b_1, b_2, \dots, b_n)$  with  $b_i | b_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ , then it is a consequence of Theorem 9.3 of [6] that  $b_i = a_i$  for each  $i$ . This can also be seen directly as follows: since  $a_r$  divides each  $r$ -rowed minor of  $\text{diag}(a_1, a_2, \dots, a_n)$ ,  $a_r$  divides  $b_r = b_1 b_2 \dots b_r$ . Similarly,  $b_r$  divides  $a_r$  and, since both  $a_r$  and  $b_r$  are idempotent,  $a_r = b_r$ ,  $r = 1, 2, \dots, n$ .

**COROLLARY.** *If  $\mathcal{R}$  is ID and  $A = A^2 \in \mathcal{R}_n$  then  $A$  has a characteristic vector corresponding to  $|A|$ .*

*Proof.* We need merely observe that if  $A \cong \text{diag}(a_1, a_2, \dots, a_n)$  with  $a_i | a_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ , then  $a_n = |A|$ .

**THEOREM 5.** *Let  $\mathcal{J}$  be the Jacobson radical of  $\mathcal{R}$ , let  $\mathcal{N}$  be the ideal of nilpotents in  $\mathcal{R}$ , and let  $\mathcal{K}$  be an arbitrary ideal in  $\mathcal{R}$ . If  $\mathcal{K} \subseteq \mathcal{J}$  and  $\mathcal{R}|\mathcal{K}$  is an ID-ring then  $\mathcal{R}$  is an ID-ring. If  $\mathcal{K} \subseteq \mathcal{N}$  then  $\mathcal{R}$  is an ID-ring if and only if  $\mathcal{R}|\mathcal{K}$  is an ID-ring.*

*Proof.* Let  $\mathcal{K} \subseteq \mathcal{J}$  and assume that  $\mathcal{R}|\mathcal{K}$  is ID. Let  $A = A^2 = (A_{ij}) \in \mathcal{R}_n$  and  $A^* = (a_{ij} + \mathcal{K})$ . Then  $(A^*)^2 = A^*$  and if  $d = |A|$  then  $d + \mathcal{K} = |A^*|$ . By the corollary to Theorem 4, we may let  $X^* = [x_1 + \mathcal{K}, x_2 + \mathcal{K}, \dots, x_n + \mathcal{K}]$  be a characteristic vector of  $A^*$  corresponding to  $d + \mathcal{K}$ . Then, if  $X = [x_1, x_2, \dots, x_n]$ ,  $XA = dX + Y$  where the components of  $Y$  are in  $\mathcal{K}$ . Since  $A^2 = A$  and  $d^2 = d$ ,  $XA = dXA + YA$ ,  $YA = (1 - d)XA = (1 - d)Y$ , and  $(X + (2d - 1)Y)A = dX + dY = d(X + (2d - 1)Y)$ . Since  $\mathcal{K} \subseteq \mathcal{J}$ ,  $u + \mathcal{K}$  is a unit of  $\mathcal{R}|\mathcal{K}$  if and only if  $u$  is a unit of  $\mathcal{R}$ . It follows, therefore, that since  $X^*$  is basal so also is  $X + (2d - 1)Y$ . By Foster's Lemma,  $\mathcal{R}$  is ID. Now let  $\mathcal{K} \subseteq \mathcal{N}$ . Since  $\mathcal{N} \subseteq \mathcal{J}$ , we need only prove that if  $\mathcal{R}$  is ID then  $\mathcal{R}|\mathcal{K}$  is ID. Hence, assume that  $\mathcal{R}$  is ID and  $A^* = (A^*)^2 = (a_{ij} + \mathcal{K}) \in (\mathcal{R}|\mathcal{K})_n$ . It will suffice to show that there exists an idempotent matrix  $F = (f_{ij}) \in \mathcal{R}_n$  such that  $f_{ij} + \mathcal{K} = a_{ij} + \mathcal{K}$ ,  $i, j = 1, 2, \dots, n$ . If  $A = (a_{ij})$  then  $A^2 = A + B$  where the components of  $B$  are in  $\mathcal{K}$ . Thus  $B$  is nilpotent. Let  $k$  be the least natural number such that  $B^k = Z = \text{zero matrix}$ . If  $k = 1$ , there is nothing left to prove. Hence, assume that  $k > 1$  and let  $C = A + (I - 2A)B$ . Then the components of  $C - A$  are in  $\mathcal{K}$  and, since  $AB = BA$ ,

$$C^2 = A^2 + 2A(I - 2A)B + (I - 2A)^2 B^2 .$$

Therefore,  $C^2 - C = B + (I - 2A)^2(B^2 - B)$ . Since  $(I - 2A)^2 = I + 4B$ ,

$C^2 = C + B^2(4B - 3I)$ . If we let  $D = B^2(4B - 3I)$ , we have  $C^2 = C + D$  where the components of  $D$  are in  $\mathcal{K}$  and, for some natural number  $l < k$ ,  $D^l = Z$ . Repeating this process, we arrive in a finite number of steps at the required matrix  $F$ .

