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A MODIFICATION OF MORITA'S CHARACTERIZATION OF DIMENSION

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A MODIFICATION OF MORITA'S CHARACTERIZATION OF DIMENSION

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Morita's characterization of dimension may be stated in the following form. Let R be a metric space. A necessary and sufficient condition that dim $R \leq n$ is that there exists a σ -locally finite base \mathcal{G} for the topology of R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} .

The main result of this paper is the following:

THEOREM. Let R be a metric space. A necessary and sufficient condition that dim $R \leq n$ is that there exists a σ -closure-preserving base \mathcal{G} for the topology of R such that $\dim(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} .

Thus the "locally finite" condition in Morita's characterization can be replaced by the weaker "closure-preserving" condition. A further result is that the "closure-preserving" condition can be replaced by the still weaker condition of "linearly-closure-preserving" provided the "base" condition is strengthened to a "star-base" condition.

Finally, several examples are given which show that the "linearly-closure-preserving" condition is weaker than the "closure-preserving" condition in important ways. In particular, the following is proved.

THEOREM. There exists a nonmetric, regular T_1 -space which has a σ -linearly-closure-preserving star-base.

If the word "linearly" is deleted from the above theorem, the resulting statement is false since Bing has proved that a regular T_1 -space with a σ -closure-preserving star-base is metrizable.

1. Introduction and results. Throughout this paper, dim R represents the usual covering dimension, and ind R represents the small inductive dimension for a topological space R. See [2; 3; 5].

Morita's well known characterization of dimension [5, Lemma 2.2, p. 351] states:

Let R be a metric space. A necessary and sufficient condition that dim $R \leq n$ is that there exists a σ -locally finite base \mathcal{G} for the topology of R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} .

The main result of this paper is to modify Morita's result to:

THEOREM 1. Let R be a metric space. A necessary and sufficient condition that dim $R \leq n$ is that there exists a σ -closure-preserving base \mathcal{G} for the topology of R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} .

Following the terminology of Michael [4], we say that a collection \mathcal{G} of subsets of a topological space is *closure-preserving* provided that for every subcollection $\mathcal{B} \subset \mathcal{G}$ it is true that

$$\cup \{\overline{B} \colon B \in \mathscr{B}\} = \overline{\cup \{B \in \mathscr{B}\}}$$

A collection \mathcal{G} of subsets is called σ -closure-preserving provided

$$\mathscr{G} = \cup \{ \mathscr{G}_i : i = 1, 2, \cdots \}$$

with each \mathcal{G}_i closure-preserving.

Instead of proving Theorem 1 directly, we shall prove a similar result, Theorem 2, which has a weaker condition, but from which Theorem 1 can be proven easily. To facilitate the discussion of this and further results, we first make the following definitions.

DEFINITION. A collection \mathcal{G} of subsets of a topological space is called *linearly-closure-preserving* provided that there exists a well ordering of $\mathcal{G} = \{G_0, G_1, \dots, G_{\alpha}, \dots; \alpha < \eta\}$ such that

$$\cup \{ ar{G}_{meta} \colon eta < lpha \} = \ \overline{\cup \{ G_{meta} \colon eta < lpha \}}$$

for all $\alpha \leq \eta$. A collection \mathcal{G} of subsets of a topological space is called σ -linearly-closure-preserving provided $\mathcal{G} = \bigcup \{ \mathcal{G}_i : i = 1, 2, \cdots \}$ with each \mathcal{G}_i linearly-closure-preserving.

DEFINITION. A collection \mathcal{G} of open subsets of a topological space R is called a σ -closure-preserving (respectively σ -linearly-closure-preserving) star-base for R provided $\mathcal{G} = \bigcup \{\mathcal{G}_i: i = 1, 2, \cdots\}$ is a σ -closure-preserving (respectively σ -linearly-closure-preserving) collection such that for every point x in R and for every open set D containing x there exists a positive integer k = k(x, D) such that

$$\phi \neq S(x, \mathcal{G}_k) \subset D,$$

where $S(x, \mathcal{G}_k) = \bigcup \{G \in \mathcal{G}_k : x \in G\}$.

