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SOME METRICAL THEOREMS IN NUMBER THEORY

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In this paper some metrical theorems on Diophantine approximation, continued fractions and θ -adic expansions are proved.

In the first part some of the common properties of the following transformations from the unit interval onto itself are investigated. Denote by $\{a\}$ the fractional part of x ,

A. $T: \alpha \rightarrow \{a\alpha\} \quad a > 1$ integer

which describes the expansion of α in the scale a

B. $T: \alpha \rightarrow \left\{ \frac{1}{\alpha} \right\}$

which describes the continued fractions

C. $T: \alpha \rightarrow \{\theta\alpha\} \quad \theta > 1$ noninteger

which describes the expansion of α as a θ -adic fraction.

The main theorem of the first part (Theorem 2) gives an estimate of the number of solutions of the system of inequalities

$$T^k \alpha \in I_k \quad 1 \leq k \leq n$$

where n is an integer, T is any of these three transformations and (I_k) is an arbitrary sequence of intervals contained in the unit interval.

It generalises and refines well known theorems on the distribution function of the sequence $(T^k \alpha)$. Theorem 2 follows from a very general theorem—a quantitative Borel-Cantelli Lemma.

It is also shown that T is strongly mixing (Theorem 1). The second part of the paper deals with the metric theory of continued fractions. Theorems of LeVeque and Bernstein are refined.

1. Frequently a real number α is represented in one of the following ways:

- A. in the scale a , where $a > 1$ is an integer,
- B. as a continued fraction,
- C. as a θ -adic fraction, where $\theta > 1$ is a noninteger.

Let us recall some of the properties of these representations:

- A. If $a > 1$ denotes an integer then every $\alpha \in [0, 1)$ can be written

as

$$\alpha = \sum_{k=1}^{\infty} \frac{c_k}{a^k} = \sum_{k=1}^n \frac{c_k}{a^k} + \frac{y_{n+1}}{a^n}$$

where the digits c_k are nonnegative integers less than a and $0 \leq y_n < 1$. The representation can be made unique.

Define on $[0, 1)$ the transformation T

$$T: \alpha \rightarrow \{a\alpha\}$$

Clearly for $n \geq 0$ we have

$$y_{n+1} = y_{n+1}(\alpha) = T^n \alpha = \{a^n \alpha\}.$$

As is well known T preserves the Lebesgue-measure and T is ergodic. (For definitions and theorems in ergodic theory see Halmos [5] pp. 5-37).

B. Every $\alpha \in (0, 1]$ can be expressed as a simple continued fraction

$$(1) \quad \alpha = [a_1, a_2, \dots]$$

where the partial quotients $a_i = a_i(\alpha)$ are positive integers. Again, the representation can be made unique. If α is given by (1) then the finite continued fraction

$$[a_1, a_2, \dots, a_n] = \frac{p_n}{q_n}$$

is called the n -th convergent. It is in its lowest terms.

Define on $(0, 1]$ the transformation T

$$T: \alpha \rightarrow \left\{ \frac{1}{\alpha} \right\}$$

or else

$$T([a_1(\alpha), a_2(\alpha), \dots]) = [a_2(\alpha), a_3(\alpha), \dots].$$

Then clearly for $n \geq 0$ we have

$$a_{n+1}(\alpha) = a_1(T^n \alpha)$$

and

$$z_{n+1}(\alpha) = T^n \alpha = [a_{n+1}(\alpha), a_{n+2}(\alpha), \dots].$$

Define on $(0, 1]$ a measure μ by setting

$$\mu(E) = \frac{1}{\log 2} \int_E \frac{dx}{1+x}$$

for every Lebesgue-measurable set E . Knopp [13] proved that T is ergodic with respect to the Lebesgue-measure. Ryll-Nardzewski [21] showed that T preserves μ and that μ is equivalent to the Lebesgue-

measure (i.e. both are absolutely continuous to each other).

C. For a fixed noninteger $\theta > 1$ define the transformation T on $[0, 1)$

$$T: \alpha \rightarrow \{\theta\alpha\} .$$

Put

$$\begin{aligned} x_1 &= \alpha, x_2 = \{\theta x_1\}, \dots, x_{n+1} = \{\theta x_n\}, \dots \\ \lambda_1 &= [\theta\alpha], \dots, \lambda_n = [\theta x_n], \dots \end{aligned}$$

For $n \geq 0$ we have $x_{n+1} = x_{n+1}(\alpha) = T^n\alpha$. Clearly $0 \leq \lambda_k < \theta, 0 \leq x_n < 1$ and

$$\alpha = \sum_{k=1}^{\infty} \frac{\lambda_k}{\theta^k} = \sum_{k=1}^n \frac{\lambda_k}{\theta^k} + \frac{x_{n+1}}{\theta^n} .$$

Rényi [19] proved that there exists a unique measure μ invariant under T and equivalent to the Lebesgue measure and furthermore that T is ergodic with respect to μ .

Write $\lambda_k(\mathbf{1}) = \lambda_k$ and $x_k(\mathbf{1}) = t_k$. Denote by $\varphi_\gamma(t)$ the characteristic function of $[0, \gamma)$. Put (Gel'fond [4])

$$\sigma(t) = \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{\varphi_{t_k}(t)}{\theta^{k-1}} \quad \tau = \sum_{k=1}^{\infty} \frac{t_k}{\theta^{k-1}} .$$

Cigler [3] showed that this unique measure μ is defined by setting

$$\mu(E) = \int_E \sigma(\alpha) d\alpha$$

for every measurable set $E \subset [0, 1)$.

