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CONTINUITY OF TRANSFORMATIONS WHICH LEAVE INVARIANT CERTAIN TRANSLATION INVARIANT SUBSPACES

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It is shown that a linear operator $T: L^2(X) \to L^2(X)$ (X a locally compact group), with the property that $TE \subset E$ for each norm closed right translation invariant subspace E of $L^2(X)$, is necessarily continuous. In § 5 the author shows that this is also true for $L^1(X)$ when X contains an element a which does not lie in any compact subgroup. An example is constructed to show that, in $l^\infty(-\infty, +\infty)$, T can be discontinuous and still leave invariant each $\sigma(l^\infty, l^1)$ closed translation invariant subspace of l^∞. If however $T: l^\infty(-\infty, +\infty) \to l^\infty(-\infty, +\infty)$ leaves invariant all norm closed translation invariant subspaces, then T must be continuous.

We shall use the notation of [3] without further explanation. Sections 3, 4 and 5 overlap with some of the results in 1.4 of Edwards' paper [3] where he shows *inter alia* that T is automatically continuous in the L^2 case if X is compact and in the L^1 case if X is finitely representable. The work in § 6 answers the conjecture in 1.7 of [3].

2. Three basic lemmas. For the first lemma let $\mathfrak{X}$ be a normal topological space and $\mathscr{A}$ a set of complex valued functions on $\mathfrak{X}$, closed with respect to pointwise multiplication, containing the constant function 1 and such that if $f \in \mathscr{A}$ then so is $1 - f$. We assume that $\mathscr{A}$ is normal in the sense that if F_0 and F_1 are closed disjoint subsets of $\mathfrak{X}$ then there is a function $f \in \mathscr{A}$ with $f(F_0) = \{0\}, f(F_1) = \{1\}$. We suppose that there is a mapping $f \to S_f$ of $\mathscr{A}$ into $\mathscr{B}(E)$, where E is some Banach space, such that $S_{fg} = S_f S_g, S_1 = I, S_{1-f} = I - S_f$. We denote the null space of S_f by N_f and suppose that T is a linear map of E into E which leaves invariant all the subspaces $N_f (f \in \mathscr{A})$. For each $f \in \mathscr{A}$ the composition of T with the quotient map $E \to E/N_f$ gives a map $T_f: E \to E/N_f$.

DEFINITION 2.1. $\lambda \in \mathfrak{X}$ is called a *discontinuity value* of T if T_f is discontinuous whenever $f \in \mathscr{A}$ and there exists a neighbourhood N of λ such that $f(\lambda') \neq 0$ for all $\lambda' \in N$.

LEMMA 2.2. *T has only a finite number of discontinuity values.*

Proof. Suppose T has a infinity of discontinuity values. Then we can find a sequence λ_i of such values and a closed neighbourhood V_i of each such that, for each i, V_i and $[\bigcup_{j\neq i} V_j]^-$ are disjoint. For each i let f_i be a function in $\mathscr{A}$ with $f_i(V_j) = \delta_{i,j}$ and let $g_i \in \mathscr{A}$ such that $g_i(\lambda) = 0$ for $\lambda \notin V_i$ and $g_i(\lambda) \neq 0$ for all λ in some neighbourhood of λ_i. Then take $\xi_i \in E$ such that $\| \xi_i \| . \| S_{f_i} \| < 2^{-i}$ and $\| T_{g_i}\xi_i \| > 2^i$. Put $\eta = \sum S_{f_i}\xi_i$. Then $S_{g_j}S_{f_i} = 0$ for $j \neq i$ so that $\sum_{i\neq j} S_{f_i}\xi_i \in N_{g_j}$, and $S_{g_j}S_{1-f_j} = 0$ so that $\xi_j - S_{f_j}\xi_j \in N_{g_j}$. Thus

$$\| T\eta \| \geqq \| T_{g_j}\eta \| = \| T_{g_j}S_{f_j}\xi_j \| = \| T_{g_j}\xi_j \| > 2^j$$

for all j, which is impossible.

