Pacific Journal of Mathematics

A MODIFICATION OF MORITA'S CHARACTERIZATION OF DIMENSION

JERRY EUGENE VAUGHAN

Vol. 20, No. 1 September 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 20, No. 1, 1967

A MODIFICATION OF MORITA'S CHARACTERIZATION OF DIMENSION

J. E. VAUGHAN

Morita's characterization of dimension may be stated in the following form. Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that dim $(\bar{G} - G) \leqq n-1$ for all G in $\mathscr{G}$.

The main result of this paper is the following:

THEOREM. Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-closure-preserving base $\mathscr{G}$ for the topology of R such that dim$(\bar{G} - G) \leqq n-1$ for all G in $\mathscr{G}$.

Thus the "locally finite" condition in Morita's characterization can be replaced by the weaker "closure-preserving" condition. A further result is that the "closure-preserving" condition can be replaced by the still weaker condition of "linearly-closure-preserving" provided the "base" condition is strengthened to a "star-base" condition.

Finally, several examples are given which show that the "linearly-closure-preserving" condition is weaker than the "closure-preserving" condition in important ways. In particular, the following is proved.

THEOREM. There exists a nonmetric, regular T_1-space which has a σ-linearly-closure-preserving star-base.

If the word "linearly" is deleted from the above theorem, the resulting statement is false since Bing has proved that a regular T_1-space with a σ-closure-preserving star-base is metrizable.

1. Introduction and results. Throughout this paper, dim R represents the usual covering dimension, and ind R represents the small inductive dimension for a topological space R. See [2; 3; 5].

Morita's well known characterization of dimension [5, Lemma 2.2, p. 351] states:

Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that dim$(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}$.

The main result of this paper is to modify Morita's result to:

THEOREM 1. *Let R be a metric space. A necessary and sufficient condition that dim $R \leqq n$ is that there exists a σ-closure-preserving base $\mathscr{G}$ for the topology of R such that* $\dim(\bar{G} - G) \leqq n - 1$ *for all G in $\mathscr{G}$.*

Following the terminology of Michael [4], we say that a collection $\mathscr{G}$ of subsets of a topological space is *closure-preserving* provided that for every subcollection $\mathscr{B} \subset \mathscr{G}$ it is true that

$$\cup\{\bar{B} : B \in \mathscr{B}\} = \overline{\cup\{B \in \mathscr{B}\}}$$

A collection $\mathscr{G}$ of subsets is called *σ-closure-preserving* provided

$$\mathscr{G} = \cup\{\mathscr{G}_i : i = 1, 2, \cdots\}$$

with each $\mathscr{G}_i$ closure-preserving.

Instead of proving Theorem 1 directly, we shall prove a similar result, Theorem 2, which has a weaker condition, but from which Theorem 1 can be proven easily. To facilitate the discussion of this and further results, we first make the following definitions.

DEFINITION. A collection $\mathscr{G}$ of subsets of a topological space is called *linearly-closure-preserving* provided that there exists a well ordering of $\mathscr{G} = \{G_0, G_1, \cdots, G_\alpha, \cdots : \alpha < \eta\}$ such that

$$\cup\{\bar{G}_\beta : \beta < \alpha\} = \overline{\cup\{G_\beta : \beta < \alpha\}}$$

for all $\alpha \leqq \eta$. A collection $\mathscr{G}$ of subsets of a topological space is called *σ-linearly-closure-preserving* provided $\mathscr{G} = \cup\{\mathscr{G}_i : i = 1, 2, \cdots\}$ with each $\mathscr{G}_i$ linearly-closure-preserving.

DEFINITION. A collection $\mathscr{G}$ of open subsets of a topological space R is called a *σ-closure-preserving* (respectively *σ-linearly-closure-preserving*) *star-base* for R provided $\mathscr{G} = \cup\{\mathscr{G}_i : i = 1, 2, \cdots\}$ is a σ-closure-preserving (respectively σ-linearly-closure-preserving) collection such that for every point x in R and for every open set D containing x there exists a positive integer $k = k(x, D)$ such that

$$\phi \neq S(x, \mathscr{G}_k) \subset D,$$

where $S(x, \mathscr{G}_k) = \cup\{G \in \mathscr{G}_k : x \in G\}$.

THEOREM 2. *Let R be a metric space. A necessary and sufficient condition that* dim $R \leqq n$ *is that there exists a σ-linearly-closure-preserving star-base $\mathscr{G}$ for the topology of R such that*

$$\dim(\bar{G} - G) \leqq n - 1$$

for all G in $\mathscr{G}$.

