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WALTER A. STRAUSS

In the discussion of weak solutions of certain kinds of partial differential equations, a crucial point, which is isolated in this paper, concerns the proof of identities of energy type and of the continuity of the solutions, which two questions are intimately related. The continuity referred to is with respect to a distinguished independent real variable t, the other variables being suppressed into some Banach space.

In § 2 a simple argument shows that an essentially bounded function of t with values in a space V is automatically weakly continuous in V provided it is weakly continuous in some larger space.

In § 3 conditions are found under which a square-integrable function $u(t)$ with values in V is strongly continuous in V (Theorem 3.3). Roughly speaking, the main condition is that there exist self-adjoint linear operators $A(t)$ coercive with respect to V such that $A(\cdot)u(\cdot)$ and du/dt lie in spaces which are dual to each other.

The proof is based on an energy identity (Theorem 3.1). Theorem 3.3 has application to certain equations of the form

$$A(t)u(t) = \text{(possibly) nonlinear terms in } du/dt \tag{1.1}$$

(cf. [5]).

The analogous strong continuity and energy problems for equations like (cf. [4])

$$d^2u/dt^2 + A(t)u(t) = \text{(possibly) nonlinear terms} \tag{1.2}$$

are considered in § 4. These results are used in [6].

Torelli [7] solved these problems in the case of linear equations (1.2). Partial results for the nonlinear case of (1.2) were obtained in [4] and [1] and for (1.1) in [5]. Here Torelli's method is generalized so as to be applicable in the nonlinear case of (1.2) as well as to (1.1). In each of these cases variants of a regularization procedure of Lions and Prodi [3] for the Navier-Stokes equations are used. Lions [2] also proves a different case of Theorem 3.2.

2. Weak continuity.

We begin with some definitions. If V is a Banach space and Ω is a real closed interval then by $L^p(\Omega; V)$ we mean the space of all strongly measurable functions u on Ω into V such that

$$|| u ||^p = \int_\Omega | u(t) |_V^p \, dt < \infty$$

if $1 \leqq p < \infty$ and

$$|| u || = \text{ess sup} \{ | u(t) |_V, \, t \in \Omega \}$$

if $p = \infty$. By $C_w(\Omega; V)$ (resp. $C_s(\Omega; V)$) we mean the subspace of $L^\infty(\Omega; V)$ consisting of those functions which are a.e. equal to weakly (resp. strongly) continuous functions with values in V. Primes on a function of t denote (weak) derivatives with respect to t. For a space V, V' denotes the "dual" space of all continuous conjugate-linear functions on V.

THEOREM 2.1. *Let V and Y be Banach spaces, V reflexive, V a dense subset of Y and the inclusion map of V into Y continuous. Then*

$$L^\infty(0, T; V) \cap C_w(0, T; Y) = C_w(0, T; V) \, .$$

Proof. Let u be weakly continuous with values in Y and $u \in L^\infty(0, T; V)$. We must show that it is also weakly continuous with values in V. (The converse is obvious.) It suffices to prove that there is a constant M such that

$$u(t) \in V \text{ and } | u(t) |_V \leqq M \text{ for all } t \in [0, T] \, . \tag{2.1}$$

Indeed, if (2.1) is true we can choose from any convergent net $t_n \to t_0$ of numbers $[0, T]$ a subnet t_m such that $u(t_m)$ converges weakly in V. Since u is weakly continuous with values in Y, the limit of this subnet must be $u(t_0)$. Therefore $u(t_n) \to u(t_0)$ weakly in V.

To prove (2.1) we define approximate delta-functions $\eta_\varepsilon(t)$ in the usual way: let $\eta_0(t)$ be a nonnegative even C^∞-function on the real line with compact support and integral one; define $\eta_\varepsilon(t) = \varepsilon^{-1}\eta_0(t/\varepsilon)$ for $\varepsilon > 0$. Now consider $0 < t < T$ so that $(\eta_\varepsilon * u)(t) \in V$ for sufficiently small ε. Let M be the norm of u in $L^\infty(0, T; V)$. Then

$$| (\eta_\varepsilon * u)(t) |_V \leqq \int \eta_\varepsilon(s) \, | u(t - s) |_V \, ds \leqq M \, .$$

So there exists a net of ε's such that $(\eta_\varepsilon * u)(t)$ converges weakly in V. On the other hand, for all $v \in Y'$,

$$((\eta_\varepsilon * u)(t) - u(t), v) \to 0 \quad \text{as} \quad \varepsilon \to 0$$

since $(u(s), v)$ is a continuous function of s. It follows that $u(t) \in V$,

$$(\eta_\varepsilon * u)(t) \to u(t) \qquad \text{weakly in } V$$

and $|u(t)|_V \leqq M$.