LEMMA 2.3. *Let A be an index set and let, for each $\alpha \in A$, E_α be a Banach space and P_α a continuous linear map from E into E_α with null space N_α. Let $N = \cap N_\alpha$ and let P be the quotient map $E \to E/N$. Let T be a linear map $E \to E$ such that $P_\alpha T$ is continuous for each $\alpha \in A$. Then PT is continuous.*

Proof. Suppose $x_n \to 0$ in E and $PTx_n \to y$ in E/N. Since $N \subset N_\alpha$ the P_α give rise to maps $Q_\alpha: E/N \to E_\alpha$ and $Q_\alpha PT = P_\alpha T$. Hence $Q_\alpha y = \lim_n Q_\alpha PTx_n = \lim_n P_\alpha Tx_n = 0$. Thus $Q_\alpha y = 0$ for all $\alpha \in A$ which implies that $y = 0$. Hence, by the closed graph theorem, PT is continuous.

LEMMA 2.4. *Let H be a Hilbert space, $\mathfrak{X}$ a locally compact Hausdorff space, $P(\,.\,)$ a spectral measure on $\mathfrak{X}$* (see [5], p. 246), *$\mathscr{G}$ a collection of open sets from $\mathfrak{X}$, $N = \bigcap_{G\in\mathscr{G}} N(P(G))$ and P_0 the orthogonal projection with null space N. Then $P_0 = P(\cup \mathscr{G})$.*

Proof. With the usual ordering of projections $P(\cup \mathscr{G}) \geqq P(G)$ for each G in $\mathscr{G}$ and so $P(\cup \mathscr{G}) \geqq P_0$. Now let $\xi_0 \in N$ and consider the measure $\mu(Y) = (P(Y)\xi_0, \xi_0)$ defined on the borel sets in $\mathfrak{X}$. Since $\mu(G) = 0$ for all G in $\mathscr{G}$, $\cup \mathscr{G}$ is locally of μ measure zero and hence since it is open, $\mu(\cup \mathscr{G}) = 0$ (see [1], p. 183). However this implies $P(\cup \mathscr{G})\xi_0 = 0$ and so $P_0 = P(\cup \mathscr{G})$.

3. $L^2(X)$, the simple case.

THEOREM 3.1. *Let X contain an element a which is not contained in any compact subgroup. Let T be a linear map $L^2(X) \to L^2(X)$ which leaves invariant each closed right translation invariant subspace of $L^2(X)$. Then T is continuous.*

Proof. Apply Lemma 2.2 with $\mathfrak{X}$ as the circle $|z| = 1$ in C and $\mathscr{A}$ the set of characteristic functions of measurable sets in $\mathfrak{X}$. The operator τ_a is a unitary operator in $L^2(X)$ and its spectral resolution $P(\,.\,)$ gives the required map from $\mathscr{A}$ into $\mathscr{B}(L^2(X))$. Since the operators corresponding to elements of $\mathscr{A}$ are weak operator limits of polynomials in τ_a and τ_a^*, they commute with all right translations. In particular the ranges and null spaces of the values of $P(\,.\,)$ are closed right translation invariant subspaces.

In the following it will be convenient not to distinguish too carefully between the quotient of a Hilbert space by a closed subspace and the orthogonal complement of the subspace.

Application of 2.2 shows that T has only finitely many discontinuity values $\lambda_1, \cdots, \lambda_k$. For each $\lambda \in \mathfrak{X}$ which is not a discontinuity value we thus have a set $Y \subset \mathfrak{X}$ containing λ as an interior point such that $P(Y)T$ is continuous; clearly Y can be taken to be open. Application of 2.3 taking A as the class of open sets G in $\mathfrak{X}$ with $P(G)T$ continuous and $P_G = P(G)$ shows that $P_0 T$ is continuous where P_0 is the orthogonal projection along the intersection of the null spaces of the $P(G)$; $G \in A$. Lemma 2.4 then shows that $P_0 = I - P(\{\lambda_1\}) - \cdots - P(\{\lambda_k\})$.