The Nagata-Smirnov [7; 9] characterization of metrizability for regular spaces (i.e., there exists a σ-locally finite base for the topology of the space) shows that Morita's result above can be modified to the following form:

Let R be a regular T_1-space. A necessary and sufficient condition that R be metrizable with dim $R \leqq n$ is that there exists a σ-locally finite base $\mathscr{G}$ for the topology of R such that

$$\dim(\bar{G} - G) \leqq n - 1$$

for all G in $\mathscr{G}$.

A similar modification of Theorem 1 is not possible. Bing has given [1, Example C, p. 180] a nonmetric, regular T_1-space which has a σ-closure-preserving base. Bing has proven, however, [1, Theorem 4, p. 179] that a necessary and sufficient condition for a regular T_1-space to be metrizable is that there exists a σ-closure-preserving star-base for the topology of the space. Thus, as a direct result of Bing's Theorem and Theorem 1, we have:

THEOREM 3. *Let R be a regular T_1-space. A necessary and sufficient condition that R be metrizable with* dim $R \leqq n$ *is that there exists a σ-closure-preserving star-base $\mathscr{G}$ for the topology of R such that* $\dim(\bar{G} - G) \leqq n - 1$ *for all G in* $\mathscr{G}$.

Theorem 3 raises the question of whether one can replace "σ-closure-preserving" by "σ-linearly-closure-preserving" in Theorem 3. This question is equivalent to the following one. Suppose a regular T_1-space R has a σ-linearly-closure-preserving star-base; does this imply that R is metrizable? The answer is in the negative as can be seen from the following example.

EXAMPLE. A nonmetric, regular T_1-space which has a σ-linearly-closure-preserving star-base. Let C denote the usual "middle third" Cantor set in [0,1], and let Q denote the set of all rational points in [0,1]. The space R, which is to be the example, is the set of points of $C \cup Q$ with the following topology: V is open in $R = C \cup Q$ if and only if $V = U \cup W$, where U is open in the usual subspace topology of R, and W is any set of irrational points in R. In this topology the irrational points of R are discrete, and the topology induced on Q is the usual subspace topology of Q. Now, R is regular and T_1, but R is not metrizable.

To construct a σ-linearly-closure-preserving star-base for R, we first enumerate the rational points of R by $r_1, r_2, \cdots, r_k, \cdots$; and define

$$\mathscr{G}_{i,j} = \{(r_i - 1/j,\, r_i + 1/j) \cap R\}$$

for all $i, j \in N$ (where N is the set of natural numbers). Since each $\mathscr{G}_{i,j}$ contains only one open set, it is trivially linearly-closure-preserving. We define one additional collection $\mathscr{G}_0 = \{G_0, G_1, \cdots, G_\alpha, \cdots\}$ where $G_0 = R - C$, and $\{G_1, G_2, \cdots, G_\alpha, \cdots\}$ is the set of irrational points in R with any well ordering. Now G_0 is an open set in R such that $G_0 \cap C = \phi$ and $\bar{G}_0 \cap C \supset Q$. From this it follows that the collection $\mathscr{G}_0$ is a linearly-closure-preserving collection of open sets. It is easily verified that the collections

$$\mathscr{G}_0 \cup (\cup\{\mathscr{G}_{i,j}\colon i, j \in N\})$$

can be ordered into a single countable sequence of collections, and as such form a σ-linearly-closure-preserving star-base for R.

Theorem 2 raises the question of whether one can replace "star-base" by "base" in Theorem 2. This question is easily answered in the negative as we now show. Roy [8] has defined a metric space Δ which has the property that $\dim \Delta = 1$ and $\operatorname{ind} \Delta = 0$. Since ind $\Delta = 0$, there exists a base $\mathscr{G}$ for Δ such that $\dim(\bar{G} - G) = -1$ for all G in $\mathscr{G}$. If $\mathscr{G}$ is given any well ordering, and if the whole space Δ is added to the collection $\mathscr{G}$ as its first element, then $\mathscr{G}$ becomes a linearly-closure-preserving base for Δ such that $\dim(\bar{G} - \mathrm{G}) = -1$ for all G in $\mathscr{G}$. Since $\dim \Delta = 1$, it is clear that "star-base" cannot be replaced by "base" in Theorem 2.

