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The present paper deals with the convergence of quasi-Hermite-Fejér interpolation series $\{S_n(x, f)\}$ satisfying the conditions

$$S_n(1, f) = f(1), S_n(x_{n\nu}, f) = f(x_{n\nu}) \quad 1 \leq \nu \leq n, S_n(-1, f) = f(-1)$$

and

$$S'_n(x_{n\nu}, f) = \beta_{n\nu} \quad 1 \leq \nu \leq n,$$

where $\beta_{n\nu}$'s are arbitrary numbers; $x_{n0} = 1, x_{n, n+1} = -1$ and $\{x_{n\nu}\}$ are the zeros of orthogonal polynomial system $\{p_n(x)\}$ belonging to the weight function $(1-x^2)^p |x|^q, 0 < p \leq \frac{1}{2}, 0 < q < 1$ (which actually vanishes at a point in the interval $[-1, +1]$). Further it has been proved that quasi-conjugate pointsystem $\{X_{n\nu}\}$ (similar to Fejér conjugate pointsystem) belonging to the fundamental pointsystem $\{x_{n\nu}\}$ lie everywhere thickly in the interval $[-1, +1]$.

Let there be given a point system

$$(1.1) \quad 1 = x_{n0} > x_{n1} > x_{n2} > \cdots > x_{nn} > x_{n, n+1} = -1 \quad (n = 1, 2, \dots)$$

on the real axis and arbitrary real numbers

$$(1.2) \quad y_{n0}, y_{n1}, y_{n2}, \dots, y_{nn}, y_{n, n+1}, \\ y_{n1}^*, y_{n2}^*, \dots, y_{nn}^*.$$

Then setting

$$(1.3) \quad \omega_n(x) = c_n(x - x_{n1})(x - x_{n2}) \cdots (x - x_{nn}) \quad (c_n \neq 0)$$

and

$$(1.4) \quad l_{n\nu}(x) = \frac{\omega_n(x)}{\omega'_n(x_{n\nu})(x - x_{n\nu})} \quad (\nu = 1, 2, \dots, n),$$

the quasi-Hermite-Fejér interpolation polynomial $S_n(x)$ [6] is given by

$$(1.5) \quad S_n(x) = \sum_{\nu=0}^{n+1} y_{n\nu} r_{n\nu}(x) + \sum_{\nu=1}^n y_{n\nu}^* \rho_{n\nu}(x)$$

where $r_{n\nu}(x)$ and $\rho_{n\nu}(x)$ are called the fundamental polynomials of the 1st and the second kind of quasi-Hermite-Fejér interpolation.

For the fundamental polynomials of the 1st kind we have

$$\begin{aligned}
 (1.6) \quad r_{n0}(x) &= \frac{1+x}{2} \cdot \frac{\omega_n(x)^2}{\omega_n(1)^2}, \\
 r_{n,n+1}(x) &= \frac{1-x}{2} \cdot \frac{\omega_n(x)^2}{\omega_n(-1)^2}, \\
 r_{n\nu}(x) &= \frac{1-x^2}{1-x_{n\nu}^2} v_{n\nu}(x) l_{n\nu}(x)^2, \quad (\nu = 1, 2, \dots, n)
 \end{aligned}$$

where

$$(1.7) \quad v_{n\nu}(x) = 1 + c_{n\nu}(x - x_{n\nu}),$$

$$(1.8) \quad c_{n\nu} = \frac{2x_{n\nu}}{1-x_{n\nu}^2} - \frac{\omega_n''(x_{n\nu})}{\omega_n'(x_{n\nu})} \quad (\nu = 1, 2, \dots, n)$$

and those of second kind

$$(1.9) \quad \rho_{n\nu}(x) = \frac{1-x^2}{1-x_{n\nu}^2} (x - x_{n\nu}) l_{n\nu}(x)^2 \quad (\nu = 1, 2, \dots, n).$$

The polynomials $S_n(x)$ are the unique polynomials of degree $\leq 2n+1$ which satisfy the requirements:

$$\begin{aligned}
 (1.10) \quad S_n(x_{n\nu}) &= y_{n\nu} \quad \nu = 0, 1, 2, \dots, n+1, \\
 S_n'(x_{n\nu}) &= y_{n\nu}^* \quad \nu = 1, 2, \dots, n.
 \end{aligned}$$