If now $\xi \neq 0$ is in the range of $P(\{\lambda_1\})$ then $\tau_a \xi = \lambda_1 \xi$. We can find a compact set K such that $\int_K |\xi(x)|^2\,dx > 1/2.\,||\xi||^2$ and an integer n such that $a^n \notin KK^{-1}$. Thus

$$\begin{aligned} 1/2.\,||\xi||^2 &< \int_K |\xi(x)|^2\,dx \\ &= \int_K |\lambda_1^n \xi(x)|^2\,dx \\ &= \int_K |\xi(a^{-n}x)|^2\,dx \\ &= \int_{a^{-n}K} |\xi(x)|^2\,dx\,. \end{aligned}$$

Since $a^{-n}K \cap K$ is void this gives

$$\begin{aligned} ||\xi||^2 &\geqq \int_K |\xi(x)|^2\,dx + \int_{a^{-n}K} |\xi(x)|^2\,dx \\ &> 1/2.\,||\xi||^2 + 1/2.\,||\xi||^2\,. \end{aligned}$$

This contradiction shows that the range of $P(\{\lambda_1\})$ is the zero subspace and $P(\{\lambda_1\}) = 0$. Similarly $P(\{\lambda_2\}) = \cdots P(\{\lambda_k\}) = 0$, $P_0 = I$ and T is continuous.

REMARK 3.2. A stronger result would be obtained if we could consider operators reduced by subspaces closed in some topology weaker than the norm topology. Since L^2 is reflexive there is no advantage in considering the weak topology. The result is untrue if we take the topology $\sigma(L^2, L_0)$ where L_0 is the subspace of L^2 consisting of functions with compact support since, in the case $X = \boldsymbol{Z}$, the additive group of the integers, there are no $\sigma(L^2, L_0)-$ closed translation invariant proper subspaces of L^2. In this case if $\xi \in L_0, \xi \neq 0$ then for η and all its translates to be in $\xi^\perp$ we must have $\xi^*\eta = 0$, or $\hat{\xi}\hat{\eta} = 0$ where $\hat{\xi}, \hat{\eta}$ are the functions in L^2 of the unit circle whose Fourier coefficients are ξ, η. Since $\hat{\xi}$ is a polynomial this implies that $\hat{\eta} = 0$ almost everywhere and so $\eta = 0$.

REMARK 3.3. If X is not unimodular or has a nonunimodular subgroup then it satisfies the condition of the theorem.

4. $L^2(X)$, the general case. The methods of [3] apply when the group X is finitely representable. To show that there is indeed ground not covered by either Edwards' method or Theorem 3.1 we take the group in the following example with the discrete topology.

EXAMPLE 4.1. *The alternating group A on $\aleph_0$ symbols* (see [4], p. 40) *is not finitely representable yet every element lies in a finite subgroup.*

Proof. We consider the permutations as acting on $1, 2, 3, \cdots$ and denote by A_n the subgroup leaving $n+1, n+2, \cdots$ invariant. A_n is isomorphic with the alternating group on n symbols in an obvious way. Suppose $\varphi: A \rightarrow GL(m)$ is an m dimensional representation of A. Then φ induces m dimensional representations of A_n for each n which must be trivial for $n > m + 1$ (see [2], p. 466). Since $A = \bigcup A_n$, φ is trivial on A. Thus there is no nontrivial finite dimensional representation of A.

THEOREM 4.2. *Let X be locally compact but not compact. Let T be a linear map $L^2(X) \rightarrow L^2(X)$ which leaves invariant each closed right translation invariant subspace of $L^2(X)$. Then T is continuous.*

Proof. We use here ideas occuring in the theory of W^*-algebras (see e.g. [5], § 41).

Let $\mathfrak{L}$ be the set of elements of $\mathscr{B}(L^2)$ commuting with all the right translation operators. The range and null space of any projection in $\mathfrak{L}$ is closed with respect to right translation and hence each such projection commutes with T. We consider Z, the centre of $\mathfrak{L}$, a commutative W^*-algebra. Let $\mathscr{M}$ be its maximal ideal space so that $\mathscr{M}$ is an extremally disconnected compact topological space (see [6], p. 294). To show that T has only finitely many discontinuity values $\lambda_1, \cdots, \lambda_k$, we then apply Lemma 2.2 with $\mathscr{M}$ as $\mathfrak{X}$, $\mathscr{A}$ as the set of characteristic functions of open closed sets in $\mathfrak{X}$ and for $f \in \mathscr{A}$,

S_f as the projection in Z whose Gelfand transform is f.