2. Proof of Theorem 2. To prove the necessity of the condition, we note by Morita's result mentioned above that $\dim R \leqq \mathrm{n}$ implies that there exists a σ-locally finite base $\mathscr{G} = \cup\{\mathscr{G}_i\colon i \in N\}$ for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}$. Since R is a metric space, we may define

$$\mathscr{G}_{i,k} = \{G \in \mathscr{G}_i\colon \text{diameter of } G < 1/k\}$$

for all $i, k \in N$. Each $\mathscr{G}_{i,k}$ is locally finite (hence, linearly-closure-preserving), and $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}_{i,k}$ since $\mathscr{G}_{i,k} \subset \mathscr{G}_i$ for all k. By well ordering $\mathscr{G}' = \cup\{\mathscr{G}_{i,k}\colon i,k \in N\}$ into a single countable sequence of collections, we have that $\mathscr{G}'$ is a σ-linearly-closure-preserving star-base for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G in $\mathscr{G}'$.

The proof of the sufficiency will be broken up into several assertions. Each assertion will be assumed to have as hypothesis the condition of Theorem 2, i.e., $\mathscr{G} = \cup\{\mathscr{G}_i\colon i \in N\}$ is a σ-linearly-closure-preserving star-base for R such that $\dim(\bar{G} - G) \leqq n - 1$ for all G

in $\mathscr{G}$. The following notation and definitions will be used in the assertions.

For any subset S of a topological space R, the *boundary* of S is defined to be $\bar{S} \cap \overline{(R - S)}$, and is denoted by Bdry (S).

Since each collection $\mathscr{G}_i$ is linearly-closure-preserving, we may write $\mathscr{G}_i = \{G_{i0}, G_{i1}, \cdots, G_{i\alpha}, \cdots : \alpha < \eta_i\}$ and define a collection of open sets by

$$\left\{H_{i\alpha} = (G_{i\alpha} - \bigcup_{\beta<\alpha} \bar{G}_{i\beta}) : \alpha < \eta_i\right\},$$

and a collection of closed sets by

$$\left\{F_{i\alpha} = \left(\bar{G}_{i\alpha} - \bigcup_{\beta\leqq\alpha} G_{i\beta}\right) : \alpha < \eta_i\right\},$$

and let

$$\mathscr{H}_i = \{H_{i\alpha} \cap (R - F) : \alpha < \eta_i\}$$

for all $i \in N$, where F is defined below.

2.1. ASSERTION. For all $i \in N$, $\bigcup\{F_{i\beta} : \beta < \alpha\}$ is a closed set in R for every $\alpha \leqq \eta_i$.

Proof. Let i be arbitrary, but fixed. Let $\alpha \leqq \eta_i$ and let x be a limit point of $\cup\{F_{i\beta} : \beta < \alpha\}$. Then

$$x \in \overline{\bigcup_{\beta<\alpha} F_{i\beta}} = \overline{\bigcup_{\beta<\alpha} (\bar{G}_{i\beta} - \bigcup_{\delta\leqq\beta} G_{i\delta})} \subset \overline{\bigcup_{\beta<\alpha} \bar{G}_{i\beta}}.$$

Since the collection $\mathscr{G}_i$ is linearly-closure-preserving by hypothesis, $x \in \bigcup\{\bar{G}_{i\beta} : \beta < \alpha\}$. Let $\sigma < \alpha$ be the first index such that $x \in \bar{G}_{i\delta}$. It is easy to see that $x \notin G_{i\sigma}$, for $G_{i\sigma}$ is an open set which does not intersect $\bigcup\{F_{i\beta} : \sigma \leqq \beta < \alpha\}$. Hence, $x \in G_{i\sigma}$ would imply that x is a limit point of $\bigcup\{F_{i\beta} : \beta < \sigma\}$. But this would imply that

$$x \in \overline{\bigcup_{\beta<\sigma} F_{i\beta}} \subset \bigcup_{\beta<\sigma} \bar{G}_{i\beta},$$

and this would mean that there exists $\delta < \sigma$ such that $x \in \bar{G}_{i\delta}$ which is impossible by the definition of σ. Hence, $x \notin G_{i\sigma}$. Thus, we have that

$$x \in \left(\bar{G}_{i\sigma} - \bigcup_{\beta\leqq\sigma} G_{i\beta}\right) = F_{i\sigma},$$

and the assertion is proven.

The following notation will be used in the succeeding assertions. Let $F_i = \bigcup\{F_{i\beta} : \beta < \eta_i\}$, and let $F = \bigcup\{F_i : i \in N\}$.

