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**ON THE SCARCITY OF LATTICE-ORDERED MATRIX RINGS**

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## ON THE SCARCITY OF LATTICE-ORDERED MATRIX RINGS

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It is well-known that the ring  $Q_n$  of  $n \times n$  matrices over a lattice-ordered ring  $Q$  may be lattice-ordered by prescribing that a matrix is positive exactly when each of its entries is positive. We conjecture in case  $Q$  is the field of rational numbers that this is essentially the only lattice-order of the matrix ring in which the multiplicative identity 1 is positive and settle the conjecture in case  $n=2$ . There are however other lattice-orders of  $Q_2$  in which 1 is not positive. A complete description of this family is obtained.

**THEOREM.** *Up to isomorphism there is exactly one lattice-order of the algebra  $Q_2$  of two-by-two matrices over the field  $Q$  of rational numbers in which the identity 1 is positive.*

*For each rational number  $B > 1$ , there is a lattice-order of  $Q_2$  in which there are distinct positive idempotents  $f_1, f_2, f_3$ , and  $f_4$  satisfying:*

(i)  $(1 - B)(f_1 + f_2) + B(f_3 + f_4) = 1$ , and

(ii)  $Q_2$  is the  $l$ -group direct sum of the subrings  $Qf_i$ ,  $1 \leq i \leq 4$ .

*These lattice-orders are not isomorphic, and each lattice-order in which 1 is not positive is isomorphic to one of them.*

*Proof.* Any lattice-order of a finite-dimensional semisimple algebra over the field of rational numbers is archimedean [1]. Hence, for any lattice-order of  $Q_2$ ,  $Q_2$ , as an  $l$ -group, is the direct sum of (at most four) totally-ordered subgroups of the real numbers [2]. We will consider and eliminate the various cases that might occur depending on the number of summands, the dimensions of the summands, and the number and sign of the nonzero coordinates of the identity matrix 1 in each such decomposition.

In each case  $\approx$  denotes  $l$ -group isomorphism.

We will begin by considering all possible lattice-orders in which 1 is positive. The reader should note that in this case the components of 1 in a decomposition of  $Q_2$  into the  $l$ -group direct sum of totally-ordered groups are pairwise disjoint mutually orthogonal idempotents.

(1) Suppose that  $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$ ,  $E_i \neq 0$ ,  $0 < 1$ , and  $1 = e_1 + e_2 + e_3 + e_4$  with  $e_i \in E_i$ .

(1a) If all of the coordinates of 1 are different from 0, then  $Q_2$

is spanned by commuting elements. This is absurd.

(1b) Suppose that exactly three coordinates of 1 are different from 0:  $e_1, e_2, e_3 > 0 = e_4$ . Let  $0 < n \in E_4$ . Then  $0 \leq e_i n \leq n$  implies  $e_i n = k_i n$  for some  $k_i \in Q^+$ . Moreover,  $e_i^2 n = k_i^2 n = k_i n$ , so  $k_i = 0$  or  $k_i = 1$ . If, for all  $i, k_i = 1$ , then  $n = e_1 n + e_2 n + e_3 n = 3n$  which is impossible. If, for some  $i, k_i = 0$ , then  $e_i Q_2 = E_i$  is a one-dimensional right ideal. However, all right ideals of  $Q_2$  have even dimension.

(1c) Suppose that exactly two coordinates of 1 are greater than 0:  $e_1, e_2 > 0, e_3 = e_4 = 0$ . In this case there is a lattice-order and we need only show that it is determined up to isomorphism. Let  $0 < n_1 \in E_3$  and  $0 < n_2 \in E_4$ . As in (1b), for each  $i$  and  $j$ , either  $e_i n_j = 0$  or  $e_i n_j = n_j$ . Moreover, by the Cayley-Hamilton theorem, there are rational numbers  $q$  and  $r$  such that

$$n_j^2 = q + r n_j = q e_1 + q e_2 + r n_j$$

Thus  $E_1 \otimes E_2 \otimes E_3$  and  $E_1 \otimes E_2 \otimes E_4$  are subalgebras of  $Q_2$ .

Let  $e_i n_1 = k_i n_1, k_i \in Q^+$ . Then  $e_1 e_2 n_1 = k_1 k_2 n_1 = 0$ , so  $k_1$  or  $k_2$  is 0. Suppose  $k_1 = 0$ . Then  $(e_1 + e_2) n_1 = n_1$ , so

- ( i )  $e_2 n_1 = n_1$  and
- ( ii )  $e_1 n_1 = 0$ .

If  $n_1 e_1 = 0$  as well, then  $n_1 e_2 = n_1$ . For some  $q, r \in Q^+, n_1^2 = q + r n_1, n_1^2 e_2 = q e_2 + r n_1$ , so  $q = 0$  and  $n_1^2 \in E_3$ . Thus  $n_1(E_1 \otimes E_2 \otimes E_3) = E_3$ . Since  $n_1 Q_2$  is at least two-dimensional,  $n_1 n_2 > 0$ . Similarly

$$e_1(E_1 \otimes E_2 \otimes E_3) = E_1$$

implies  $e_1 n_2 > 0$ , so  $e_1 n_2 = n_2$ .

Then  $0 = n_1 e_1 n_2 = n_1 n_2 > 0$ . Hence

- ( iii )  $n_1 e_1 = n_1$ ,
- ( iv )  $n_1 e_2 = 0$ , and
- ( v )  $n_1^2 = n_1 e_1 n_1 = 0$ .