Applying 2.3 with A as the set of open closed sets G in $\mathscr{M}$ for which $P(G)T$ is continuous, where $P(G)$ is the element of Z whose Gelfand transform is the characteristic function of G, we obtain a projection $P \geqq P(G)$ for all G in A, such that PT is continuous. From the way in which it is constructed, $P \in Z$, and so $P = P(G_0)$ where G_0 is the maximum element in A which must be $\mathscr{M}\backslash\{\lambda_1, \cdots, \lambda_k\}$. This shows that $\{\lambda_1, \cdots, \lambda_k\}$ is open closed and, since $\mathscr{M}$ is Hausdorff, that if T is discontinuous then $\mathscr{M}$ has isolated points.

Let now $\lambda_0 \in \{\lambda_1, \cdots, \lambda_k\}$ and let H_0 be the range of the corresponding projection $P_0 = P(\{\lambda_0\})$.

The algebras $\mathfrak{L}_0$, $\mathfrak{L}'_0$ formed by restricting the operators in $\mathfrak{L}$, $\mathfrak{L}'$ to H_0 form a factored pair (see [5], p. 450). Now let C be a maximal commutative subalgebra of $\mathfrak{L}_0$. Application of the argument in the previous paragraphs to C in place of Z shows that the maximal ideal space of C contains at least one isolated point μ. Let M_0 be the range of the corresponding projection Q. Since $Q \in \mathfrak{L}_0$, M_0 is a subspace adjoined to $\mathfrak{L}_0$ and is moreover a minimal subspace ([5], p. 451). It follows that the factor $\mathfrak{L}_0$ is of type I. The structure of such factors is as follows ([5], p. 480). H_0 is a direct sum of orthogonal subspaces $M_\alpha; \alpha \in A$, each isomorphic with M_0 (and which we shall consider as copies of M_0). Relative to this decomposition of H_0 the matrix of an element of $\mathfrak{L}'_0$ is $[\delta_{\alpha,\beta}A_0]$ where $A_0 \in \mathscr{B}(M_0)$. This is an isomorphism between $\mathfrak{L}'_0$ and $\mathscr{B}(M_0)$. The matrix of an element of $\mathfrak{L}_0$ is $[a_{\alpha,\beta}I_0]$ where the $a_{\alpha,\beta}$ are complex numbers and I_0 is the identity operator in M_0. This is an isomorphism between $\mathfrak{L}_0$ and those matrices of the form $[a_{\alpha,\beta}I_0]$ giving bounded transformations.

Applying the results in § 2 a third time we see that there is only a finite number of values $\alpha_1, \cdots, \alpha_n$ of α for which the restriction of T to M_α is not continuous. Let, if possible, α_0 be a value of α different from $\alpha_1, \cdots, \alpha_n$. Let P be the operator in $\mathfrak{L}_0$ whose matrix $[a_{\alpha,\beta}I_0]$ is given by $a_{\alpha_1,\alpha_0} = a_{\alpha_1,\alpha_1} = 1$, $a_{\alpha,\beta} = 0$ otherwise. P is a non-orthogonal projection and $PT_0 = T_0P$ (where T_0 is the restriction of T to H_0) so that $PT_0M_{\alpha_1} = T_0PM_{\alpha_1} = T_0M_{\alpha_0}$ which is impossible since P maps M_{α_1} isometrically onto M_{α_0} whilst T_0 is bounded on M_{α_0} and unbounded on M_{α_1}. Thus $A = \{\alpha_1, \cdots, \alpha_n\}$.

Now let $\xi_0 \in M_0$ with $\|\xi_0\| = 1$ and let $\xi_1, \cdots, \xi_n$ be its copies in $M_{\alpha_1}, \cdots, M_{\alpha_n}$. The subspace of $L^2(X)$ generated by $\xi_1, \cdots, \xi_n$ is closed with respect to all operators in $\mathfrak{L}$. Let K be a compact set in X such that

$$\int_K |\xi_i(t)|^2 \, dt > 1 - (2n)^{-2}$$

for $i = 1, \cdots, n$ and let $a \in X\backslash KK^{-1}$ (such an element exists since KK^{-1} is compact and X is not). Then $\tau_a\xi_1 = a_1\xi_1 + \cdots + a_n\xi_n$ where