The explicit formula for the invariant measure has been also found, independently by Parry [17] who additionally remarked that T is even weakly mixing.

Now it is necessary to say a few words about the notation. In the remainder of the §1 and in §4 T always means any one of the three transformations and μ always stands for the invariant measure associated with T as described in Sections 1 ABC. For example Theorem 2 ABC in fact consists of three theorems and should be interpreted to mean that Theorem 2 holds for each of the three transformations and further that in Theorem 2A $\mu(I)$ — where $I = (a, b)$ is an interval—stands for $b - a$, that in Theorem 2B $\mu(I)$ means

$$\frac{1}{\log 2} \int_a^b \frac{dx}{1+x} = \frac{1}{\log 2} (\log(1+b) - \log(1+a))$$

and that likewise in Theorem 2C $\mu(I)$ denotes $\int_a^b \sigma(t) dt$. Throughout

the paper "almost all" always means all except a set of Lebesgue-measure 0.

In the first part of this paper some of the common properties of these transformations are investigated. I shall prove:

THEOREM 1 A, B, C. *T is strongly mixing.*

Theorem 1A is well known. It holds even for a compact connected abelian group (Hartman, Ryll-Nardzewski [9], p. 169).

This paper was already typed when Professor Krickeberg in a letter kindly called my attention to a paper of Rohlin [20]. Rohlin showed that a wide class of transformations are—what he calls—exact endomorphisms and consequently that they are mixing of every degree which implies strongly mixing. Since the proof I give is different from Rohlin's proof and since Theorem 1 is a straightforward application of some lemmas used to prove Theorem 2, I did not withdraw Theorem 1. Furthermore its proof can be easily extended to show the mixing property of every degree.

In analogy to some results on Diophantine approximation we get:

THEOREM 2A, B, C. *Let T be any of the three transformation associated with its invariant measure μ as described in A, B, C. Let (I_n) be an arbitrary sequence of intervals contained in the unit interval. For any positive integer N and $x \in [0, 1]$ denote by $A(N, x)$ the number of positive integers $n \leq N$ such that $T^n x \in I_n$. Put*

$$\phi(N) = \sum_{n \leq N} \mu(I_n).$$

Then

$$A(N, x) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N)) \quad \epsilon > 0$$

for almost all $x \in [0, 1)$.

LeVeque [15] has proved theorems of the same type as Theorem 2A for arbitrary sequences (a_n) of integers instead of (a^n) under certain assumptions on the intervals I_n . Recently, his results have been extended by Walker [26]. The novelty in Theorem 2A is the arbitrariness of the intervals I_n ; in particular that we can dispense the assumption that the sequence $(\mu(I_n))$ is decreasing. This is not a contradiction to a theorem of Cassels ([1], p. 215) since with Cassels' notation every subsequence of (a^n) is again a Σ -sequence and so the method of proof of Cassels' theorem does not apply to our case.

Theorem 2 is a generalization and refinement of some well known results on distribution functions of certain sequences. $\mu(x)$ is called

the distribution function or distribution measure of the sequence (x_n) ($0 \leq x_n \leq 1$) if for all $0 \leq x \leq 1$

$$\lim_{n \rightarrow \infty} \frac{A(n, x)}{n} = \mu(x).$$

Here $A(n, x)$ denotes the number of positive integers $k \leq n$ such that $x_k < x$ (see Cigler und Helmberg [4], § 7). In each of the cases ABC the individual ergodic theorem implies at once that the sequence $(T^n \alpha)$ has the distribution function $\mu(x)$ for almost all α — $\mu(x)$ is the measure of the interval $(0, x)$ — μ invariant under T . These results are well known (H. Weyl [27], Ryll-Nardzewski [21], Gel'fond [7]). The case A follows also from the fact that $(a^n \alpha)$ is uniformly distributed for almost all α . Putting in Theorem 2 $I_n = I = (0, x)$ for $n = 1, 2, \dots$ we get at once

COROLLARY ABC. For $0 \leq x \leq 1$ denote by $A(n, \alpha, x)$ the number of positive integers $k \leq n$ such that $T^k \alpha < x$. Then for almost all α

$$A(n, \alpha, x) = n\mu(x) + O(n^{1/2} \log^{3/2+\varepsilon} n) \quad \varepsilon > 0$$

where $\mu(x)$ denotes the μ -measure of the interval $(0, x)$ — μ invariant under T .

2. A quantitative Borel-Cantelli Lemma. Throughout the rest of the paper $|E|$ denotes the measure of E in the underlying measure-space.

THEOREM 3. Let $(E_n, n \geq 1)$ be a sequence of measurable sets in an arbitrary probability space. Denote by $A(N, x)$ the number of integers $n \leq N$ such that $x \in E_n$. Put

$$\phi(N) = \sum_{n \leq N} |E_n|.$$

Suppose that there exists a convergent series $\sum C_k$ with $C_k \geq 0$ such that for all integers $n > m$ we have

$$(2) \quad |E_n \cap E_m| \leq |E_n| |E_m| + |E_n| C_{n-m}.$$

Then

$$A(N, x) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\varepsilon} \phi(N)) \quad \varepsilon > 0$$

for almost all x .