**COROLLARY.** *Let  $\mathcal{N}$  be the ideal of nilpotents in  $\mathcal{R}$  and let  $x_1, x_2, \dots, x_k$  be indeterminates. Then  $\mathcal{R}[x_1, x_2, \dots, x_k]$  is ID if and only if  $(\mathcal{R}/\mathcal{N})[x_1, x_2, \dots, x_k]$  is ID.*

*Proof.* The corollary follows by observing that  $\mathcal{N}[x_1, x_2, \dots, x_k]$  is the ideal of nilpotents in  $\mathcal{R}[x_1, x_2, \dots, x_k]$  and that

$$\mathcal{R}[x_1, x_2, \dots, x_k]/\mathcal{N}[x_1, x_2, \dots, x_k] \approx (\mathcal{R}/\mathcal{N})[x_1, x_2, \dots, x_k].$$

**3. Classes of ID-rings.** As an immediate consequence of Theorem 1, we have:

**THEOREM 6.** *An elementary divisor ring is an ID-ring.*

**THEOREM 7.** *Let  $\mathcal{R}$  be a Hermite ring with Jacobson radical  $\mathcal{J}$ . If  $\mathcal{R}$  has the property that  $ab = 0$  implies either  $(a) = (a^2)$  or  $a \in \mathcal{J}$  or  $b \in \mathcal{J}$  then  $\mathcal{R}$  is an ID-ring.*

*Proof.* Let  $A = A^2 = (a_{ij}) \in \mathcal{R}_n$  and let  $Q$  be an invertible matrix such that  $QA = B = (b_{ij})$  is triangular; i.e.,  $b_{ij} = 0$  if  $i < j$ . Let  $Q^{-1} = (p_{ij})$ . Then  $X = [b_{11}p_{11}, b_{11}p_{12}, \dots, b_{11}p_{1n}]$  is the first row of  $QAQ^{-1}$ . If  $(b_{11}) = (b_{11}^2)$  then there is an idempotent  $e$  such that  $b_{11} \sim e$ . By Theorem 3.9 of [6], there are vectors  $X_2, X_3, \dots, X_n$  such that

$$\begin{bmatrix} X \\ X_2 \\ \vdots \\ X_n \end{bmatrix} = e. \text{ If } C = \begin{bmatrix} X \\ eX_1 \\ \vdots \\ eX_2 \end{bmatrix} \text{ then } |C + (1 - e)I| = 1. \text{ Thus, the vector}$$

$Y = [b_{11}p_{11} + 1 - e, b_{11}p_{12}, \dots, b_{11}p_{1n}]$  is basal and  $Y(QAQ^{-1}) = X = eX = eY$ ; i.e,  $Y$  is a characteristic vector of  $QAQ^{-1}$  corresponding to  $e$ . If  $b_{11} \in \mathcal{J}$  then  $1 - b_{11}p_{11}$  is a unit of  $\mathcal{R}$  and

$$[1 - b_{11}p_{11}, -b_{11}p_{12}, \dots, -b_{11}p_{1n}]$$

is a characteristic vector of  $QAQ^{-1}$  corresponding to 0. Suppose now that neither of these assumptions on  $b_{11}$  is true. From the equation,  $BA = QA^2 = QA = B$ , we obtain  $b_{11}(1 - a_{11}) = 0$ . By the hypothesis on  $\mathcal{R}$ ,  $1 - a_{11} \in \mathcal{J}$ ,  $a_{11}$  is a unit of  $\mathcal{R}$ , and  $[a_{11}, a_{12}, \dots, a_{1n}]$  is a characteristic vector of  $A$  corresponding to 1. In any event,  $A$  has a characteristic vector and Foster's Lemma completes the proof.

**THEOREM 8.** *A  $\pi$ -regular ring is an ID-ring.*

*Proof.* Let  $\mathcal{R}$  be  $\pi$ -regular with Jacobson radical  $\mathcal{J}$ . Then  $\mathcal{R}/\mathcal{J}$  is regular and, therefore an elementary divisor ring (see [3], p. 365). The conclusion follows from Theorems 5 and 6.

**THEOREM 9.** *A quasi-semi-local ring is an ID-ring.*

*Proof.* Let  $\mathcal{R}$  be quasi-semi-local with Jacobson radical  $\mathcal{J}$ . Since, by definition,  $\mathcal{R}$  has only a finite number of maximal ideals,  $\mathcal{R}/\mathcal{J}$  is a finite direct sum of fields. Theorem 5 completes the proof.