$|a_1|^2 + \cdots + |a_n|^2 = 1$. Let c be the characteristic function of aK. Then

$$\begin{aligned} \|(\tau_a\xi_1)c\|^2 &= \int_K |\xi_1(t)|^2\,dt \\ &> 3/4 \end{aligned}$$

whereas

$$\begin{aligned} \|\xi_i c\|^2 &\leqq \int_{X\setminus K} |\xi_i(t)|^2\,dt \\ &\leqq (2n)^{-2} \end{aligned}$$

so that

$$\begin{aligned} \|(\tau_a\xi_1)c\| &= \|a_1\xi_1\cdot c + \cdots + a_n\xi_n\cdot c\| \\ &\leqq |a_1|\cdot\|\xi_1\cdot c\| + \cdots + |a_n|\cdot\|\xi_n\cdot c\| \\ &\leqq 1/2 \end{aligned}$$

which is a contradiction, proving the theorem.

5. $L^1(X)$, the simple case.

THEOREM 5.1. *Let X contain an element a which is not contained in any compact subgroup. Let T be a linear operator from $L^1(X)$ into $L^1(X)$ such that $\mu^*Tf = 0$ whenever $\mu \in M_{bd}(X)$, $f \in L^1(X)$ with $\mu^*f = 0$. Then T is continuous.*

Proof. As in 3.1 take $\mathfrak{X}$ as the circle $|z| = 1$ and $\mathscr{A}$ as the set of functions f on $\mathfrak{X}$ with absolutely convergent Fourier series $\sum a_n e^{in\theta}$. We take for S_f the operator $g \to \mu_f{}^*g$ where, for $f \in \mathscr{A}$ $\mu_f = \sum a_i(\delta_a)^i$. By Lemma 2.2, T has only finitely many discontinuity values $\lambda_1, \cdots, \lambda_n$. Let $\mathscr{A}_0$ be the subset of $\mathscr{A}$ consisting of those elements f for which T_f is continuous. For each $\lambda \in \mathfrak{X}$ different from $\lambda_1, \cdots, \lambda_n$ there is $f \in \mathscr{A}_0$ such that T_f is continuous and $f(\lambda) \neq 0$. Put $N = \cap N_f$ $(f \in \mathscr{A}_0)$. If $\{\beta_j\} \in l^1(-\infty, +\infty)$ with $\hat{\beta}f = 0$ for all f in $\mathscr{A}_0$ where $\hat{\beta} = \sum \beta_j z^j$ then $\beta_j = 0$ for all j and using the fact that $\mathfrak{X}\setminus\{\lambda_1, \cdots, \lambda_n\}$ is σ-compact, we can find a countable subset $\mathscr{A}_*$ of $\mathscr{A}_0$ with this property. Now if $g \in N$, then for all $\sum a_n z^n \in \mathscr{A}_0$, $(\mu^*g)(t) = \sum a_n g(a^{-n}t) = 0$ for almost all t in X. Putting $\beta_{t,k} = g(a^k t)$, $k = \cdots, -1, 0, 1, \cdots$ we have, for each $f \in \mathscr{A}_0$, $\hat{\beta}_t f = 0$ for almost all t in $\mathfrak{X}$. Since $\mathscr{A}_* \subset \mathscr{A}_0$ and $\mathscr{A}_*$ is countable, we have that for almost all t in $\mathfrak{X}$, $\hat{\beta}_t f = 0$ for all $f \in \mathscr{A}_*$. Thus $\beta_t = 0$ for almost all t in $\mathfrak{X}$ and, in particular, $\beta_{t,0} = g(t) = 0$ a.e. in $\mathfrak{X}$ so that $N = \{0\}$. Using 2.3 with $A = \mathscr{A}_0$ and P_f as the quotient map $L^1(X) \to L^1(X)/N_f$, the proof is complete.

REMARK 5.2. The author has not been able to extend this result to the general case.

REMARK 5.3. The condition $\mu^* Tf = 0$ whenever $\mu^* f = 0$ is of course equivalent to saying that the spaces $N_\mu = \{f; f \in L^1(X), \mu^* f = 0\}$ are closed with respect to T. These spaces are right translation closed and also closed in a variety of topologies including $\sigma(L^1, C_0)$ or any other topology $\sigma(L^1, F)$ where F is a subspace of L^∞ total over L^1 and closed with respect to convolution with bounded measures.