THEOREM 2. Let R be a metric space. A necessary and sufficient condition that dim $R \leq n$ is that there exists a σ -linearly-closurepreserving star-base \mathcal{G} for the topology of R such that

$$\dim(\bar{G}-G) \leq n-1$$

for all G in \mathcal{G} .

The Nagata-Smirnov [7;9] characterization of metrizability for regular spaces (i.e., there exists a σ -locally finite base for the topology of the space) shows that Morita's result above can be modified to the following form:

Let R be a regular T_1 -space. A necessary and sufficient condition that R be metrizable with dim $R \leq n$ is that there exists a σ -locally finite base \mathcal{G} for the topology of R such that

$$\dim(\bar{G} - G) \leq n - 1$$

for all G in \mathcal{G} .

A similar modification of Theorem 1 is not possible. Bing has given [1, Example C, p. 180] a nonmetric, regular T_1 -space which has a σ -closure-preserving base. Bing has proven, however, [1, Theorem 4, p. 179] that a necessary and sufficient condition for a regular T_1 space to be metrizable is that there exists a σ -closure-preserving starbase for the topology of the space. Thus, as a direct result of Bing's Theorem and Theorem 1, we have:

THEOREM 3. Let R be a regular T_1 -space. A necessary and sufficient condition that R be metrizable with dim $R \leq n$ is that there exists a σ -closure-preserving star-base \mathcal{G} for the topology of R such that dim $(\bar{G} - G) \leq n - 1$ for all G in \mathcal{G} .

Theorem 3 raises the question of whether one can replace " σ -closurepreserving" by " σ -linearly-closure-preserving" in Theorem 3. This question is equivalent to the following one. Suppose a regular T_1 space R has a σ -linearly-closure-preserving star-base; does this imply that R is metrizable? The answer is in the negative as can be seen from the following example.

EXAMPLE. A nonmetric, regular T_1 -space which has a σ -linearlyclosure-preserving star-base. Let C denote the usual "middle third" Cantor set in [0,1], and let Q denote the set of all rational points in [0,1]. The space R, which is to be the example, is the set of points of $C \cup Q$ with the following topology: V is open in $R = C \cup Q$ if and only if $V = U \cup W$, where U is open in the usual subspace topology of R, and W is any set of irrational points in R. In this topology the irrational points of R are discrete, and the topology induced on Q is the usual subspace topology of Q. Now, R is regular and T_1 , but R is not metrizable. To construct a σ -linearly-closure-preserving star-base for R, we first enumerate the rational points of R by $r_1, r_2, \dots, r_k, \dots$; and define

$${\mathscr G}_{i,\,j}=\{(r_i-1/j,\,r_i+1/j)\cap R\}$$

for all $i, j \in N$ (where N is the set of natural numbers). Since each $\mathscr{G}_{i,j}$ contains only one open set, it is trivially linearly-closure-preserving. We define one additional collection $\mathscr{G}_0 = \{G_0, G_1, \dots, G_{\alpha}, \dots\}$ where $G_0 = R - C$, and $\{G_1, G_2, \dots, G_{\alpha}, \dots\}$ is the set of irrational points in R with any well ordering. Now G_0 is an open set in R such that $G_0 \cap C = \phi$ and $\overline{G}_0 \cap C \supset Q$. From this it follows that the collection \mathscr{G}_0 is a linearly-closure-preserving collection of open sets. It is easily verified that the collections

$$\mathscr{G}_0 \cup (\cup \{\mathscr{G}_{i,j}: i, j \in N\})$$

can be ordered into a single countable sequence of collections, and as such form a σ -linearly-closure-preserving star-base for R.