If  $e_1 n_2 = 0$  as well, then  $e_1 Q_2 = E_1$ , so

- ( vi )  $e_1 n_2 = n_2$ ,
- ( vii )  $e_2 n_2 = 0$ , and, as above,
- ( viii )  $n_2 e_1 = 0$
- ( ix )  $n_2 e_2 = n_2$ , and
- ( x )  $n_2^2 = 0$ .

To complete a multiplication table for  $Q_2$  it suffices to calculate  $n_1 n_2$  and  $n_2 n_1$ :

$n_1 n_2 = a e_1 + b e_2 + c n_1 + d n_2$  for some  $a, b, c, d \in Q^+$ . Then  $n_1^2 n_2 = 0 = a n_1 + d n_1 n_2$  implies  $a = d = 0$ , while  $n_1 n_2^2 = 0 = c n_1 n_2$  implies  $c = 0$ , so  $n_1 n_2 = b e_2$ . If  $n_1 n_2 = 0$ , then  $n_1 Q_2$  is one dimensional, so  $b > 0$ . Observe

that replacing  $n_1$  by  $b^{-1}n_1$  does not change the validity of any of the equations (i)-(x), so we may suppose

$$(xi) \quad n_1n_2 = e_2.$$

Similarly,  $n_2n_1 = ce_1$  for some  $c > 0$ . Using the relations already obtained it is now easy to check that  $n_1n_2 + n_2n_1 = e_2 + ce_1$  commutes with  $e_1, e_2, n_1,$  and  $n_2$  and hence is in the center of  $Q_2$ . Thus  $c = 1$ , and

$$(xii) \quad n_2n_1 = e_1.$$

The equations (i)-(xii) uniquely determine a multiplication table for  $Q_2$ . This lattice-order is evidently the usual order for  $Q_2$ .

(1d) Suppose that exactly one coordinate of 1 is greater than 0:  $e_1 = e_2 = e_3 = 0, e_4 = 1 > 0$ . Let  $0 < n_i \in E_i, i = 1, 2, 3$ . Observe that

$$0 \leq n_i n_j \leq (n_i + n_j)^2 = a + b(n_i + n_j) \text{ for some } a, b \in Q^+$$

implies that each  $E_i \otimes E_4$  and each  $E_i \otimes E_j \otimes E_4$  is a subalgebra of  $Q_2$ . We will consider and successively eliminate several cases depending upon the location of idempotents in the summands.

(1d<sub>1</sub>) Suppose that  $E_1, E_2,$  and  $E_3$  contain no nonzero idempotents. Assume that one of the  $n_i$ 's, say  $n_1$ , is invertible. Then  $n_1^2 = q + rn_1; q, r \in Q^+, q > 0$ . We have  $n_1n_2 = a + bn_1 + cn_2$  for some  $a, b, c \in Q^+$ . Since  $E_1 \otimes E_4$  is an algebra containing  $n_1^{-1}, c > 0$  and  $n_1n_2 > 0$ . Then  $qn_2 + rn_1n_2 = n_1^2n_2 = bq + (a + br)n_1 + cn_1n_2,$  and  $(r - c)n_1n_2 = bq + (a + br)n_1 - qn_2,$  so  $bq \leq 0, a + br \leq 0$ . Thus  $a = b = 0$  and  $n_1n_2 = cn_2 > 0$ . Now, if  $n_2^2 = s + tn_2,$  then  $cn_2^2 = n_1n_2^2 = sn_1 + tn_2 = cs + ctn_2,$  and  $s = 0$ . If  $t > 0,$  then  $t^{-1}n_2$  is a nonzero idempotent, so  $n_2^2 = 0$ . Similarly  $n_3^2 = 0$ .

If none of the  $n_i$ 's are invertible, then again  $n_2^2 = n_3^2 = 0$ . Recalling that  $n_2n_3$  and  $n_3n_2$  belong to  $E_2 \otimes E_3 \otimes E_4$  one can quickly compute  $n_2n_3 = n_3n_2 = 0$  so that  $E_2 \otimes E_3$  is a two-dimensional nilpotent subalgebra of  $Q_2$ . This is absurd.

(1d<sub>2</sub>) Suppose that at least two summands other than  $E_4$  contain nonzero idempotents: say  $0 < n_1 = n_1^2 \in E_1$  and  $0 < n_2 = n_2^2 \in E_2$ . We have  $n_1n_2 = q + un_1 + vn_2$  for some  $q, u, v \in Q^+; n_1n_2 = n_1n_2^2 = un_1n_2 + (q + v)n_2,$  so  $un_1n_2 = un_1,$  and similarly  $vn_1n_2 = vn_2$ . Suppose, for example, that

$$(*) \quad n_1n_2 = n_1.$$

Calculate  $n_1n_3 = a + bn_1 + cn_3$  for some  $a, b, c \in Q^+, n_1^2n_3 = (a + b)n_1 + cn_1n_3 = a + bn_1 + cn_3,$  whence  $cn_1n_3 = a + cn_3 - an_1$  and  $a = 0$ . If  $c = 0,$  then  $n_1Q_2 = E_1,$  so  $b = 0$  and