Theorems of this kind have been proved by LeVeque [15] and W.M. Schmidt [22] for particular sequences of sets on the real line.

In an earlier draft I obtained the error term $O(\phi^{2/s}(N) \log^{1/2+\epsilon} \phi(N))$ using a well known device of H. Weyl [27]. However, it was pointed out to me that W.M. Schmidt's [22] modification of Rademacher's method for orthogonal sums gives a better estimate. In the following proof I shall use Schmidt's method.

Proof. In case that $\phi(\infty) < \infty$ the theorem follows from the convergence part of the Borel-Cantelli lemma even without assuming (2). So we may assume that $\phi(N) \rightarrow \infty$. Denote by $\psi_n(x)$ the characteristic function of E_n . For $m < n$ put

$$A(m, n, x) = \sum_{i=m+1}^n \psi_i(x)$$

and

$$\phi(m, n) = \int A(m, n, x) d\mu(x).$$

Then clearly $\phi(0, N) = \phi(N)$ and $A(0, N, x) = A(N, x)$. Using (2) we obtain

$$\begin{aligned} & \int (A(m, n, x) - \phi(m, n))^2 d\mu(x) \\ &= 2 \sum_{m < i < j \leq n} |E_i \cap E_j| + \phi(m, n) - \phi^2(m, n) \\ (3) \quad & \leq \phi(m, n) - \phi^2(m, n) + 2 \sum_{m < i < j \leq n} |E_i| |E_j| + 2 \sum_{m < i < j \leq n} |E_j| C_{j-i} \\ & \leq \phi(m, n) + 2 \sum_{m < j \leq n} |E_j| \cdot \sum_{m < i < j} C_{j-i} \\ & = O(\phi(m, n)). \end{aligned}$$

For integer $u \geq 0$ we define N_u to be the largest integer N such that $\phi(N) < u$. Denote by L_r the set of intervals $(u, v]$ with $u = t \cdot 2^s$, $v = (t+1)2^s \leq 2^r$ where $s \geq 0$, $t \geq 0$ and $r \geq 1$ are integer. Now

$$\sum \phi(N_u, N_v) \leq \phi(N_{2^r}) < 2^r$$

where the summation is extended over all $(u, v] \in L_r$ corresponding to a fixed s since these intervals cover $(0, 2^r]$ exactly once and thus the corresponding intervals $(N_u, N_v]$ cover $(0, N_{2^r}]$ exactly once. Clearly $s \leq r$ and so

$$(4) \quad \sum_{(u, v] \in L_r} \phi(N_u, N_v) < (r+1)2^r.$$

Put

$$Z_r = Z(r, x) = \sum_{(u, v] \in L_r} (A(N_u, N_v, x) - \phi(N_u, N_v))^2.$$

Then (3) and (4) imply

$$\int Z_r d\mu(x) = O(r2^r)$$

or

$$\int \frac{Z^r}{2^r r^{2+\varepsilon}} d\mu(x) = O(r^{-1-\varepsilon}).$$

Therefore

$$(5) \quad Z_r = O(2^r r^{2+\varepsilon})$$

for almost all x . If w is an integer and $2^{r-1} < w \leq 2^r$ then $(0, w]$ can be represented as the union of at most r intervals of L_r and thus so can $(0, N_w]$. Hence

$$A(N_w, x) - \phi(N_w) = \sum A(N_u, N_v, x) - \phi(N_u, N_v)$$

where the sum is over at most $r + 1$ intervals $(u, v] \in L_r$. This equation together with (5) and Cauchy's inequality yields

$$(A(N_w, x) - \phi(N_w))^2 = O(2^r r^{3+\varepsilon})$$

almost everywhere. Hence the theorem is true for all $N = N_w$. If N is arbitrary find w such that $N_w \leq N < N_{w+1}$. We obviously have

$$A(N_w, x) \leq A(N, x) \leq A(N_{w+1}, x)$$

and

$$\phi(N_w) \leq \phi(N) \leq \phi(N_{w+1}).$$

Since

$$\phi(N_{w+1}) \leq \phi(N_w) + O(1)$$

the result follows.

3. The overlap estimates.

3A. This section deals with the situation as described in Section 1A.

LEMMA 1. *Let $P > 1$ be an integer and let $E = (x_1, x_2)$ be an interval and F be a measurable set both contained in $[0, 1]$. Then*

$$|E \cap S^{-1}F| = |E| |F| + 2P^{-1} |F| \eta$$

where $|\eta| \leq 1$ and S denotes the transformation $\alpha \rightarrow \{P\alpha\}$.

Proof. If $\rho(t)$ denotes the characteristic function of F with period 1 we have

$$\begin{aligned} |E \cap S^{-1}F| &= \int_{x_1}^{x_2} \rho(Pt) dt = P^{-1} \int_{Px_1}^{Px_2} \rho(t) dt \\ &= |E| |F| + 2P^{-1} |F| \eta \quad |\eta| \leq 1. \end{aligned}$$

3B. We use now the notation introduced in Section 1B. We begin with a lemma which is essentially due to Khintchine [10, 11] (see also [12] p. 89).