To prove (2.1) for the case $t = 0$, we apply a similar argument with $(\eta_\varepsilon * u)(t)$ replaced by $(\eta_\varepsilon * u)(\varepsilon)$; and for $t = T$, we consider $(\eta_\varepsilon * u)(T - \varepsilon)$.

COROLLARY 2.1. *Let V and W be Banach spaces, V reflexive, both contained in some fixed linear space, and $V \cap W$ dense in V and in W. If $u \in L^\infty(0, T; V)$ and $u' = du/dt \in L^1(0, T; W)$, then there exists a weakly continuous function on $[0, T]$ with values in V which is equal to u almost everywhere.*

Proof. The space $Y = V + W$ satisfies the conditions of Theorem 1. The assumptions on u imply, in particular, that $u \in L^1(0, T; Y)$ and $u' \in L^1(0, T; Y)$. Therefore u is weakly continuous with values in Y, and Theorem 2.1 may be applied.

3. **Strong continuity.** We are given a reflexive Banach space V and a family of hermitian linear operators $A(t): V \to V'$ $(0 \leqq t \leqq T)$ which are weakly continuously differentiable functions of t. If $u \in L^1(0, T; V)$, we denote by Au the function $t \to A(t)u(t)$ and by $A'u$ the function $t \to [dA(t)/dt]u(t)$. By the uniform boundedness principle, $A(t)$ and $A'(t) = dA(t)/dt$ are uniformly bounded operators from V to V'. The pairing between elements of V and V' is denoted by $(\ ,\ )$.

Secondly, we are given another Banach space W as well as a linear space containing both V and W such that $V \cap W$ is dense in V and W.[1] In addition, Z is a Banach space such that

$$Z \subset L^1(0, T; W), \qquad Z' \subset L^1(0, T; W')$$

as dense subsets with continuous inclusion. If $f \in Z'$, $u \in Z$ and $\langle f, u \rangle$ indicates the pairing, we assume that $(f(t), u(t))$ [pairing in W] is integrable and

$$\langle f, u \rangle = \int_0^T (f(t), u(t))dt \,. \tag{3.1}$$

Furthermore, we assume that multiplications by step functions map Z into Z and that translations in t are continuous in the strong operator topology on Z (where functions are extended to be zero outside $[0, T]$ when necessary).

THEOREM 3.1. *Let $u \in L^\infty(0, T; V)$,[2] $u' = du/dt \in Z$ and $Au \in Z'$. Then for every $t \in [0, T]$*

[1] Elements of V' and W' are identified if they agree on $V \cap W$.

[2] Whence $Au \in L^\infty(0, T; V')$ and u' is a distribution with values in V.

$$(3.2)\qquad (A(t)u(t), u(t)) - (A(0)u(0), u(0)) = \int_0^t (A'u, u) + 2\mathrm{Re}\int_0^t (Au, u') .$$

Proof. By Corollary 2.1 we may assume that u is weakly continuous with values in V.[3] We shall prove (3.2) for a fixed $t = t_0$, $0 < t_0 \leqq T$. Let θ_0 be the characteristic function of the interval $[0, t_0]$; for small $\delta > 0$, let $\theta(t) = \theta_\delta(t)$ be 1 for $t \in [\delta, t_0 - \delta]$, zero for $t \notin (0, t_0)$ and linear in the intervals $[0, \delta]$ and $[t_0 - \delta, t_0]$. Next $\eta(t) = \eta_\varepsilon(t)$ is defined as in the proof of Theorem 1 as an approximate delta-function.

For any C^∞-function v with values in V with compact support, we have

$$0 = \int_{-\infty}^{\infty} d(Av, v) = \langle A'v, v\rangle + 2\mathrm{Re}\langle Av, v'\rangle .$$

Putting $v = \eta * (\theta u)$ and noting that

$$v' = \eta' * \theta u = \eta * (\theta' u) + \eta * (\theta u') ,$$

we have

$$(3.3)\qquad \begin{cases} 0 = \langle A'(\eta * \theta u), \eta * \theta u\rangle \\ \quad + 2\mathrm{Re}\langle \eta * \theta Au, \eta * \theta u'\rangle + 2\mathrm{Re}\langle \eta * \theta Au, \eta * \theta' u\rangle \\ \quad + 2\mathrm{Re}\langle A(\eta * \theta u) - \eta * \theta Au, \eta' * \theta u\rangle . \end{cases}$$

We shall examine each term in (3.3) separately as $\theta = \theta_\delta \to \theta_0$ and η is fixed. (The idea of *first* letting $\delta \to 0$ is due to Torelli [7].)