From the unicity of the polynomials $S_n(x)$ it follows that for each polynomial $\Pi(x)$ of degree $\leq 2n$

$$(1.11) \quad \Pi(x) = \sum_{\nu=0}^{n+1} \Pi(x_{n\nu}) r_{n\nu}(x) + \sum_{\nu=1}^n \Pi'(x_{n\nu}) \rho_{n\nu}(x)$$

holds. For the special case $\Pi(x) \equiv 1$, we have

$$(1.12) \quad \sum_{\nu=0}^{n+1} r_{n\nu}(x) \equiv 1.$$

2. Let $f(x)$ be continuous in $-1 \leq x \leq 1$ and $f(x_{n\nu})$ its values at the points $x_{n\nu} (\nu = 0, 1, 2, \dots, n+1)$. Further let $y_{n\nu}^* (\nu = 1, 2, \dots, n)$ be arbitrary real numbers then the polynomials $S_n(x)$ in (1.5) written as

$$(2.1) \quad S_n(x, f) = \sum_{\nu=0}^{n+1} f(x_{n\nu}) r_{n\nu}(x) + \sum_{\nu=1}^n y_{n\nu}^* \rho_{n\nu}(x)$$

are called the generalised quasi-Hermite-Fejér interpolation polynomials. For $y_{n\nu}^* = 0$, they are called quasi-step parabolas. In this case for $\omega_n(x) = P_n(x)$, where $P_n(x)$ stands for the n th Legendre polynomial,

the interpolatory polynomials

$$(2.2) \quad R_n(x) = f(1) \frac{1+x}{2} P_n(x)^2 + f(-1) \frac{1-x}{2} P_n(x)^2 + \sum_{\nu=1}^n f(x_{n\nu}) \frac{1-x^2}{1-x_{n\nu}^2} \left(\frac{P_n(x)}{P_n'(x_{n\nu})(x-x_{n\nu})} \right)^2$$

have been obtained by E. Egerváry and P. Turán [2]. They have shown that if $f(x)$ is a function continuous in the closed interval $[-1, 1]$, then the polynomials in (2.2) converge uniformly to $f(x)$ in $[-1, 1]$. The convergence of the polynomials $S_n(x, f)$ in (2.1) constructed on the roots of $P_n(x)$ has been investigated by P. Szász [6]. He has shown that assuming $f(x)$ to be continuous and $|y_{n\nu}^*| < \Delta$, where Δ is a constant independent of n and ν the series $S_n(x, f)$ in (2.1) converges uniformly to $f(x)$ in $[-1, 1]$.

In this paper we answer the question of P. Turán for the quasi-Hermite-Fejér interpolation polynomials $S_n(x, f)$ which Balázs has answered [1] in the case of Hermite-Fejér interpolation polynomials.

Does there exist in $[-1, 1]$ an orthogonal polynomial system $\{g_n(x)\}$ whose weight function vanishes some where in this interval while the series $\{S_n(x, f)\}$ in (2.1) constructed on the roots of $\{g_n(x)\}$ converges uniformly to the continuous function $f(x)$ in the closed interval $[-1, 1]$ provided $\{y_{n\nu}^*\}$ are bounded?

The answer to this question is explained in our Theorem 1.

3. Similar to the normal and strongly normal point system due to L. Fejér [3, 4], the notion of quasi-normal and strongly quasi-normal point systems have been defined by Szász [6]. Thus an infinite sequence of point system,

$$(3.1) \quad x_{n1}, x_{n2}, \dots, x_{nn}, \quad (n = 1, 2, \dots)$$

lying in $-1 < x < 1$, is called strongly quasi-normal if by the notation of (1.3) and (1.7)

$$(3.2) \quad 1 + c_{n\nu}(x - x_{n\nu}) \geq \rho > 0, \quad -1 \leq x \leq 1 \\ (\nu = 1, 2, \dots, n; n = 1, 2, \dots)$$

where ρ is a positive number independent of x , ν and n .

If $X_{n\nu}$ indicates a zero of $v_{n\nu}(x)$ in (1.7), then

$$(3.3) \quad X_{n\nu} = x_{n\nu} + \frac{1}{c_{n\nu}}, \quad \nu = 1, 2, \dots, n.$$

These points will be called quasi-conjugate points similar to the conjugate points due to L. Fejér [4]. The quasi-conjugate points lie outside $[-1, 1]$ when the fundamental point system is quasi-strongly

normal. In this connection we shall answer another question of P. Turán for the case of quasi-Hermite-Fejér interpolation polynomials which Balázs [1] has answered for the Hermite-Fejér interpolation polynomials.

Is it possible to assume in the interval $[-1, 1]$ a fundamental point system whose quasi-conjugate points (3.3) lie thickly in $[-1, 1]$ and the interpolation series $\{S_n(x, f)\}$ belonging to this fundamental point-system converges uniformly to the continuous function $f(x)$ in $[-1, 1]$ provided the numbers $\{y_{n\nu}^*\}$ are bounded.

In Theorem II we answer this in affirmative.