LEMMA 2. Let $E = \{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k\}$ and let F be a measurable set. Then¹

$$|E \cap T^{-n-k}F| = |E| |F| (1 + O(q^{\sqrt{n}})) \quad q < 1.$$

Proof. For $0 < x \leq 1$ denote by $\varphi_n(x)$ the (μ -)measure of the set—Khintchine used the Lebesgue-measure—

$$(6) \quad \{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k, T^{n+k}\alpha < x\}.$$

Then $\varphi_n(x)$ satisfies the functional equation

$$(7) \quad \varphi_n(x) = \sum_{g=1}^{\infty} \left(\varphi_{n-1}\left(\frac{1}{g}\right) - \varphi_{n-1}\left(\frac{1}{g+x}\right) \right).$$

In fact, $T^{n+k}\alpha < x$ is equivalent with

$$0 \leq \frac{1}{T^{n+k-1}\alpha} - \left[\frac{1}{T^{n+k-1}\alpha} \right] < x$$

or else with

$$\frac{1}{g+x} < T^{n+k-1}\alpha \leq \frac{1}{g}$$

for $g = 1, 2, \dots$. But this implies (7).

Now we are going to compute $\varphi_0(x)$ and its derivatives; $\varphi_0(x)$ is the measure of the set

$$M = \{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k, T^k\alpha < x\}.$$

Consequently M is just the interval with the endpoints p_k/q_k , $(p_k + p_{k-1}x)/(q_k + q_{k-1}x)$ where $p_k/q_k = [r_1, \dots, r_k]$. This follows from the fact that $\alpha \in M$ is equivalent with

$$\alpha = [r_1, r_2, \dots, r_k + T^k\alpha], T^k\alpha < x.$$

Hence, by computation

¹ Throughout the rest of the paper q, ρ and the constant implied by 0 are absolute constants unless otherwise stated.

$$\begin{aligned} \log 2 \cdot \varphi_0(x) &= (-1)^k \left(\log \left(1 + \frac{p_k + p_{k-1}x}{q_k + q_{k-1}x} \right) - \log \left(1 + \frac{p_k}{q_k} \right) \right) \\ \log 2 \cdot \varphi_0'(x) &= ((p_k + q_k)q_k + x((p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1}) \\ &\quad + q_{k-1}(p_{k-1} + q_{k-1})x^2)^{-1} \\ \log 2 \cdot \varphi_0''(x) &= \\ &= - \frac{(p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1} + 2q_{k-1}(p_{k-1} + q_{k-1})x}{((p_k + q_k)q_k + x((p_{k-1} + q_{k-1})q_k + (p_k + q_k)q_{k-1}) + q_{k-1}(p_{k-1} + q_{k-1})x^2)^2} \end{aligned}$$

and

$$\log 2 \cdot |E| = (-1)^k \log \left(1 + \frac{(-1)^k}{(p_k + q_k)(q_k + q_{k-1})} \right).$$

Write

$$\psi_n(x) = |E|^{-1} \varphi_n(x).$$

Then $\psi_n(x)$ also satisfies (7) and by the last set of formulas we obtain

$$0 < |\psi_n'(x)| \leq 4 \quad \text{and} \quad |\psi_n''(x)| < 32 \quad x \in (0, 1].$$

Hence a theorem of Kuzmin yields (see [10] or [12] p. 78)

$$\left| \psi_n'(x) - \frac{1}{\log 2} \cdot \frac{1}{1+x} \right| \leq c_1 q^{\sqrt{n}}$$

where c_1 and $q < 1$ are absolute positive constants. Integrating from a to b we obtain

$$|\psi_n(b) - \psi_n(a) - |F|| \leq c_1(b-a)q^{\sqrt{n}} \leq c |F| q^{\sqrt{n}}.$$

This proves the lemma for the case that $F = (a, b)$ is an interval. Now $\psi_n(x)$ defines a normed measure ψ_n on $[0, 1)$ in the usual way. We rewrite the last inequality as

$$C'_n \mu(I) \leq \psi_n(I) \leq C_n \mu(I)$$

where C'_n, C_n are constants. It follows that this inequality holds for arbitrary measurable sets $F \subset [0, 1)$.

Note. If $\varphi_n(x)$ is defined to be the Lebesgue-measure of the set (6) a theorem of Szűsz [24] gives a sharper estimate. Strangely enough the hypotheses of Szűsz' theorem are not satisfied in our case.

LEMMA 3. *Let F be a measurable set and let $E = (p_k/q_k, p'_k/q'_k)$ where $p_k/q_k, p'_k/q'_k$ are k th convergents. Then*

$$|E \cap T^{-(n+k)}F| = |E| |F| (1 + O(q^{\sqrt{n}})) \quad q < 1.$$

Proof. This is an immediate consequence of Lemma 2 since E is the union of at most countably many disjoint intervals for which the partial quotients $a_j(\alpha)$, $j \leq k$ are constant.