2.2. ASSERTION. Dim $F \leqq n-1$.

Proof. By Assertion 2.1, F_i is closed for all $i \in N$. Hence, it suffices by the usual sum theorem [5, Theorem 5.2, p. 355] to prove that $\dim F_i \leqq n-1$ for all i. Let i be arbitrary, but fixed. Then by the subset theorem [5, Theorem 5.1, p. 355] we have that $\dim F_{i\alpha} \leqq n-1$ because

$$F_{i\alpha} \subset (\bar{G}_{i\alpha} - G_{i\alpha})$$

and $\dim(\bar{G}_{i\alpha} - G_{i\alpha}) \leqq n-1$ by hypothesis. By Assertion 2.1

$$\{F_{i\alpha}: \alpha < \eta_i\}$$

is a linearly-closure-preserving collection such that $\dim F_{i\alpha} \leqq n-1$ for all $\alpha < \eta_i$. Hence, the collection $\{F_{i\alpha}: \alpha < \eta_i\}$ satisfys the hypothesis of a sum theorem of Nagami [6, Theorem 1, p. 82]. Thus,

$$\dim(\bigcup \{F_{i\alpha}: \alpha < \eta_i\}) \leqq n-1$$

and the assertion is proven.

To complete the proof of Theorem 2, we need only prove that $\dim(R-F) \leqq 0$ by [5, Theorem 5.4, p. 355]. To prove that

$$\dim(R-F) \leqq 0$$

it suffices by Morita's characterization of dimension to demonstrate a σ-discrete base for $R-F$ each member of which has an empty boundary in $R-F$.

2.3. ASSERTION. The collections $\mathscr{H}_i$ are discrete in the subspace $R-F$ for all $i \in N$.

Proof. Let i be arbitrary, but fixed. We shall show that for every x in $R-F$ there exists an open neighborhood of x in $R-F$ which intersects at most one of the sets $H_{i\alpha} \cap (R-F)$. Let $x \in R-F$. If $x \notin \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ then $R - \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ is an open neighborhood of x in R which intersects none of the $H_{i\alpha}$, hence, none of the

$$H_{i\alpha} \cap (R-F) .$$

If, in the other case, $x \in \bigcup \{\bar{G}_{i\alpha}: \alpha < \eta_i\}$ let $\sigma < \eta_i$ denote the first index such that $x \in \bar{G}_{i\sigma}$. We may assume that $x \in G_{i\sigma}$, for otherwise,

$$x \in \left(\bar{G}_{i\sigma} - \bigcup_{\beta \leqq \sigma} G_{i\beta}\right) = F_{i\sigma} \subset F ,$$

which is impossible because $x \in R-F$. By the definition of σ we see that

$$x \in \left(G_{i\sigma} - \bigcup_{\beta<\sigma} \bar{G}_{i\beta}\right) \subset H_{i\sigma} .$$

Clearly, $H_{i\sigma}$ is an open neighborhood of x which does not intersect any $H_{i\alpha}$ for $\alpha \neq \sigma$. Hence, $H_{i\sigma} \cap (R - F)$ is the required neighborhood of x. This completes the proof of Assertion 2.3.

2.4. ASSERTION. The collection $\bigcup\{\mathscr{H}_i: i \in N\}$ is a base for the subspace $R - F$.

Proof. Let $x \in R - F$. Let D be any open set in $R - F$ which contains x. Let D' be an open set in R such that $D = D' \cap (R - F)$. By hypothesis there exists an integer k such that $\phi \neq S(x, \mathscr{G}_k) \subset D'$. Let $\sigma < \eta_k$ be the first index such that $x \in G_{k\sigma}$, then $G_{k\sigma} \subset D'$. Now, $x \notin \bigcup\{\bar{G}_{k\beta}: \beta < \sigma\}$ for otherwise, $x \in \bigcup\{\bar{G}_{k\beta}: \beta < \sigma\}$ would imply that there exists an index $\delta < \sigma$ such that $x \in \bar{G}_{k\delta}$. Since $\delta < \sigma$, we would have that

$$x \in \left(\bar{G}_{k\delta} - \bigcup_{\beta \leqq \delta} G_{k\beta}\right) = F_{k\delta} \subset F .$$

This is impossible since $x \in R - F$. Thus

$$x \in \left(G_{k\sigma} - \bigcup_{\beta<\sigma} \bar{G}_{k\beta}\right) = H_{k\sigma} .$$

Hence, $x \in H_{k\sigma} \cap (R - F)$, which is an open neighborhood of x in $R - F$ and a subset of D. Assertion 2.4 is, therefore, proven.