$$(**) \quad n_1n_3 = n_3.$$

As above,  $n_2n_3 = yn_2 + zn_3$  and  $zn_2n_3 = zn_3$ . If  $z \neq 0$ , then  $Q_2n_3 = E_3$ , so  $n_2n_3 = yn_2$  for some  $y \in Q^+$ . However, by (\*) and (\*\*), this yields  $(n_1n_2)n_3 = n_1n_3 = n_3 = n_1(n_2n_3) = yn_1n_2 = yn_1$ . Hence (\*) is false and  $n_1n_2 = 0$ . Similarly  $n_2n_1 = 0$ . Calculate, as above,  $n_1n_3 = xn_1 + yn_3$  and  $yn_1n_3 = yn_3$ . If  $y = 0$ , then  $n_1Q_2 = E_1$ , so  $n_1n_3 = n_3$ . Similarly  $n_3n_1 = n_3n_2 = n_2n_3 = n_3$ ; so  $n_3$  belongs to the center of  $Q_2$ , which is impossible.

(1d<sub>3</sub>) Suppose that  $0 < n_1 = n_1^2 \in E_1$ , but  $E_2$  and  $E_3$  do not contain nonzero idempotents. As in (1d<sub>2</sub>) either  $n_1n_2 = kn_1$  or  $n_1n_2 = n_2$ ; either  $n_1n_3 = mn_1$  or  $n_1n_3 = n_3$ . We cannot have both  $n_1n_2$  and  $n_1n_3$  in  $E_1$ , for then  $n_1Q_2 = E_1$ . We cannot have both  $n_1n_2 = n_2$  and  $n_1n_3 = n_3$  for then  $n_1Q_2$  is three-dimensional. Thus we may assume that  $n_1n_2 = n_2$  and  $n_1n_3 = kn_1$  for some  $k \in Q^+$ . If  $k > 0$  we can replace  $n_3$  by  $k^{-1}n_3$ , obtaining the possible cases:

(i)  $n_1n_2 = n_2$  and  $n_1n_3 = 0$ , or

(ii)  $n_1n_2 = n_2$  and  $n_1n_3 = n_1$ .

Consider (i). Calculate  $n_3^2 = a + bn_3$  for some  $a, b \in Q^+$ . Then  $n_1n_3 = 0$  implies  $a = 0$ , and the fact that  $E_3$  contains no nonzero idempotents implies  $b = 0$ ; i.e.,  $n_3^2 = 0$ . From this we can show  $n_2n_3 = 0$ , which yields  $Q_2n_3 = E_3$ .

Consider (ii). As in the first part of the argument for (1d<sub>3</sub>),  $n_3n_1 = n_3$  or  $n_3n_1 = kn_1$  for some  $k \in Q^+$ . If  $n_3n_1 = n_3$ , then  $n_3^2 = n_3n_1n_3 = n_3n_1 = n_3$ , although  $E_3$  contains no idempotents. Thus  $n_3n_1 = kn_1$ ; moreover,  $k = 1$ , so  $n_3$  commutes with  $n_1$ . Similarly  $n_2n_1 = kn_1$  or  $n_2n_1 = n_2$ . In the first case,  $Q_2n_1 = E_1$ . In the second case,  $n_1$  is in the center of  $Q_2$ , which is false.

This completes the proof that there is no lattice-order of  $Q_2$  satisfying the hypotheses of (1d).

(2) Suppose that  $Q \approx E_1 \otimes E_2 \otimes E_3$ ,  $1 > 0$ ,  $E_1$  is two-dimensional, and  $E_i \neq 0$ . Let  $1 = e_1 + e_2 + e_3$ ,  $e_i \in E_i$ .

(2a) If all  $e_i > 0$ , then each  $E_i$  is an ideal.

(2b) Suppose that  $e_1, e_2 > 0 = e_3$ . Let  $0 < n \in E_3$ . As in (1b), for  $i = 1$  or  $2$ ,  $e_i n = n$  or  $e_i n = 0$ . If  $e_2 n = 0$ , then  $e_1 n = n$  and  $E_1 n = E_3$ . Since  $n^2 = a + bn$  implies  $e_2 n^2 = 0 = ae_2$ , we also get  $n^2 \in E_3$ , so  $Q_2 n = E_3$ . Thus  $e_2 n = n$ ,  $e_1 n = 0$ , and again  $Q_2 n = E_3$ .

(2c) Suppose that  $e_1 = 0 < e_2, e_3$ . Let  $0 < n \in E_1$ . Since  $e_i n$  and  $n e_i$  belong to  $E_1^+$ , we can show that  $E_1$  is an ideal if it is a subalgebra. Either  $e_2 n$  or  $e_3 n$ , say  $e_2 n$ , is different from 0. Then  $(e_2 n)^2 = a + be_2 n$ , so  $e_3(e_2 n)^2 = 0 = ae_3$  implies  $(e_2 n)^2$  and hence  $(E_1)^2$  is contained in  $E_1$ .

(2d) Suppose that  $e_1 > 0 = e_2 = e_3$ . Let  $1$  and  $y$  be a positive basis for  $E_1$ . Then  $y^2 = a + by \in E_1$  implies  $E_1$  is a totally-ordered ring. If  $a = 0$ , then either  $E_1$  is a zero-ring or  $E_1$  is an archimedean totally-ordered ring with two linearly independent idempotents. Since both of these cases are impossible [2],  $y$ , and hence each nonzero element of  $E_1$ , is invertible. From this it is easy to see that  $E_2$  is two-dimensional, a contradiction.