Theorem 2 raises the question of whether one can replace "starbase" by "base" in Theorem 2. This question is easily answered in the negative as we now show. Roy [8] has defined a metric space Δ which has the property that dim $\Delta = 1$ and ind $\Delta = 0$. Since ind $\Delta = 0$, there exists a base \mathcal{G} for Δ such that dim $(\bar{G} - G) = -1$ for all G in \mathcal{G} . If \mathcal{G} is given any well ordering, and if the whole space Δ is added to the collection \mathcal{G} as its first element, then \mathcal{G} becomes a linearly-closure-preserving base for Δ such that dim $(\bar{G} - G)$ = -1 for all G in \mathcal{G} . Since dim $\Delta = 1$, it is clear that "star-base" cannot be replaced by "base" in Theorem 2.

2. Proof of Theorem 2. To prove the necessity of the condition, we note by Morita's result mentioned above that dim $R \leq n$ implies that there exists a σ -locally finite base $\mathcal{G} = \bigcup \{ \mathcal{G}_i : i \in N \}$ for R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} . Since $R \cdot is$ a metric space, we may define

$$\mathcal{G}_{i,k} = \{G \in \mathcal{G}_i: \text{diameter of } G < 1/k\}$$

for all $i, k \in N$. Each $\mathcal{G}_{i,k}$ is locally finite (hence, linearly-closure-preserving), and dim $(\overline{G} - G) \leq n - 1$ for all G in $\mathcal{G}_{i,k}$ since $\mathcal{G}_{i,k} \subset \mathcal{G}_i$ for all k. By well ordering $\mathcal{G}' = \bigcup \{ \mathcal{G}_{i,k} : i, k \in N \}$ into a single countable sequence of collections, we have that \mathcal{G}' is a σ -linearly-closure-preserving star-base for R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G}' .

The proof of the sufficiency will be broken up into several assertions. Each assertion will be assumed to have as hypothesis the condition of Theorem 2, i.e., $\mathscr{G} = \bigcup \{ \mathscr{G}_i : i \in N \}$ is a σ -linearly-closurepreserving star-base for R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathcal{G} . The following notation and definitions will be used in the assertions.

For any subset S of a topological space R, the boundary of S is defined to be $\overline{S} \cap (\overline{R-S})$, and is denoted by Bdry (S).

Since each collection \mathscr{G}_i is linearly-closure-preserving, we may write $\mathscr{G}_i = \{G_{i0}, G_{i1}, \dots, G_{i\alpha}, \dots; \alpha < \eta_i\}$ and define a collection of open sets by

$$\left\{ H_{ilpha} = (G_{ilpha} - oldsymbol{igcup}_{eta < lpha} ar{G}_{ieta}) {:} \, lpha < \eta_i
ight\}$$
 ,

and a collection of closed sets by

$$\left\{F_{ilpha}=\left(ar{G}_{ilpha}-igcup_{eta\leqlpha}G_{ieta}
ight)\!\!:lpha<\eta_i
ight\},$$

and let

$$\mathscr{H}_i = \{H_{ilpha} igcap (R-F)
angle lpha < \eta_i\}$$

for all $i \in N$, where F is defined below.

2.1. ASSERTION. For all $i \in N$, $\bigcup \{F_{i\beta}: \beta < \alpha\}$ is a closed set in R for every $\alpha \leq \eta_i$.

Proof. Let *i* be arbitrary, but fixed. Let $\alpha \leq \eta_i$ and let *x* be a limit point of $\bigcup \{F_{i\beta}; \beta < \alpha\}$. Then

$$x\in \overline{igcup_{eta .$$

Since the collection \mathcal{G}_i is linearly-closure-preserving by hypothesis, $x \in \bigcup \{\overline{G}_{i\beta}: \beta < \alpha\}$. Let $\sigma < \alpha$ be the first index such that $x \in \overline{G}_{i\delta}$. It is easy to see that $x \notin G_{i\sigma}$, for $G_{i\sigma}$ is an open set which does not intersect $\bigcup \{F_{i\beta}: \sigma \leq \beta < \alpha\}$. Hence, $x \in G_{i\sigma}$ would imply that x is a limit point of $\bigcup \{F_{i\beta}: \beta < \sigma\}$. But this would imply that

$$x\in \overline{igcup_{eta<\sigma}F_{ieta}}\subset igcup_{eta<\sigma}ar{G}_{ieta}$$
 ,

and this would mean that there exists $\delta < \sigma$ such that $x \in \overline{G}_{i\delta}$ which is impossible by the definition of σ . Hence, $x \notin G_{i\sigma}$. Thus, we have that

$$x \in \left(ar{G}_{i\sigma} \, - oldsymbol{\bigcup}_{eta \leq \sigma} G_{ieta}
ight) = F_{i\,\sigma} \; ,$$

and the assertion is proven.