6. The space l^∞. We now show that the continuity hypothesis in 1.6 of [3] is necessary in the case $X = \boldsymbol{Z}$ (the additive group of integers). We use the results in Ch. VII, § 1 of [7].

EXAMPLE 6.1. The elements $a = \{a_n\}$ of $l^\infty(-\infty, +\infty)$ can be considered as Fourier coefficients of certain distributions $\hat{a}$ on the circle group $|z| = 1$ and since $\{a_n(1 + n^2)^{-1}\} \in l^1(-\infty, +\infty)$ these distributions are, at worst, of order two. If $b = \{b_n\}$ is a sequence of rapid decrease it is the sequence of Fourier coefficients of an infinitely differentiable function $\hat{b}$ on the circle and we have $a^*b \in l^\infty$, $(a^*b)^\wedge = \hat{a}\cdot\hat{b}$. It follows from this and the ideas of spectral synthesis, that the constant sequence 1 lies in the $\sigma(l^\infty, l^1)$-closed linear subspace of l^∞ spanned by the translates of a if and only if 0 lies in the support of $\hat{a}$. For each positive integer n let $E_n = \{a; a \in l^\infty, d(0, \operatorname{supp} \hat{a}) \geqq 1/n\}$. The E_n are norm closed subspaces of l^∞, $E_n \subsetneqq E_{n+1}$ so that $E_\infty = \cup E_n$ is of first category with the induced l^∞ topology and so is not closed in l^∞. We can thus find a discontinuous functional f on l^∞ which is 0 on E_∞. If we then put $Ta = f(a)1$ we then have a discontinuous linear transformation T on l^∞ such that Ta is in the $\sigma(l^\infty, l^1)$-closure of linear combinations of translates of a.

6.2. The conjecture in 1.7 of [3] can only break down in a way similar to that in 6.1. This can be seen by application of the results in § 2 taking $\mathfrak{X}$ as $|z| = 1$ and $\mathscr{A}$ as the set of infinitely differentiable functions on $\mathfrak{X}$. We see that if T is a linear transformation $l^\infty \to l^\infty$ such that Ta is a $\sigma(l^\infty, l^1)$-limit of linear combinations of translations of a then there is a subspace N such that PT is continuous where P is the quotient map $l^\infty \to l^\infty/N$. The space N is the intersection of null spaces corresponding to a subset $\mathscr{A}_0$ of $\mathscr{A}$, the elements of $\mathscr{A}_0$ having only a finite number of common zeros $\lambda_1, \cdots, \lambda_n$. Thus N consists only of elements a with $\operatorname{Supp} \hat{a} \subset \{\lambda_1, \cdots, \lambda_l\}$ and since a is of order $\leqq 2$ we have

$$a_n = \alpha_l \lambda_l^n + \cdots + \alpha_l \lambda_l^n + n(\beta_l \lambda_l^n + \cdots) .$$

The coefficient of n is almost periodic so that for a to be bounded the β_i are all zero. Hence N is a space of linear combinations of the λ_j^n, $j = 1, \cdots, l$.

6.3. If T is assumed to leave invariant all norm closed translation invariant subspaces of l^∞ then it is continuous. To show this we apply the argument in 6.2 and then put

$$l^+ = \{a; a \in l^\infty, a_n = 0 \quad \text{for} \quad n < 0\}$$
$$l^- = \{a; a \in l^\infty, a_n = 0 \quad \text{for} \quad n \geqq 0\} .$$

Let K be the norm of PT. If $a \in l^\infty$ then $a = a^+ + a^-$ with $a^+ \in l^+$, $a^- \in l^-$. Then $\| PTa^+ \| \leqq K \| a^+ \| \leqq K \| a \|$ and so there is $b \in N$ such that $\| Ta^+ - b \| \leqq 2K \| a \|$. However since Ta^+ is a uniform limit of linear combinations of translates of a^+, and hence $(Ta^+)_n \to 0$ as $n \to -\infty$, and b is almost periodic we have $\| b \| \leqq \| Ta^+ - b \|$. From this we see that $\| Ta^+ \| \leqq \| Ta^+ - b \| + \| b \| \leqq 4K \| a \|$, and applying the same argument to $\| Ta^- \|$ we get a similar conclusion, the two together giving $\| Ta \| \leqq 8K \| a \|$.

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