(2e) Suppose that  $e_1 = e_2 = 0 < e_3$ . Let  $p_1$  and  $p_2$  be positive linearly independent elements of  $E_1$ , and let  $0 < n \in E_2$ .

Calculate  $p_i^2 = q_i + r_i p_i$  for some  $q_i, r_i \in Q^+$ . Since  $E_1$ , if a subalgebra, can neither be nilpotent nor contain linearly independent idempotents, neither  $q_i$  is 0, so both  $p_1$  and  $p_2$  are invertible in  $Q_2$ . Calculate

$$p_1 n = a + (bp_1 + cp_2) + dn$$

for some  $a, d \in Q^+; b, c \in Q; bp_1 + cp_2 \geq 0$ . Then

$$q_1 n + r_1 p_1 n = p_1^2 n = ap_1 + p_1(bp_1 + cp_2) + dp_1 n$$

and

$$(r_1 - d)p_1 n = ap_1 + p_1(bp_1 + cp_2) - q_1 n .$$

Before proceeding, observe that  $p_1 p_2 \leq (p_1 + p_2)^2 = x + y(p_1 + p_2)$  implies that  $E_1 \otimes E_2$  is a subalgebra. Since  $q_1 > 0, p_1(a + bp_1 + cp_2) = 0, a + bp_1 + cp_2 = 0$ , and hence  $a = b = c = 0$ . Thus  $p_1 n$ , and similarly  $p_2 n$ , belong to  $E_2$ . Since  $p_1$  and  $p_2$  are invertible, this implies that  $E_2$  is two-dimensional which is a contradiction.

(3) Suppose that  $Q_2 \approx E_1 \otimes E_2, E_i \neq 0$ , and  $1 = e_1 + e_2 > 0, e_i \in E_i$ .

(3a) If both coordinates of  $1$  are greater than  $0$ , then each  $E_i$  is an ideal.

(3b) In case  $E_1$  is three-dimensional and  $1 \in E_1$ , see the argument of (2d).

(3c) Suppose that  $E_1$  is three-dimensional,  $E_2$  is one-dimensional, and  $1 \in E_2$ . Let  $0 < f \in E_1$  and  $f^2 = a + bf$  for some  $a, b \in Q^+$ . Since  $E_1$  cannot be a right ideal,  $a > 0$  and  $f$  is invertible. Let  $h$  be an element of  $E_1$  which is bigger than but not a rational multiple of  $f$ . Then  $h^2 = x + yh$ . Define

$$L = \{r \in Q^+ : rf \leq h\}$$

and

$$U = \{s \in Q^+ : sf \geq h\} .$$

Define  $t = \sup L = \inf U$ .

If  $y = 0$ , then  $fh \in E_2$ . In such a case, for each  $r \in L$  and  $s \in U$ ,  $rf^2 \leq fh \leq sf^2$ , so  $ta = fh$  and  $t$  is rational. Since this is impossible,  $y \neq 0$ . For each  $r \in L$  and  $s \in U$ ,  $r^2f^2 \leq h^2 \leq s^2f^2$  whence  $t^2a = x$  and  $t^2$  is rational. Then  $t^2bf = yh$  and  $h$  is a rational multiple of  $f$ .

(3d) Suppose that  $E_1$  and  $E_2$  are two-dimensional and  $1 \in E_1$ . Let  $0 < f \in E_2, f^2 = a + bf, a$  and  $b$  in  $Q^+$ . Observe that  $a$  must be nonzero in order to prevent  $E_2$  from being an ideal.

If  $e$  is a positive element of  $E_1$  which is linearly independent of  $1$ , consider  $e^2 = x + ye, x$  and  $y$  in  $Q^+$ . If  $x = 0$ , then  $y > 0$  and  $y^{-1}e$  is a nonzero idempotent of  $E_1$  different from  $1$ . Since this is impossible,  $E_1$  is a field. The remainder of the argument for this case resembles that of (3c).

(4) Suppose that  $Q_2 = E_1$ . Since the field of rational complex numbers is a subalgebra of  $Q_2$  which has no total order, this is impossible.

We now consider the possible lattice orders of  $Q_2$  in which  $1$  is not positive. Their description is obtained in (7b).

(5) Suppose that  $Q_2 \approx E_1 \otimes E_2, 1 = e_1 + e_2, e_i \in E_i$ , and  $e_1 < 0 < e_2$ . One of the summands, say  $E_1$ , has dimension bigger than  $1$ . Calculate  $e_1^2 = a + be_1 = (a + b)e_1 + ae_2$ . If  $a = 0$ , then  $e_1e_2 = e_1 - e_1^2 = e_2e_1 \in E_1$  and  $E_1$  is an ideal. Thus  $a > 0$ .

Let  $0 < f$  be any positive element of  $E_1$  which is linearly independent of  $e_1$ . Let  $L = \{p \in Q^+ : -pe_1 \leq f\}$ , let  $U = \{q \in Q^+ : -qe_1 \geq f\}$ , and let  $r_f$  be the common least upper bound of  $L$  and greatest lower bound of  $U$  in the set of real numbers. Calculate  $f^2 = x + yf$  for some  $x, y \in Q$ . For any  $p$  in  $L$  and  $q$  in  $U$ ,  $p^2a \leq x \leq q^2a$ , so  $r_f^2 = xa^{-1}$ . However,  $r_f$  and  $r_{f-e_1}$  cannot both have rational squares.