Since $\eta^{(j)} \in L^\infty(R)$ and $\theta \to \theta_0$ in $L^1(R)$ as $\delta \to 0$, we have $\eta^{(j)} * \theta u \to \eta^{(j)} * \theta_0 u$ in $L^\infty(R, V)$ strongly ($j = 0, 1$). Therefore, as $\delta \to 0$, θ may be replaced by θ_0 in the first and last terms in (3.3). The same is true of the second term in view of Lebesgue's convergence theorem because $u \in L^1(0, T; W)$ and $Au \in L^1(0, T; W')$.

As for the third term in (3.3), we note that $\eta * \eta * [(\theta - \theta_0)Au] \to 0$ in $L^\infty(R, V')$ strongly while

$$\int |\, \theta'(t)\,|\, dt = 2 .$$

Thus $\langle \eta * (\theta - \theta_0)Au, \eta * \theta' u\rangle \to 0$ as $\delta \to 0$. On the other hand, since $((\eta * \eta * \theta_0 Au)(t), u(t))$ is a continuous function of t, $\langle \eta * \eta * \theta_0 Au, \theta' u\rangle$ tends to

$$((\eta * \eta * \theta_0 Au)(0), u(0)) - ((\eta * \eta * \theta_0 Au(t_0), u(t_0)) .$$

Hence the third term in (3.3) also tends to the latter expression. Writing $F = Au$, we conclude that

[3] Thus in (3.2), we are referring to the weakly continuous function.

$$(3.4)\qquad \begin{cases} 0 = \langle A'(\eta * \theta_0 u), \eta * \theta_0 u\rangle + 2\mathrm{Re}\,\{\langle \eta * \theta_0 F, \eta * \theta_0 u'\rangle \\ \quad + ((\eta * \eta * \theta_0 Au)(0), u(0)) - ((\eta * \eta * \theta_0 Au)(t_0), u(t_0)) \\ \quad + \langle A(\eta * \theta_0 u) - \eta * \theta_0 Au, \eta' * \theta_0 u\rangle\} . \end{cases}$$

We now consider the convergence of each term in (3.4) separately as $\varepsilon \to 0$ ($\eta = \eta_\varepsilon$). Since $\eta * \theta_0 u \to \theta_0 u$ strongly in $L^2(R, V)$, the first term tends to $\langle A'\theta_0 u, \theta_0 u\rangle$. By a standard argument using the assumption that translations in t are continuous in Z, $\eta * \eta * \theta_0 u' \to \theta_0 u'$ strongly in Z. Thus the second term tends to $2\mathrm{Re}\langle \theta_0 F, \theta_0 u'\rangle$. Now if we let $\rho = \rho_\varepsilon = \eta_\varepsilon * \eta_\varepsilon$, then $0 \leqq \rho(t) = O(\varepsilon^{-1})$, the support of ρ is an interval of length $O(\varepsilon)$, and

$$\int_0^\infty \rho(t)dt = \frac{1}{2}\int_{-\infty}^{\infty} \rho(t)dt = \frac{1}{2} .$$

Therefore for sufficiently small ε,

$$((\rho * \theta_0 Au)(t_0) - \frac{1}{2}(Au)(t_0), u(t_0))$$

$$= \int_0^{t_0} \rho(t)((Au)(t_0 - t) - (Au)(t_0), u(t_0))dt .$$

Since the inner product within this integral is a continuous function of t, we conclude that as $\varepsilon \to 0$,

$$((\rho_\varepsilon * \theta_0 Au)(t_0), u(t_0)) \to \frac{1}{2}(A(t_0)u(t_0), u(t_0)) .$$

In a similar way we find the same result with t_0 replaced by 0.

The proof is terminated once we show that the last term in (3.4) tends to zero with ε. Since

$$\int |\, \eta_\varepsilon'(t)\,|\, dt = O(\varepsilon^{-1}) ,$$

$\{\varepsilon\eta_\varepsilon' * v\}$ is bounded in $L^2(R, V)$, where we have put $v = \theta_0 u$. But for fixed ε_1, $\varepsilon\eta_\varepsilon' * (\eta_{\varepsilon_1} * v) = \varepsilon(\eta_\varepsilon * \eta_{\varepsilon_1}' * v)$ tends to zero strongly in $L^2(R, V)$ as $\varepsilon \to 0$. It follows that

$$\varepsilon\eta_\varepsilon' * v \to 0 \qquad \text{strongly in } L^2(R, V) .$$

On the other hand,

$$[A(\eta_\varepsilon * v) - \eta_\varepsilon * (Av)](t) = \int [A(t) - A(t - s)]\eta_\varepsilon(s)v(t - s)ds .$$

Therefore

$$\|\, A(\eta_\varepsilon * v) - \eta_\varepsilon * (Av)\,\|_{L^2(V')} = O(\varepsilon\,\|\,\eta_\varepsilon * v\,\|_{L^2(V)}) = O(\varepsilon) .$$

Thus the last term in (3.4) tends to zero with ε.