2.5. ASSERTION. For each i, $\mathrm{Bdry}\,(H_{i\alpha}) \subset F_i$ for all $\alpha < \eta_i$.

Proof. Let i be fixed, and let $\alpha < \eta_i$. Since $\mathscr{G}_i$ is a linearly-closure-preserving collection of open sets,

$$\mathrm{Bdry}\,(H_{i\alpha}) = \mathrm{Bdry}\left(G_{i\alpha} - \bigcup_{\beta<\alpha} \bar{G}_{i\beta}\right) \subset \bigcup\{\mathrm{Bdry}\,(G_{i\beta}): \beta \leqq \alpha\} .$$

Let $x \in \mathrm{Bdry}\,(H_{i\alpha})$. Since $\bigcup\{G_{i\beta}: \beta < \alpha\}$ is an open set which does not intersect $H_{i\alpha}$, we have that $x \notin \bigcup\{G_{i\beta}: \beta < \alpha\}$. Let $\delta \leqq \alpha$ be the first index such that $x \in \mathrm{Bdry}\,(G_{i\delta})$. Then

$$x \in \left(\bar{G}_{i\delta} - \bigcup_{\beta \leqq \delta} G_{i\beta}\right) = F_{i\delta} \subset F_i .$$

2.6. ASSERTION. $\mathrm{Bdry}\,(H_{i\alpha} \cap (R - F)) = \phi$ in the subspace $R - F$ for all $i \in N$, and for all $\alpha < \eta_i$.

Proof. This assertion follows from Assertion 2.5 and the fact that the boundary of $(H_{i\alpha} \cap (R-F))$ with respect to the subspace $R-F$ is a subset of the boundary of $H_{i\alpha}$ with respect to the space R.

By Assertions 2.3, 2.4, and 2.6 we have shown that

$$\mathscr{H} = \bigcup \{\mathscr{H}_i : i \in N\}$$

is a σ-discrete base for $R-F$ such that $\dim(\bar{H}-H) = -1$ for all H in $\mathscr{H}$. Hence, $\dim(R-F) \leqq 0$, and Theorem 3 is completely proven.

3. Proof of Theorem 1. The proof of the necessity of the condition is trivial.

To prove the sufficiency, let $\mathscr{G}$ be the σ-closure-preserving base for R such that $\dim(\bar{G}-G) \leqq n-1$ for all G in $\mathscr{G}$. By the same method as was used in the proof of the necessity of Theorem 2, $\mathscr{G}$ may be "rearranged" into a σ-closure-preserving star-base. Thus the condition of Theorem 2 is satisfied. We may, therefore, conclude that $\dim R \leqq n$, and Theorem 1 is proven.

The author would like to thank Dr. J. H. Roberts and Dr. Keiô Nagami for their guidance in the preparation of this paper.

REFERENCES

1. R. H. Bing, *Metrization of topological spaces,* Canad. J. Math. **3** (1951), 175-186.
2. W. Hurewitz and H. Wallman, *Dimension theory,* Princeton, 1955.
3. M. Katetov, *On the dimension of non-separable spaces* I, Cehoslovack Mat. Z. **2**(77) (1953), 333-368.
4. E. Michael, *Another note on paracompact spaces,* Proc. Amer. Math. Soc. **8** (1957), 822-828.
5. Kiiti Morita, *Normal families and dimension theory for metric spaces,* Math. Annalen **128** (1954/1955), 350-362.
6. Keiô Nagami, *Some theorems in dimension theory for non-separable spaces,* J. Math. Soc. Japan **9** No. 1 (1957), 80-92.
7. J. Nagata, *On a necessary and sufficient condition of metrizability,* J. Inst. Polytech., Osaka City Univ., Sec. A. Math., Vol. 1, (1950), 93-100.
8. Prabir Roy, *Failure of equivalence of dimension concepts for metric spaces,* Bull. Amer. Math. Soc. **68** (1962), 609-613.
9. Yu. Smirnov, *A necessary and sufficient condition for metrizability of a topological space,* Doklady Akad. Nauk. SSSR. N. S. **77** (1951), 197-200.

Received December 15, 1964. This work was supported by the National Science Foundation, Grant GP-2065, and is taken from the author's doctorial dissertation, Duke University, 1965.

DUKE UNIVERSITY

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics

Vol. 20, No. 1 September, 1967