The following notation will be used in the succeeding assertions. Let $F_i = \bigcup \{F_{i\beta} : \beta < \eta_i\}$, and let $F = \bigcup \{F_i : i \in N\}$. 2.2. Assertion. Dim $F \leq n - 1$.

Proof. By Assertion 2.1, F_i is closed for all $i \in N$. Hence, it suffices by the usual sum theorem [5, Theorem 5.2, p. 355] to prove that dim $F_i \leq n-1$ for all *i*. Let *i* be arbitrary, but fixed. Then by the subset theorem [5, Theorem 5.1, p. 355] we have that dim $F_{i\alpha} \leq n-1$ because

$${F}_{ilpha} \subset (ar{G}_{ilpha} - G_{ilpha})$$

and dim $(\bar{G}_{i\alpha} - G_{i\alpha}) \leq n - 1$ by hypothesis. By Assertion 2.1

 $\{F_{i\alpha}: \alpha < \eta_i\}$

is a linearly-closure-preserving collection such that dim $F_{i\alpha} \leq n-1$ for all $\alpha < \eta_i$. Hence, the collection $\{F_{i\alpha}: \alpha < \eta_i\}$ satisfys the hypothesis of a sum theorem of Nagami [6, Theorem 1, p. 82]. Thus,

$$\dim \left(\bigcup \left\{ F_{i\alpha} : \alpha < \eta_i \right\} \right) \leq n - 1$$

and the assertion is proven.

To complete the proof of Theorem 2, we need only prove that $\dim (R - F) \leq 0$ by [5, Theorem 5.4, p. 355]. To prove that

$$\dim\left(R-F\right) \leq 0$$

it suffices by Morita's characterization of dimension to demonstrate a σ -discrete base for R - F each member of which has an empty boundary in R - F.

2.3. ASSERTION. The collections \mathscr{H}_i are discrete in the subspace R - F for all $i \in N$.

Proof. Let *i* be arbitrary, but fixed. We shall show that for every *x* in R - F there exists an open neighborhood of *x* in R - Fwhich intersects at most one of the sets $H_{i\alpha} \cap (R - F)$. Let $x \in R - F$. If $x \notin \bigcup \{\overline{G}_{i\alpha} : \alpha < \eta_i\}$ then $R - \bigcup \{\overline{G}_{i\alpha} : \alpha < \eta_i\}$ is an open neighborhood of *x* in *R* which intersects none of the $H_{i\alpha}$, hence, none of the

$$H_{i\alpha} \cap (R-F)$$
.

If, in the other case, $x \in \bigcup \{\overline{G}_{i\alpha}: \alpha < \eta_i\}$ let $\sigma < \eta_i$ denote the first index such that $x \in \overline{G}_{i\alpha}$. We may assume that $x \in G_{i\sigma}$, for otherwise,

$$x \in \left(ar{G}_{i\sigma} - igcup_{eta \leq \sigma} G_{ieta}
ight) = F_{i\sigma} \subset F$$
 ,

which is impossible because $x \in R - F$. By the definition of σ we see

that

$$x \in \left(G_{i\sigma} - igcup_{eta < \sigma} ar{G}_{ieta}
ight) \subset H_{i\sigma}$$
 .

Clearly, $H_{i\sigma}$ is an open neighborhood of x which does not intersect any $H_{i\alpha}$ for $\alpha \neq \sigma$. Hence, $H_{i\sigma} \cap (R - F)$ is the required neighborhood of x. This completes the proof of Assertion 2.3.