(6) Suppose that  $Q_2 \approx E_1 \otimes E_2 \otimes E_3, E_i \neq 0; e_i \in E_i$ , and  $1 = e_1 + e_2 + e_3$  is not positive. Let  $E_1$  be the two-dimensional summand.

(6a) Suppose  $e_1 < 0 < e_2, e_3$ . Then  $e_2^2 = a + be_2 = ae_1 + (a + b)e_2 + ae_3 \geq 0$  implies  $a = 0$ . Thus  $e_2^2 = k_2e_2$  and  $e_3^2 = k_3e_3$  for some  $k_i \in Q^+$ . Since  $E_1$  cannot be a nilpotent subalgebra,  $e_1^2 = x + ye_1 > 0$ . If  $x = 0$ , then  $e_11 = e_1 = ye_1 + e_1e_2 + e_1e_3$ , and  $e_1e_2 + e_1e_3 \in E_1$ , so  $e_1e_2$  and  $e_1e_3 \in E_1$ . However  $1e_2 = e_2 = e_1e_2 + e_2^2 + e_3e_2, e_3e_2 > e_3$ , and  $e_2^2 \in E_2$  gives rise to a

contradiction. Thus  $x > 0$ ,  $e_1$  is invertible, and each element of  $E_1$  is invertible. This means that  $T = E_1(e_2 + e_3)$  is a two-dimensional totally ordered subspace of  $Q_2$  and hence equals  $E_1$ , although  $e_1(e_2 + e_3) = e_1 - e_1^2 = (1 - x - y)e_1 - x(e_2 + e_3)$  belongs to  $T$ .

(6b) Suppose that  $e_1, e_2 > 0 > e_3$ . Then  $e_1^2 = ke_1$ , so  $E_1$  is a subalgebra of  $Q_2$ . Since  $E_1$  cannot be nilpotent,  $k > 0$ . Moreover, if  $0 < f \in E_1$  is linearly independent of  $e_1$ , then  $f^2 = tf$  for some  $t \in Q$ ,  $t \neq 0$ . Unfortunately, this yields linearly independent idempotents  $t^{-1}f$  and  $k^{-1}e_1$  of a subring of the real field.

(6c) Suppose that  $e_1, e_2 < 0 < e_3$ . Argue as in (6b).

(6d) Suppose that  $e_1 > 0 > e_2, e_3$ . Argue as in (6a) to obtain  $e_1e_2$  in  $E_1$ . Then  $e_3e_2 = (e_2 - e_2^2) - e_1e_2$  and  $e_2 - e_2^2 \in E_2$  implies  $e_2 \geq e_2^2$  which is absurd.

(6e) Suppose that  $e_1 = 0, e_2 < 0 < e_3$ . Let  $0 < f \in E_1$ . Then  $f^2 = kf$  for some  $k \in Q$ . Since  $E_1$  cannot be a nilpotent algebra,  $k > 0$ . In this way we can produce linearly independent idempotents of the archimedean ordered ring  $E_1$ .

(6f). Suppose that  $e_1 < 0 < e_2, e_3 = 0$ . Let  $0 < n \in E_3$ . In the usual manner it can be shown that  $E_1 \otimes E_2$  is a subalgebra of  $Q_2$ .

Now  $n^2 = a + bn = ae_1 + ae_2 + bn$  implies  $a = 0$ . Assume that  $n^2 = 0$ . If, in addition,  $e_1n = 0$ , then  $e_2n = n$  and  $Q_2n = E_3$ . Thus  $e_1n = g + xe_2 + yn \neq 0$  for some  $g \in E_1; x, y \in Q$ . Then  $e_1n^2 = 0 = gn + xe_2n$ . Since  $g \leq 0$  and  $x \leq 0$ ,  $gn = xe_2n = 0$ . Thus  $x = 0, g = 0$ , and  $Q_2n = E_3$ .

Hence  $n^2 = bn$  for some  $b > 0$ . Without loss of generality we may assume that  $n$  is idempotent. Again,  $e_1n = g + xe_2 + yn, e_1n^2 = e_1n = gn + xe_2n + yn$ , and  $((1 + x)e_1 - g)n = (x + y)n$ . Since  $e_1n \notin E_3$ , it follows that  $x + y = 0, x = y = 0$ , and  $g = e_1$ , so  $e_1n = e_1$  and  $e_2n = n - e_1$ . Since  $Q_2n$  cannot be three-dimensional, if  $f$  is an element of  $E_1$  which is linearly independent of  $e_1$ , then  $fn = te_1$  for some  $t \in Q, t \neq 0$ . Whence  $(e_1 - t^{-1}f)n = 0$ , which implies  $e_1n = 0$ , a contradiction.

(6g) Suppose that  $e_1 > 0 > e_2, e_3 = 0$ . Proceed as in (6f) down to the point where it is concluded that  $x + y = 0$ . From the two equations for  $e_1n$  we calculate  $(g - xe_1)n = g + xe_2 - xn = e_1n$ , so  $g = (1 + x)e_1$ . We have  $e_1n = (1 + x)e_1 + xe_2 - xn$  and

$$e_2n = -(1 + x)e_1 - xe_2 + (1 + x)n$$

which yields  $0 \leq 1 + x \leq 0$ . Thus  $e_1n = -e_2 + n$  and  $e_2n = e_2$ .