LEMMA 4. *Let E be an interval and F be a measurable set. Then*

$$|E \cap T^{-n}F| = |E| |F| + |F| O(q^{\sqrt{n}}) \quad q < 1.$$

Proof. Put $k = [n/2]$ and let $E = (x, y)$. Then there exist convergents $p_k/q_k, p'_k/q'_k$ such that x and y are contained in intervals with endpoints

$$x \in \left(\frac{p_k}{q_k}, \frac{p_k + p_{k-1}}{q_k + q_{k-1}} \right), \quad y \in \left(\frac{p'_k}{q'_k}, \frac{p'_k + p'_{k-1}}{q'_k + q'_{k-1}} \right)$$

respectively. Call the intersection of E with these two intervals E_1 and E_2 . Then $E = E_0 \cup E_1 \cup E_2$ where E_0 is of type described in Lemma 3. Hence by Lemma 3 and using the T preserves μ

$$||E \cap T^{-n}F| - |E| |F|| \leq C |E_0| |F| q^{\sqrt{k}} + |F| (|E_1| + |E_2|).$$

But for $i = 1, 2$

$$|E_i| \leq \frac{1}{q_k^2} \leq \frac{3}{2} \left(\frac{2}{3} \right)^k$$

Observing that $q < 1$ implies $q^{2^{-1/2}} < 1$ we get the result.

Later we need

LEMMA 5. *Let $E = \{\alpha | a_1(\alpha) \geq M\}$ and $F = \{\alpha | a_1(\alpha) \geq N\}$ for positive integers M, N . Then*

$$|E \cap T^{-n}F| = |E| |F| (1 + O(q^{\sqrt{n}})) \quad q < 1.$$

Proof. This follows at once from Lemma 2 since for example

$$E = \bigcup_{i=M}^{\infty} \{\alpha | a_1(\alpha) = i\}.$$

3C. In this section we use the notation introduced in section 1C. In the next lines $\lambda(E)$ stands for the Lebesgue-measure of E .

LEMMA 6. (Gel'fond [7]) *For $t \leq 1$ let $E = [0, t)$ be an interval and let F be a measurable set. Then*

$$\lambda(E \cap T^{-n}F) = \lambda(E) |F| + |F| O(\rho^{-n}) \quad 1 < \rho < \theta.$$

Proof. Apart from the factor $|F|$ before the 0 symbol this is

just formula (12) in Gel'fond's paper. But inspection of its proof shows that we may pull out the factor $|F|$ of the 0-symbol and this symbol still has the required properties.

LEMMA 7. *Let E be an interval and let F be a measurable set. Then*

$$|E \cap T^{-n}F| = |E||F| + |F|O(\rho^{-n}) \quad \rho > 1.$$

Proof. It is enough to show the lemma in case that $E = [0, t)$. Let f be the characteristic function of F . Using Lemma 6 we obtain

$$\begin{aligned} |E \cap T^{-n}F| &= \int_0^1 \varphi_t(\alpha) f(x_{n+1}(\alpha)) \sigma(\alpha) d\alpha \\ &= \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{1}{\theta^{k-1}} \int_0^1 \varphi_t(\alpha) f(x_{n+1}(\alpha)) \varphi_{t_k}(\alpha) d\alpha \\ &= \frac{1}{\tau} \sum_{k=1}^{\infty} \frac{1}{\theta^{k-1}} \int_0^{\min(t, t_k)} f(x_{n+1}(\alpha)) d\alpha \\ &= \frac{1}{\tau} |F| \sum_{k=1}^{\infty} \frac{\min(t, t_k)}{\theta^{k-1}} + |F|O(\rho^{-n}) \\ &= |E||F| + |F|O(\rho^{-n}). \end{aligned}$$

4. Proof of Theorem 1 and 2.

4.1 *Proof of Theorem 1.* We have to show that for every pair of measurable sets $A_1, A_2 \subset [0, 1]$

$$\lim_{n \rightarrow \infty} |A_1 \cap T^{-n}A_2| = |A_1||A_2|.$$

We approximate A_i ($i = 1, 2$) in measure by a finite union E_i ($i = 1, 2$) of disjoint intervals arbitrarily closely: Given $\varepsilon > 0$ we can find E_i such that

$$|A_i \Delta E_i| < \varepsilon \quad (i = 1, 2).$$

By Lemmas 1, 4, 7 we easily get for all sufficiently large n

$$\| |E_i \cap T^{-n}E_2| - |E_1||E_2| \| < \varepsilon.$$

Using that T preserves μ we have

$$\begin{aligned} \| |A_1 \cap T^{-n}A_2| - |A_1||A_2| \| &\leq \| |A_1 \cap T^{-n}A_2| - |E_1 \cap T^{-n}A_2| \| \\ &+ \| |E_1 \cap T^{-n}A_2| - |E_1 \cap T^{-n}E_2| \| \\ &+ \| |E_1 \cap T^{-n}E_2| - |E_1||E_2| \| + \| |E_1||E_2| - |A_1||A_2| \| < 5\varepsilon \end{aligned}$$

for all sufficiently large n .

Rohlin's theorem that T is mixing of every degree follows in the same way from:

LEMMA 8ABC. Let n_1, \dots, n_r be positive integers and let I_0, I_1, \dots, I_r be intervals all contained in the unit interval. Then

$$\begin{aligned} & |I_0 \cap T^{-n_1}I_1 \cap \dots \cap T^{-(n_1+\dots+n_r)}I_r| \\ &= \prod_{i=0}^r |I_i| + |I_r| O(q^{\min \sqrt{n_j}}) \quad q < 1 \end{aligned}$$

where the constant implied by O only depends on r .