2.4. ASSERTION. The collection $\bigcup \{\mathscr{H}_i : i \in N\}$ is a base for the subspace R - F.

Proof. Let $x \in R - F$. Let D be any open set in R - F which contains x. Let D' be an open set in R such that $D = D' \cap (R - F)$. By hypothesis there exists an integer k such that $\phi \neq S(x, \mathscr{G}_k) \subset D'$. Let $\sigma < \eta_k$ be the first index such that $x \in G_{k\sigma}$, then $G_{k\sigma} \subset D'$. Now, $x \notin \bigcup \{\overline{G}_{k\beta}: \beta < \sigma\}$ for otherwise, $x \in \bigcup \{\overline{G}_{k\beta}: \beta < \sigma\}$ would imply that there exists an index $\hat{\sigma} < \sigma$ such that $x \in \overline{G}_{k\delta}$. Since $\delta < \sigma$, we would have that

$$x \in \left(ar{G}_{k\delta} - igcup_{eta \leq \delta} G_{keta}
ight) = F_{k\delta} \subset F$$
 .

This is impossible since $x \in R - F$. Thus

$$x \in \left(G_{k\sigma} - igcup_{eta < \sigma} ar{G}_{keta}
ight) = H_{k\sigma}$$
 .

Hence, $x \in H_{k\sigma} \cap (R - F)$, which is an open neighborhood of x in R - F and a subset of D. Assertion 2.4 is, therefore, proven.

2.5. ASSERTION. For each *i*, Bdry $(H_{i\alpha}) \subset F_i$ for all $\alpha < \eta_i$.

Proof. Let *i* be fixed, and let $\alpha < \eta_i$. Since \mathcal{G}_i is a linearlyclosure-preserving collection of open sets,

$$\operatorname{Bdry} \left(H_{i\alpha} \right) = \operatorname{Bdry} \left(G_{i\alpha} - \bigcup_{\beta < \alpha} \overline{G}_{i\beta} \right) \subset \bigcup \left\{ \operatorname{Bdry} \left(G_{i\beta} \right) : \beta \leqq \alpha \right\}.$$

Let $x \in Bdry(H_{i\alpha})$. Since $\bigcup \{G_{i\beta}: \beta < \alpha\}$ is an open set which does not intersect $H_{i\alpha}$, we have that $x \notin \bigcup \{G_{i\beta}: \beta < \alpha\}$. Let $\delta \leq \alpha$ be the first index such that $x \in Bdry(G_{i\delta})$. Then

$$x \, \epsilon \left(ar{G}_{i\delta} - oldsymbol{\bigcup}_{eta \leq \delta} \, G_{ieta}
ight) = \, F_{i\delta} \subset F_i \; .$$

2.6. ASSERTION. Bdry $(H_{i\alpha} \cap (R-F)) = \phi$ in the subspace R-F for all $i \in N$, and for all $\alpha < \eta_i$.

Proof. This assertion follows from Assertion 2.5 and the fact that the boundary of $(H_{i\alpha} \cap (R-F))$ with respect to the subspace R-F is a subset of the boundary of $H_{i\alpha}$ with respect to the space R.

By Assertions 2.3, 2.4, and 2.6 we have shown that

$$\mathcal{H} = \bigcup \left\{ \mathcal{H}_i : i \in N \right\}$$

is a σ -discrete base for R - F such that dim $(\overline{H} - H) = -1$ for all H in \mathcal{H} . Hence, dim $(R - F) \leq 0$, and Theorem 3 is completely proven.

3. Proof of Theorem 1. The proof of the necessity of the condition is trivial.

To prove the sufficiency, let \mathscr{G} be the σ -closure-preserving base for R such that dim $(\overline{G} - G) \leq n - 1$ for all G in \mathscr{G} . By the same method as was used in the proof of the necessity of Theorem 2, \mathscr{G} may be "rearranged" into a σ -closure-preserving star-base. Thus the condition of Theorem 2 is satisfied. We may, therefore, conclude that dim $R \leq n$, and Theorem 1 is proven.

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