Since  $Q_2n$  cannot be three-dimensional, if  $f$  is an element of  $E_1$  which is linearly independent of  $e_1$ , then  $fn = ae_2 + bn$  for some  $a, b \in Q$ . We have  $(f + ae_1)n = (a + b)n$ , whence  $e_1n \in E_3$ , a contradiction.

(7) Suppose that  $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$ ,  $E_i \neq 0$ ,  $e_i \in E_i$ , and  $1 = e_1 + e_2 + e_3 + e_4$  is not comparable to 0.

(7a) Suppose that  $e_1 < 0 < e_2, e_3, e_4$ . Then  $e_2^2 = a + be_2 = ae_1 + (a + b)e_2 + ae_3 + ae_4$  implies  $a = 0$  and  $e_2^2 \in E_2$ . Similarly,  $e_3^2 \in E_3$ ,  $(e_2 + e_3)^2 \in E_2 \otimes E_3$ , etc. Thus  $E_2 \otimes E_3 \otimes E_4$  is a subalgebra of  $Q_2$ . Now calculate

$$\begin{aligned} 0 \leq e_1^2 &= (1 - (e_2 + e_3 + e_4))^2 = 1 - 2(e_2 + e_3 + e_4) + (e_2 + e_3 + e_4)^2 \\ &= e_1 + f \end{aligned}$$

for some  $f \in E_2 \otimes E_3 \otimes E_4$ , although  $e_1 < 0$ .

(7b) Suppose that  $e_1, e_2 < 0 < e_3, e_4$ . There are lattice-orders of  $Q_2$  in which this situation is realized.

For each  $i$  there exists  $k_i \in Q$  such that  $e_i^2 = k_i e_i$ . In addition,  $(e_j - e_i)^2 = t(e_j - e_i)$  for some  $t \in Q$  as long as  $j = 3$  or  $4$  and  $i = 1$  or  $2$ , in which case  $E_i \otimes E_j$  is a subalgebra of  $Q_2$ . Calculate  $e_1 e_3 = ae_1 + be_3$  for some  $a, b \in Q$ ,  $e_1^2 e_3 = k_1 e_1 e_3 = ak_1 e_1 + be_1 e_3$ , and  $e_1 e_3^2 = k_3 e_1 e_3 = ae_1 e_3 + bk_3 e_3$ , which yield  $be_1 e_3 = bk_1 e_3$  and  $ae_1 e_3 = ak_3 e_1$ . Either  $e_1 e_3 = 0$ , or  $e_1 e_3 = k_1 e_3$ , or  $e_1 e_3 = k_3 e_1$ . Similar results hold for  $e_j e_i$  and  $e_i e_j$  as long as  $i = 1$  or  $2$  and  $j = 3$  or  $4$ .

Assume that one such product is 0; e.g.,  $e_1 e_3 = 0$ . Then  $e_1 = e_1 1 = e_1^2 + e_1 e_2 + e_1 e_3 + e_1 e_4$ , and  $e_1 e_2 = (1 - k_1)e_1 - e_1 e_4$ . If  $e_1 e_4 = 0$  or  $e_1 e_4 = k_4 e_1$ , then  $e_1 Q_2 = E_1$  is one-dimensional. If  $e_1 e_4 = k_1 e_4$ , then  $e_1 e_2 = (1 - k_1)e_1 - k_1 e_4$  implies  $1 \leq k_1 \leq 0$  which is absurd. Thus no such product is 0.

Suppose that

( i )  $e_1 e_3 = k_1 e_3, k_1 < 0$ .

(The case  $e_1 e_3 = k_3 e_1$  will be discussed separately.) Then  $e_1 = e_1 1$  yields  $e_1 e_2 = (1 - k_1)e_1 - k_1 e_3 - e_1 e_4$ . If  $e_1 e_4 = k_1 e_4$ , then  $1 - k_1 \leq 0$  which contradicts  $k_1 < 0$ , so

( ii )  $e_1 e_4 = k_4 e_1, k_4 > 0$ , and

$$e_1 e_2 = (1 - k_1 - k_4)e_1 - k_1 e_3 .$$

Calculating  $e_3 = 1e_3$  we get  $e_4 e_3 = (1 - k_1 - k_3)e_3 - e_2 e_3$ . If  $e_2 e_3 = k_2 e_3$ , then  $Q_2 e_3$  is one-dimensional, so

( iii )  $e_2 e_3 = k_3 e_2, k_3 > 0$  and

$$e_4 e_3 = (1 - k_1 - k_3)e_3 - k_3 e_2 .$$

Now  $e_1e_2e_3 = k_3e_1e_2 = k_3(1 - k_1 - k_4)e_1 - k_3k_1e_3 = (1 - k_1 - k_4)e_1e_3 - k_1e_3^2 = k_1(1 - k_1 - k_4)e_3 - k_1k_3e_3$ , whence

(iv)  $1 = k_1 + k_4$  and

(v)  $e_1e_2 = -k_1e_3$ .

Calculate  $e_1e_2^2 = k_2e_1e_2 = -k_1k_2e_3 = -k_1e_3e_2$ , so

(vi)  $e_3e_2 = k_2e_3$ ,

and  $e_4e_2 = e_2 - e_1e_2 - e_2^2 - e_3e_2 = (1 - k_2)e_2 + (k_1 - k_2)e_3$ , whence

(vii)  $k_1 = k_2$ .