COROLLARY 3.1. *$(A(t)u(t), u(t))$ is a continuous function of t.*

THEOREM 3.2. *Assume in addition that ($c > 0$, λ real):*

$$(A(t)v, v) + \lambda \mid v \mid_X^2 \geqq c \mid v \mid_V^2 \tag{3.5}$$

for all $v \in V$, where X is some Banach space containing both V and W with continuous inclusions. Then the function u in Theorem 3.1 is strongly continuous with values in V.

Proof. Suppose $t_n \to t$ in $[0, T]$. Then $\mid u(t_n) - u(t) \mid_X \to 0$ since both u and u' lie in $L^1(0, T; X)$. By Corollary 3.1,

$$(A(t)u(t_n), u(t_n)) \to (A(t)u(t), u(t)) .$$

Therefore, by the weak continuity

$$(A(t)(u(t_n) - u(t)), u(t_n) - u(t)) \to 0 .$$

By (3.5), $\mid u(t_n) - u(t) \mid_V \to 0$.

THEOREM 3.3. *In Theorem 3.2 the assumption $u \in L^\infty(0, T; V)$ can be weakened to $u \in L^2(0, T; V)$.*

Proof. We begin with identity (3.3), but fix $\theta = \theta_\delta$ and let $\varepsilon \to 0$ first. By arguments similar to those following (3.4), (cf. [4, 5]), we find

$$0 = \langle \theta A'u, \theta u \rangle + 2\mathrm{Re}\langle \theta F, \theta u' \rangle + 2\mathrm{Re}\langle \theta Au, \theta' u \rangle . \tag{3.6}$$

Next, as $\delta \to 0$, the first two terms in (3.6) approach the right hand side of (3.2) with $t = t_0$. However, the third term may not converge since the function $t \to (A(t)u(t), u(t))$ is not known to be continuous. We alter the definition of θ as follows: $\theta(t) = \theta_\delta(t)$ is 1 in $[t_1 + \delta, t_0 - \delta]$, zero outside of (t_1, t_0) and otherwise linear, where $0 < t_1 < t_0 < T$. Then letting $\delta \to 0$ in (3.4) gives for *almost every* t_1 and t_0.

$$\begin{aligned}(A(t_0)u(t_0), u(t_0)) - (A(t_1)u(t_1), u(t_1)) \\ = \int_{t_1}^{t_0} \{(A'u, u) + 2\mathrm{Re}\,(Au, u')\} .\end{aligned}$$

In particular, the function $t \to (A(t)u(t), u(t))$ is essentially bounded. From (3.5) it follows that $u \in L^\infty(0, T; V)$ and so the situation is reduced to that of Theorem 3.2.

4. Solutions of hyperbolic equations. The assumptions are the same as in § 3 except that we are also given a Hilbert space H containing V, where V is dense in H and the inclusion is continuous;

H and W are contained in some fixed linear space. The analogue to Theorem 3.1 is:

THEOREM 4.1. *Let* $u \in L^\infty(0, T; V)$, $u' \in Z \cap L^\infty(0, T; H)$ *and*

$$u'' + Au = F \in Z' + L^1(0, T; H) \tag{4.1}$$

where $u' = du/dt$, $u'' = d^2u/dt^2$. *Then for every* $t \in [0, T]$

$$\begin{aligned} &|u'(t)|_H^2 + (A(t)u(t), u(t)) - |u'(0)|_H^2 - (A(0)u(0), u(0)) \\ &\quad = \int_0^t (A'u, u) + 2\mathrm{Re}\int_0^t (F, u') \,. \end{aligned} \tag{4.2}$$