**THEOREM 10.** *Let  $\mathcal{R}$  be an ID-ring and let  $\mathcal{S}$  be a subring of  $R[[x]]$  which contains  $\mathcal{R}$ . If  $\mathcal{S}$  has the property that  $u \in \mathcal{S}$  and  $u$  is a unit of  $\mathcal{R}[[x]]$  imply that  $u$  is a unit of  $\mathcal{S}$  then  $\mathcal{S}$  is an ID-ring.*

*Proof.* Let  $A = A^2 \in \mathcal{S}_n$  and let  $A'$  be the matrix in  $\mathcal{S}_n$  obtained from  $A$  by suppressing all positive powers of  $x$ . If  $A' = Z =$  zero matrix and  $A \neq Z$ , let  $k$  be the highest power of  $x$  which divides (in  $R[[x]]$ ) each entry in  $A$ . Then we may write  $A = x^k B$ ; and some entry in  $B$  is not divisible by  $x$ . Since  $A$  is idempotent  $x^k B = x^{2k} B^2$ . Thus,  $B = x^k B^2$  and, since  $k > 0$ , we have arrived at a contradiction. Again, let  $A = A^2 \in \mathcal{S}_n$ . Then  $(A')^2 = A'$  and, since  $\mathcal{R}$  is ID, it follows from Theorem 4 that the entries of  $A'$  generate in  $\mathcal{R}$  a principal ideal  $(e)$  where  $e$  is idempotent. Then  $(1 - e)A$  is idempotent and  $((1 - e)A)' = Z$ . Thus,  $(1 - e)A = Z$ . Let  $P$  be an invertible matrix in  $\mathcal{S}_n$  such that  $PA'P^{-1} = \text{diag}(a_1, a_2, \dots, a_n)$  where  $a_i | a_{i+1}$ ,  $i = 1, 2, \dots, n - 1$ . Therefore,  $a_1 = e$  and  $PAP^{-1} = B = (b_{ij})$  with  $b_{11} = e + r_1 x + r_2 x^2 + \dots$ . Then, if  $Y = [1 - e + b_{11}, b_{12}, \dots, b_{1n}]$ ,  $(1 - e)B = Z$  implies  $YB = eY$ . Since  $1 - e + b_{11}$  is a unit in  $R[[x]]$ , by the hypothesis on  $\mathcal{S}$ ,  $Y$  is a characteristic vector corresponding to  $e$ . The theorem follows from Foster's lemma.

Theorem 10 shows for example that the domain of complex valued functions of a complex variable which are analytic at some point  $z_0$  in the complex plane is an ID-ring, or that the domain of real valued functions of a real variable analytic at some real number  $r_0$  is ID. It is also true that the domain of entire functions is ID. This has, however, nothing to do with Theorem 10; but it is rather a consequence of Theorem 7 in conjunction with a theorem proved in [4] to the effect that in the domain of entire functions every finitely generated ideal is principal.

The problem of determining, given a ring  $\mathcal{R}$ , whether or not  $\mathcal{R}[x]$  is *ID* is a difficult one. An important result in this area is due to Seshadri who proved in [8] that if  $\mathcal{R}$  is a principal ideal domain then  $\mathcal{R}[x]$  is *ID*. In particular,  $\mathcal{K}[x, y]$ , where  $\mathcal{K}$  is a field, is *ID*. The character of  $\mathcal{K}[x, y, z]$  is open. Horrocks showed ([5]), p. 718) that if  $\mathcal{R}$  is a regular local ring of dimension 2 with a field of coefficients then  $\mathcal{R}[x]$  is *ID*. Chase, on the other hand, has constructed an example (unpublished) of a complete local domain  $\mathcal{R}$  such that  $\mathcal{R}[x]$  is not *ID*. The ring in Chase's example has dimension 1, is not a regular local ring, and in fact is not integrally closed.

**THEOREM 11.** *Let  $\mathcal{R}$  be a ring with  $\mathcal{N}$  its ideal of nilpotents. (1) If  $\mathcal{R}/\mathcal{N}$  is a principal ideal ring then  $\mathcal{R}[x]$  is *ID*; (2) if  $\mathcal{R}/\mathcal{N}$  is a Boolean ring then  $\mathcal{R}[x, y]$  is *ID*; and (3) if  $\mathcal{R}$  is a  $\pi$ -regular ring with finitely many idempotents then  $\mathcal{R}[x, y]$  is *ID*.*

*Proof.* The assertions of this theorem are a consequence of applying the Corollary to Theorem 5 to Seshadri's result. First, assume that  $\mathcal{R}/\mathcal{N}$  is a principal ideal ring. It is a consequence of the result on page 66 of [7] that  $\mathcal{R}/\mathcal{N}$  is a finite direct sum of principal ideal domains. Thus (1) has been established. Now assume that  $\mathcal{R}/\mathcal{N}$  is a Boolean ring and let  $A = A^2 \in ((\mathcal{R}/\mathcal{N})[x, y])_n$ . Then the set of coefficients of the entries in  $A$  together with 1 generate a finite Boolean subring  $\mathcal{S}$  of  $\mathcal{R}/\mathcal{N}$  whose unit element is the unit element of  $\mathcal{R}/\mathcal{N}$ . Since  $\mathcal{S}$  is the finite direct sum of fields,  $A$  is diagonal and (2) has been proved. Finally, assume that  $\mathcal{R}$  is a  $\pi$ -regular ring with finitely many idempotents. Then  $\mathcal{R}/\mathcal{N}$  is the finite direct sum of fields. This completes the proof of (3).

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