For a proof we only have to apply Lemmas 1, 4, 7 several times.

4.2. In order to prove Theorem 2 we put $E_n = T^{-n}I_n$. Using Lemmas 1, 4, 7 and the fact that T preserves μ we obtain for $n > m$

$$\begin{aligned} |E_n \cap E_m| &= |I_m \cap T^{-(n-m)}I_n| \\ &\leq |I_m| |I_n| + |I_n| \cdot O(q^{\sqrt{n-m}}) \\ &= |E_m| |E_n| + |E_n| \cdot O(q^{\sqrt{n-m}}) \quad q < 1. \end{aligned}$$

Observing that all the measures involved are equivalent to the Lebesgue-measure we get Theorem 2 as an application of Theorem 3.

5. Some metrical theorems on continued fractions. In this section some metrical theorems on continued fractions are proved. I use the same notation as in sections 1B and 3B. The main result (Theorem 4) is a refinement of a theorem of Khintchine [11]. LeVeque [14] has outlined a proof of a weaker form of Theorem 4. Several applications of Theorem 4 are given. Finally a well known theorem (e.g. see [12], p. 67) of Bernstein is sharpened.

Again it will be illustrated that it is more natural to use the measure μ invariant under T rather than the Lebesgue-measure. In the theorems though "almost all" always means "all except a set of Lebesgue-measure 0." But μ and the Lebesgue-measure have the same null-sets and hence-considered from this point of view it makes no difference which of them we use.

5.1. The general theorem.

THEOREM 4. Let $f(u_1, \dots, u_k)$ be a nonnegative function defined for all k -tuples (u_1, \dots, u_k) with positive entries and satisfying

$$(8) \quad \int_0^1 (f(a_1(x), \dots, a_k(x)))^2 dx < \infty.$$

Then for integers $n \geq 1$ and fixed $k \geq 1$

$$(9) \quad \begin{aligned} & \frac{1}{n} \sum_{i=0}^{n-1} f(a_1(T^i x), \dots, a_k(T^i x)) \\ &= \frac{1}{\log 2} \int_0^1 f(a_1(x), \dots, a_k(x)) \frac{dx}{1+x} + O(n^{-1/2} \log^{3/2+\varepsilon} n) \end{aligned}$$

almost everywhere.

If k is not fixed but if instead n and k are linked by

$$(10) \quad 2^k \leq n < 2^{k+1}$$

then (9) holds with $n^{-1/2} \log^{2+\varepsilon} n$ instead of $n^{-1/2} \log^{3/2+\varepsilon} n$ in the error term.

Proof. Since T preserves μ we have for integer $i \geq 0$

$$(11) \quad \int_0^1 f(a_1(T^i x), \dots, a_k(T^i x)) d\mu(x) = \alpha_i = \alpha < \infty$$

by (8). Further for integers $0 \leq i < j$ and $m \geq 0$

$$\begin{aligned} I(i, j) &= \int_0^1 f(a_1(T^{i+m} x), \dots, a_k(T^{i+m} x)) \cdot f(a_1(T^{j+m} x), \dots, \\ & \quad a_k(T^{j+m} x)) d\mu(x) = \sum_1^\infty f(r_1, \dots, r_k) \cdot f(r'_1, \dots, r'_k) \\ & \quad \cdot |E \cap T^{-(j-i)} F| \end{aligned}$$

where the summation is extended over all the r 's and

$$E = \{\alpha: a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k\}$$

and

$$F = \{\alpha: a_1(\alpha) = r'_1, \dots, a_k(\alpha) = r'_k\}.$$

By Lemma 2 we obtain for $j > i + k$

$$\begin{aligned} I(i, j) &= \sum_1^\infty f(r_1, \dots, r_k) |E| \cdot \sum_1^\infty f(r'_1, \dots, r'_k) |F| \cdot (1 + O(q^{\sqrt{j-i-k}})) \\ &= \alpha^2 (1 + O(q^{\sqrt{j-i-k}})). \end{aligned}$$

By (8) we clearly have for $j \leq i + k$

$$I(i, j) \leq \alpha^2.$$

Hence we obtain by (11)

$$\begin{aligned}
& \int_0^1 \left(\sum_{i=0}^{n-1} f(a_1(T^{i+m}x), \dots, a_k(T^{i+m}x)) - \alpha n \right)^2 d\mu(x) \\
(12) \quad &= \sum_{i,j=0}^{n-1} \int_0^1 f(a_1(T^{i+m}x), \dots, a_k(T^{i+m}x)) \cdot f(a_1(T^{j+m}x), \dots, \\
& \quad a_k(T^{j+m}x)) d\mu(x) - \alpha^2 n^2 \leq \alpha^2 \cdot O\left(\sum_{\substack{i,j=0 \\ j>k+i}}^{n-1} q^{\sqrt{j-i-k}} \right) + \alpha^2 kn = O(n)
\end{aligned}$$

if k is fixed. If n and k are linked by (10) then we get in (12) only the estimate $O(n \log n)$. In both cases Theorem 6 (p. 649) of Gál and Koksma [6] gives the result.