Now  $e_4e_1 = e_4 - e_4e_2 - e_4e_3 - e_4^2 = (1 - k_4)e_4 + (k_3 - k_4)e_2 - (1 - k_1 - k_3)e_3$ , whence  $1 = k_1 + k_3$

(viii)  $k_3 = k_4$ , and

(ix)  $e_4e_1 = k_2e_4$ .

Since  $e_3e_4 = e_3 - e_3e_2 - e_3^2 - e_3e_1 = -e_3e_1$  and simultaneously  $e_3e_4 = e_4 - e_1e_4 - e_2e_4 - e_4^2 = (1 - k_4)e_4 - k_4e_1 - e_2e_4$ , we must have

(x)  $e_2e_4 = k_2e_4$ .

Let  $\alpha = k_1 = k_2, \beta = k_3 = k_4, f_i = k_i^{-1}e_i$ . Then  $\alpha < 0 < \beta, \alpha + \beta = 1$ , and the  $f_i$ 's are nonzero linearly independent idempotents different from the identity and satisfying

$$\alpha(f_1 + f_2) + \beta(f_3 + f_4) = 1.$$

Moreover, equations (i)–(x) together with the fact that  $e_1 + e_2 + e_3 + e_4 = 1$  yield the following multiplication table.

	$f_1$	$f_2$	$f_3$	$f_4$
$f_1$	$f_1$	$-\beta\alpha^{-1}f_3$	$f_3$	$f_1$
$f_2$	$-\beta\alpha^{-1}f_4$	$f_2$	$f_2$	$f_4$
$f_3$	$f_1$	$f_3$	$f_3$	$-\alpha\beta^{-1}f_1$
$f_4$	$f_4$	$f_2$	$-\alpha\beta^{-1}f_2$	$f_4$

Thus such a lattice-order would be determined up to isomorphism by the choice of  $\beta$ . The matrices

$$f_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} -\beta\alpha^{-1} & 1 \\ -\beta\alpha^{-2} & \alpha^{-1} \end{pmatrix},$$

$$f_3 = \begin{pmatrix} 1 & -\alpha\beta^{-1} \\ 0 & 0 \end{pmatrix}, \quad f_4 = \begin{pmatrix} 1 & 0 \\ \alpha^{-1} & 0 \end{pmatrix}$$

fulfill all of the requirements.

Clearly distinct  $\beta$ 's yield nonisomorphic lattice-orders.

Finally, suppose that  $e_1e_3 = k_3e_1$  (rather than  $k_1e_3$ ). Now  $e_1e_2 = e_1 - e_1e_3 - e_1e_4 - e_1^2 = (1 - k_3 - k_1)e_1 - e_1e_4$ . If  $e_1e_4 = k_4e_1$ , then  $e_1Q_2$  is

one-dimensional. Thus  $e_1e_4 = k_1e_4$ . This indicates that the lattice-order must be isomorphic to one of those already considered.

(7c) Suppose that  $e_1, e_2, e_3 < 0 < e_4$ . Proceed as in (7a). Then  $e_i^2 = k_i e_i$  for  $i = 1, 2, 3$  and  $E_1 \otimes E_2 \otimes E_3$  is a subalgebra of  $Q_2$ . Calculate  $e_4^2 = e_4 + f$  for some  $f \in E_1 \otimes E_2 \otimes E_3$ . Let  $e_i e_4 = f_i + d_i e_4$  where  $f_i \in E_1 \otimes E_2 \otimes E_3$  and  $d_i \in Q^-$ . Moreover,  $(e_1 + e_2 + e_3)e_4 = e_4 - e_4^2 = (f_1 + f_2 + f_3) + (d_1 + d_2 + d_3)e_4$ . Since  $e_4 - e_4^2 = -f$ ,  $d_1 = d_2 = d_3 = 0$ . This implies that  $E_1 \otimes E_2 \otimes E_3$  is a three-dimensional right ideal.

(7d) Suppose that  $e_4 = 0$ , the other  $e_i$ 's are not 0, and  $e_1$  and  $e_2$  have the same sign opposite that of  $e_3$ . Let  $0 < n \in E_4$ . Then  $e_1^2 = k_1 e_1$ ,  $e_2^2 = k_2 e_2$ ,  $n^2 = kn$  and  $k_1$  and  $k_2$  have the same sign. Moreover  $E_1 \otimes E_4$ ,  $E_2 \otimes E_4$ , and  $E_1 \otimes E_2 \otimes E_3$  are subalgebras of  $Q_2$ .

Let  $e_2 n = a e_2 + b n$ . Then  $e_2^2 n = k_2 e_2 n = a k_2 e_2 + b e_2 n$  and  $e_2 n^2 = k e_2 n = a e_2 n + b k n$ , so  $b e_2 n = b k_2 n$  and  $a e_2 n = a k e_2$ . Thus  $e_2 n = 0$  or  $e_2 n = k e_2$ , or  $e_2 n = k_2 n$ . Similarly for  $e_1 n, n e_2$ , and  $n e_1$ .