Proof. By Corollary 2.1, $u \in C_w(0, T; V)$. By (4.1),

$$u'' \in Z' + C_w(0, T; V') \subset L^1(0, T; W' + V') \,.$$

Thus Corollary 2.1 may be applied again to infer that $u' \in C_w(0, T; H)$. Using the notation $\theta = \theta_\delta$ and $\eta = \eta_\varepsilon$ as before, we have

$$\begin{aligned} 0 &= 2\mathrm{Re}\langle \eta * \theta u', \eta' * \theta u' \rangle \\ &= 2\mathrm{Re}\{\langle \eta * \theta u', \eta * \theta u'' \rangle \\ &\quad + \langle \eta * (\theta - \theta_0)u', \eta * \theta' u' \rangle + \langle \eta * \theta_0 u', \eta * \theta' u' \rangle\} \,. \end{aligned} \tag{4.3}$$

Since $\eta \in L^\infty(R)$, $\theta \to \theta_0$ in $L^1(R)$ and θ' is bounded in $L^1(R)$ as $\delta \to 0$ in addition to the fact that $u' \in C_w(0, T; H)$, the last term in (4.3) tends to

$$2\mathrm{Re}\{((\eta * \eta * \theta_0 u')(0), u'(0)) - ((\eta * \eta * \theta_0 u')(t_0), u'(t_0))\} \tag{4.4}$$

and the next-to-last term tends to zero as $\delta \to 0$. Thus, if we add (4.3) to (3.3) and let $\delta \to 0$, we find that (3.4) holds except for the addition of expression (4.4) to its right-hand side, where $F = u'' + Au$ now.

Next we let $\varepsilon \to 0$. Since

$$\begin{aligned} &\left((\rho_\varepsilon * \theta_0 u')(t_0) - \frac{1}{2}u'(t_0), u'(t_0)\right) \\ &\quad = \int_0^{t_0} \rho_\varepsilon(t)(u'(t_0 - t) - u'(t_0), u'(t_0))\, dt \end{aligned}$$

tends to zero with ε, the terms (4.4) approach

$$|u'(0)|_H^2 - |u'(t_0)|_H^2 \,,$$

Thus, referring to the proof of Theorem 3.1, we find (4.2) (with $t = t_0$).

COROLLARY 4.1. $|u'(t)|_H^2 + (A(t)u(t), u(t))$ *is a continuous function of* t.

THEOREM 4.2. *Assume in addition that* (3.5) *holds for all* $v \in V$, *with* $X = H$. *Then* $u \in C_s(0, T; V)$ *and* $u' \in C_s(0, T; H)$.

Proof. If $t_n \to t$ in $[0, T]$ then

$$|u'(t_n)|_H^2 + (A(t)u(t_n), u(t_n)) \to |u'(t)|_H^2 + (A(t)u(t), u(t)) .$$

Hence by the weak continuity,

$$|u'(t_n) - u'(t)|_H^2 + (A(t)(u(t_n) - u(t)), u(t_n) - u(t)) \to 0 .$$

But since $u \in C_s(0, T; H)$ and in view of (3.5),

$$\overline{\lim}\, (A(t)(u(t_n) - u(t)), u(t_n) - u(t)) \geqq c\, \overline{\lim}\, |u(t_n) - u(t)|_V^2 .$$

Hence $|u'(t_n) - u'(t)|_H \to 0$ and $|u(t_n) - u(t)|_V \to 0$.

THEOREM 4.3. *In Theorem* 4.2 *the assumptions* $u \in L^\infty(0, T; V)$ *and* $u' \in L^\infty(0, T; H)$ *may be weakened to* $u \in L^2(0, T; V)$ *and* $u' \in L^2(0, T; H)$, *if* $F \in Z' + L^2(0, T; H)$.

Proof. (cf. Theorem 3.3.) Letting $\varepsilon \to 0$ first, we have (3.6) but with the additional term

$$2\mathrm{Re}\langle \theta u', \theta' u' \rangle .$$

Then letting $\delta \to 0$ we conclude that the function

$$t \to |u'(t)|_H^2 + (A(t)u(t), u(t))$$

is essentially bounded. Since $u \in C_s(0, T; H)$, it follows that $u' \in L^\infty(0, T; H)$ and $u \in L^\infty(0, T; V)$.

REMARK. A slight variation of the proof shows that under the same conditions,

$$\begin{aligned} &\varphi(t)[|u'(t)|_H^2 + (A(t)u(t), u(t)] \\ &\qquad - \varphi(0)[|u'(0)|_H^2 + (A(0)u(0), u(0))] \\ &= \int_0^t [\varphi(A'u, u) + \varphi'(Au, u) + \varphi' |u'|_H^2 + 2\varphi\, \mathrm{Re}\,(F, u')] , \end{aligned} \tag{4.5}$$

for $0 \leqq t \leqq T$ and any C^1-function φ on $[0, T]$

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