De Vroedt [25] has obtained earlier a result of this type also for a narrow class of functions only. His theorem and mine have a non-empty overlap: To see this put in Theorem 4 $k=1$ and $f(u_i) = f(u) = \log u$. We get

COROLLARY 1 (*de Vroedt*). *Almost everywhere*

$$(a_1(x) \cdots a_n(x))^{1/n} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2 + 2n} \right)^{\log n / \log 2} + O(n^{-1/2} \log^{3/2+\varepsilon} n).$$

This is a refinement of a well known theorem of Khintchine [10]. Applying Theorem 1 for the characteristic function of the set $\{x: a_i(x) = p\}$ we get a refinement of a theorem of Lévy [19] on the frequency of the digit p .

COROLLARY 2 (*de Vroedt*). *Denote by $h(n, p, x)$ the number of positive integers $k \leq n$ such that $a_k(x) = p$. Then for almost all x*

$$h(n, p, x) = \frac{1}{\log 2} \log \left(1 + \frac{1}{p(p+2)} \right) + O(n^{-1/2} \log^{3/2+\varepsilon} n).$$

Doebelin [5] using quite different methods sketched a proof of the law of the iterated logarithm in both cases. Recently, Stackelberg [23] has announced a proof different from that of Doebelin.

As another application of Theorem 4 I shall prove:

THEOREM 5. *Denote by $q_n(x)$ the denominator of the n th convergent to x . Then*

$$\sqrt[n]{q_n(x)} = \exp(\pi^2/12 \log 2) + O(n^{-1/2} \log^{2+\varepsilon} n)$$

almost everywhere.

This improves slightly LeVeque's [14] refinement of a well known theorem of Khintchine [11] and Lévy [17]. Furthermore the proof will not depend on Lévy's result. For this reason we need some lemmas.

5.2. *Some lemmas.* The first one can be proved by induction or by taking the transposed matrices.

LEMMA 9. *If*

$$\begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_k & q_k \\ p_{k-1} & q_{k-1} \end{pmatrix}$$

then

$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q_{k-1} & q_k \\ p_{k-1} & p_k \end{pmatrix}.$$

If we consider a_1, \dots, a_k as partial quotients of a continued fraction then p_k/q_k is just the k th convergent. So the lemma describes the relation between the two fractions $[a_1, \dots, a_k]$ and $[a_k, \dots, a_1]$. This is used in

LEMMA 10. *Let*

$$E = \{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k\}$$

and

$$F = \{\alpha \mid a_1(\alpha) = r_k, \dots, a_k(\alpha) = r_1\}$$

then

$$|E| = |F|.$$

Proof. Put $[r_1, \dots, r_k] = p_k/q_k$ and $[r_k, \dots, r_1] = p'_k/q'_k$ then

$$\log 2 \cdot |E| = (-1)^k \log \left(1 + \frac{(-1)^k}{(p_k + q_k)(q_k + q_{k-1})} \right)$$

and similarly for $|F|$ with primes. But Lemma 9 yields $q'_k = q_k$, $p'_k = q_{k-1}$, and $q'_{k-1} = p_k$. Hence the result.

LEMMA 11. *Define for $k \geq 1$*

$$\begin{aligned} f_k(x) &= a_k(x) + [a_{k-1}(x), \dots, a_1(x)] \text{ if} \\ & x = [a_1(x), \dots, a_k(x), \dots] \text{-irrational} \\ &= 0 \qquad \qquad \qquad \text{if } x \text{ is rational} \end{aligned}$$

then

$$\lim_{k \rightarrow \infty} \int_0^1 \log f_k(x) d\mu(x) = \frac{\pi^2}{12 \log 2}.$$

Proof. Using Lemma 10 we obtain

$$\begin{aligned}
 \lambda_k &= \int_0^1 \log f_k(x) d\mu(x) \\
 &= \sum_{r_1, \dots, r_k \geq 1} \log(r_k + [r_{k-1}, \dots, r_1]) \mu\{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k\} \\
 &= \sum_{r_1, \dots, r_k \geq 1} \log(r_1 + [r_2, \dots, r_k]) \mu\{\alpha \mid a_1(\alpha) = r_1, \dots, a_k(\alpha) = r_k\} \\
 &= \int_0^1 \log(a_1(x) + [a_2(x), \dots, a_k(x)]) d\mu(x).
 \end{aligned}$$

But $\log(a_1(x) + [a_2(x), \dots, a_k(x)]) \leq \log(a_1(x) + 1)$ which is integrable. Hence

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_0^1 \log f_k(x) d\mu(x) &= \int_0^1 \lim_{k \rightarrow \infty} \log(a_1(x) + [a_2(x), \dots, a_k(x)]) d\mu(x) \\
 &= \frac{1}{\log 2} \int_0^1 \log \frac{1}{x} \cdot \frac{dx}{1+x} = \frac{\pi^2}{12 \log 2}.
 \end{aligned}$$

5.3. Now we can finish the proof of Theorem 5. Essentially, we use the argument of Khintchine [11]. Let $k \geq 1$ be an integer. Define for $n \geq k$

$$\begin{aligned}
 f_n^{(k)}(x) &= a_n(x) + [a_{n-1}(x), \dots, a_{n-k+1}(x)] \text{ if} \\
 & \quad x = [a_1(x), \dots, a_n(x), \dots] \text{ irrational} \\
 &= 0 \quad \quad \quad \text{if } x \text{ is rational.}
 \end{aligned}$$

Then $f_n^{(k)}(Tx) = f_{n+1}^{(k)}(x)$. Hence for $n \geq k$

$$f_n^{(k)}(x) = f_k^{(k)}(T^{n-k}x) = f_k(T^{n-k}x)$$

in the notation of Lemma 11.