(i) Suppose that  $e_2 n = 0$ . If  $e_1 n = 0$  or  $e_1 n = k_1 n$ , then  $Q_2 n = E_4$ ; so  $e_1 n = k e_1$ ,  $k \neq 0$ , and  $e_3 n = n - k e_1$ . For some  $x, y, z \in Q$ ,  $e_1 e_2 = x e_1 + y e_2 + z e_3$ . Then  $e_1 e_2 n = 0 = k(x - z)e_1 + z n$ ,  $z = x = 0$ , and  $e_1 e_2 \in E_2$ . By a similar calculation  $n e_2 \in E_2$ , whence  $Q_2 e_2 = E_2$ .

(ii) Suppose that  $e_2 n = k e_2$ . Then  $e_1 n = k e_1$  would make  $Q_2 n$  three-dimensional, so  $e_1 n = k_1 n$ . Both  $k$  and  $k_1$ , by (i), are different from 0. Now

$$\begin{aligned} e_1 e_2 &= x e_1 + y e_2 + z e_3, \\ e_1 e_2 n &= k e_1 e_2 = k(y - z)e_2 + (z - z k_1 + x k_1)n, \quad x = z = 0, \end{aligned}$$

and  $e_1 e_2 = y e_2$ . If  $n e_2 = k e_2$ , then  $Q_2 e_2 = E_2$ , so  $n e_2 = k_2 n$ . Finally,  $n e_1 = k e_1$ , which yields  $n e_1 e_2 = k e_1 e_2 = y n e_2 = y k_2 n = k y e_2$ , and  $e_1 e_2 = 0$ . By symmetry,  $e_2 e_1 = 0$ , whence  $e_2 Q_2 = E_2$ .

(iii) Suppose that  $e_2 n = k_2 n$ . Then  $e_1 n = k_1 n$  would make  $Q_2 n = E_4$ ; so  $e_1 n = k e_1$ , and we are back to case (ii).

(7e) Suppose that  $Q_2 \approx E_1 \otimes E_2 \otimes E_3 \otimes E_4$ ,  $E_i \neq 0$ ,  $1 = e_1 + e_2$ ,  $e_1 < 0 < e_2$ , and  $e_i \in E_i$ . Let  $0 < n_3 \in E_3$  and  $0 < n_4 \in E_4$ .

Then  $n_i^2 = k_i n_i$ , and we may assume  $k_i = 0$  or  $k_i = 1$ . Suppose, for example, that  $n_3^2 = 0$ . Since  $E_3 \otimes E_4$  is a subalgebra of  $Q_2$ ,  $n_4 n_3 = a n_3 + b n_4$  for some  $a, b \in Q$ ;  $0 = n_4 n_3^2 = b n_4 n_3$  yields  $n_4 n_3 \in E_3$ . Since  $E_1 \otimes E_2 \otimes E_3$  is a subalgebra of  $Q_2$ ,  $e_1 n_3 = x e_1 + y e_2 + z n_3$ , and  $e_1 n_3^2 = 0 = x e_1 n_3 + y e_2 n_3$ . Since  $x e_1 \leq 0$  and  $y e_2 \leq 0$ ,  $x e_1 n_3 = y e_2 n_3 = 0$ . In particular,  $x = 0$ . If  $e_2 n_3 = 0$ , then  $0 \geq e_1 n_3 = n_3 > 0$ , so  $y = 0$  also. Thus  $Q_2 n_3 = E_3$ .

We may thus assume that  $n_3$  and  $n_4$  are idempotents. This time

$n_3n_4 = an_3 + bn_4$  yields  $bn_3n_4 = bn_4$  and  $an_3n_4 = an_3$ . Either  $a = 0$  or  $b = 0$ . Suppose  $a = 0$ . Calculate  $e_1n_4 = xe_1 + ye_2 + zn_4$ ,  $e_1n_4^2 = e_1n_4 = xe_1n_4 + ye_2n_4 + zn_4$ , so  $(1 - x + y)e_1n_4 = (y + z)n_4$ . Since  $n_3n_4 \in E_4$ ,  $e_1n_4 \notin E_4$ , so  $y = z = 0$  and  $x = 1$ ; i.e.,  $e_1n_4 = e_1$ .

If  $n_3n_4 \neq 0$ , then  $n_3n_4 = n_4$ . Calculate  $e_1n_3 = ae_1 + be_2 + cn_3$ , from which  $e_1n_3n_4 = e_1n_4 = e_1 = ae_1 + b(n_4 - e_1) + cn_4$ . This yields  $b = c = 0$  and  $e_1n_3 = e_1$ . Similarly  $e_1e_2 = ae_1 + be_2$ , from which  $e_1e_2n_4 = e_1(1 - e_1) = ae_1n_4 + be_2n_4 = ae_1 + b(n_4 - e_1)$ . Since  $e_1^2 \in E_1 \otimes E_2$ ,  $b = 0$ . Thus  $e_1e_2$  and  $e_1^2 = e_1 - e_1e_2 \in E_1$ , whence  $e_1Q_2$  is one-dimensional.

We must have  $n_3n_4 = 0$ , and, similarly,  $n_4n_3 = 0$ . Now  $e_1n_4n_3 = e_1n_3 = 0$ , although, as in the calculation for  $e_1n_4$ ,  $e_1n_3 = e_1$ .

The referee is responsible for an important change in the statement of the theorem. Having detected an error in the original version of (7b), he suggested as a counter example the matrices  $f_i$  now listed there.

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