We observe

$$(13) \quad \frac{q_n(x)}{q_{n-1}(x)} = f_n(x).$$

For $n \geq k$

$$(14) \quad |\log f_n(x) - \log f_n^{(k)}(x)| \leq |f_n(x) - f_n^{(k)}(x)| \leq 2^{-(k-3)}$$

by Lemma 4, p. 7 in Cassel's book [2]. Using again this lemma and the first part of the proof of Lemma 11 we get

$$\begin{aligned}
 (15) \quad \left| \lambda_k - \frac{\pi^2}{12 \log 2} \right| &\leq \int_0^1 |\log(a_1(x) + [a_2(x), \dots, a_k(x)]) - \log(a_1(x) \\
 & \quad + [a_2(x), \dots])| d\mu(x) \\
 &\leq \int_0^1 |[a_2(x), \dots, a_k(x)] - [a_2(x), \dots]| d\mu(x) \\
 &\leq 2^{-(k-4)}.
 \end{aligned}$$

From now on n and k are linked by (10). Since

$$\int_0^1 \left(n^{1/2} \sum_{i=1}^{k-1} \log f_i(x) \right)^2 dx = O(n \log^2 n)$$

we obtain by the mentioned theorem of Gál and Koksma

$$(16) \quad \frac{1}{n} \sum_{i=1}^{k-1} \log f_i(x) = O(n^{-1} \log^{5/2+\varepsilon} n)$$

almost everywhere. Hence by Theorem 4, (10), (14), (15), (16)

$$\begin{aligned} & \left| \frac{1}{n+k-1} \sum_{i=1}^{n+k-1} \log f_i(x) - \frac{\pi^2}{12 \log 2} \right| \\ & < \frac{k-1}{n} \frac{\pi^2}{12 \log 2} + \left| \frac{1}{n} \sum_{i=1}^{k-1} \log f_i(x) \right| + \frac{1}{n} \sum_{i=k}^{n+k-1} |\log f_i(x) \\ & \quad - \log f_i^{(k)}(x)| + \left| \frac{1}{n} \sum_{i=k}^{n+k-1} \log f_i^{(k)}(x) - \lambda_k \right| + \left| \lambda_k - \frac{\pi^2}{12 \log 2} \right| \\ & = O(n^{-1/2} \log^{2+\varepsilon} n) \end{aligned}$$

almost everywhere. Since obviously every positive integer $N > 2$ can be written as $N = n + k$ (n, k subject to (10)) (13) yields the result.

It might be interesting to remark that Khintchine-Lévy's theorem, namely that $\sqrt[n]{q_n(x)} \rightarrow \exp(\pi^2/12 \log 2)$ almost everywhere, is another interesting application of the individual ergodic theorem. The proof is apart from a few simplifications the same as that of Theorem 2.

COROLLARY. *If $p_n(x)$ denotes the numerator of the n th convergent to x then almost everywhere*

$$\sqrt[n]{p_n(x)} = \sqrt[n]{x} \cdot \exp(\pi^2/12 \log 2) + O(n^{-1/2} \log^{2+\varepsilon} n)$$

This follows from the inequality

$$\left| \sqrt[n]{p_n(x)} - \sqrt[n]{xq_n(x)} \right| \leq \frac{2}{n}$$

for $0 < x \leq 1$.

5.4. I shall prove now a refinement of a well known theorem of Bernstein (e.g. see [12] p. 67). However, Doeblin [5] states with some misprints that even the law of the iterated logarithm holds in case that $\varphi(n) \rightarrow \infty$.

THEOREM 6. *Let $(\varphi(n))$ be any sequence of positive integers and denote by $A(N, x)$ the number of positive integers $n \leq N$ such that $a_n(x) \geq \varphi(n)$. Put*

$$\phi(N) = \frac{1}{\log 2} \sum_{n \leq N} \log \left(1 + \frac{1}{\varphi(n)} \right).$$

Then for almost all x

$$A(N, x) = \phi(N) + O(\phi^{1/2}(N) \log^{3/2+\varepsilon} \phi(N)).$$

Proof. Put

$$E_n = \{\alpha \mid a_n(\alpha) \geq \varphi(n)\} \quad \text{and} \quad U_n = T^{-(n-1)} E_n.$$

Then

$$U_n = \{\alpha \mid a_1(\alpha) \geq \varphi(n)\}.$$

Using Lemma 5 and the fact that T preserves μ we get for $n > m$

$$\begin{aligned} |E_n \cap E_m| &= |U_m \cap T^{-(n-m)} U_n| \leq |U_m| |U_n| (1 + O(q^{\sqrt{n-m}})) \\ &= |E_m| |E_n| (1 + O(q^{\sqrt{n-m}})). \end{aligned}$$

But

$$|E_n| = |U_n| = \frac{1}{\log 2} \int_0^{1/\varphi(n)} \frac{dx}{1+x} = \frac{1}{\log 2} \log \left(1 + \frac{1}{\varphi(n)} \right).$$

Since μ is equivalent to the Lebesgue-measure Theorem 6 follows from Theorem 3.

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