4. K. V Laščenov [5] has defined orthogonal polynomials

$$p_n^{(p,q)}(x) = \alpha_n x^n + \alpha_{n-2} x^{n-2} + \dots, \quad \alpha_n \neq 0, \quad p > -1, \quad q > -1$$

over the interval $[-1, 1]$ with respect to the weight function $(1 - x^2)^p |x|^q$ which are constant multiples of

$$(4.1) \quad p_n^{(p,q)}(x)^1 = \begin{cases} P_m^{(p,q-1/2)}(2x^2 - 1), & n = 2m \\ x P_m^{(p,q+1/2)}(2x^2 - 1), & n = 2m + 1 \end{cases}$$

$P_n^{(\alpha,\beta)}(t)$ being the classical Jacobi polynomial of degree n with parameters α and β satisfying the differential equation

$$(4.2) \quad (1 - t^2)y'' + [\beta - \alpha - (\alpha + \beta + 2)t]y' + n(n + \alpha + \beta + 1)y = 0.$$

The position of the roots of (4.1) is given by

$$(4.3) \quad -1 < x_{nm+1} < x_{nm+2} < \dots < x_{nn} < 0 < x_{n1} < \dots < x_{nm} < 1$$

for $n = 2m$

and

$$(4.4) \quad -1 < x_{nm+2} < x_{nm+3} < \dots < x_{nn} < 0 = x_{nm+1} < x_{n1} < \dots < x_{nm} < 1$$

for $n = 2m + 1$.

Since the roots are symmetrical, we have

$$(4.5) \quad x_{n\nu} + x_{n,n+1-\nu} = 0, \quad \nu = 1, 2, \dots [n/2].$$

We shall prove the following:

THEOREM 1. *The quasi-Hermite-Fejér interpolation series $\{S_n(x, f)\}$, constructed on the point system*

$$(4.5) \quad 1 = x_{n0}, x_{n1}, \dots, x_{n,n-1}, x_{nn}, x_{nn+1} = -1 \quad n = 1, 2, \dots$$

¹ From now onward we shall write $p_n(x)$ to mean $p_n^{(p,q)}(x)$.

where $x_{n\nu} (\nu = 1, 2, \dots, n)$ are the zeros of the orthogonal polynomial system belonging to the weight function²

$$(1 - x^2)^p |x|^q \quad 0 < p \leq \frac{1}{2}, \quad 0 < q < 1,$$

converges uniformly to the continuous function $f(x)$ in $[-1, 1]$ when $|y_{n\nu}^*| \leq cn^\nu, 1 > \delta/2 > \eta \geq 0$ and $\delta = \min(2p, q)$.

THEOREM 2. *The quasi-conjugate points (3.3)*

$$(4.6) \quad x_{n\nu} = x_{n\nu} + \frac{1}{c_{n\nu}} \quad \nu = 1, 2, \dots, n; \quad n = 1, 2, \dots,$$

belonging to the fundamental point system (4.5) lie thickly in the interval $[-1, 1]$.

5. Preliminaries. We shall use some well-known facts about Jacobi polynomials. We have

$$(5.1) \quad P_m^{(\alpha, \beta)}(1) = \binom{m + \alpha}{m}$$

$$(5.2) \quad P_m^{(\alpha, \beta)}(-1) = (-1)^m P_m^{(\alpha, \beta)}(1) = (-1)^m \binom{m + \beta}{m}$$

$$(5.3) \quad P_m^{(\alpha, \beta)}(t) = (-1)^m P_m^{(\beta, \alpha)}(-t).$$

Further we have for $-1 < x < 1$

$$(5.4) \quad P_m^{(\alpha, \beta)}(x) = O(n^{-1/2}), \quad \alpha, \beta > -1$$

$$(5.5) \quad P_m^{(\alpha+1, \beta)}(x) = \frac{2}{(2m + \alpha + \beta + 2)} \frac{(m + \alpha + 1)P_m^{(\alpha, \beta)}(x) - (m + 1)P_{m-1}^{(\alpha, \beta)}(x)}{(1 - x)}$$

and

$$(5.6) \quad \frac{d}{dt} P_m^{(\alpha, \beta)}(t) = \frac{1}{2} (m + \alpha + \beta + 1) P_{m-1}^{(\alpha+1, \beta+1)}(t).$$

Further let $t_\nu = \cos \theta_\nu$ be the root of the polynomial

$$P_m^{(\alpha, \beta)}(t) = P_m^{(\alpha, \beta)}(\cos \theta)$$

then for $-1/2 \leq \alpha \leq 1/2, -1/2 \leq \beta \leq 1/2,$

$$(5.7) \quad \frac{2\nu - 1}{2m + 1} \pi \leq \theta_\nu \leq \frac{2\nu}{2m + 1} \pi \quad (\nu = 1, 2, \dots, m).$$

² $(1 - x^2)^p |x|^q$ for $0 < p \leq \frac{1}{2}, 0 < q < 1$, actually vanishes at $x = 0$.

For $0 < \theta_\nu \leq \pi/2$ we have

$$(5.8) \quad P_m^{(\alpha, \beta)}(\cos \theta_\nu) \geq c_1 \nu^{-\alpha-3/2} m^{\alpha+2}$$

where c_1 is positive numerical constant.

6. In this section we shall obtain certain estimations for the polynomial $p_n(x)$.

We shall first prove:

LEMMA 6.1. For $-1 \leq x \leq 1$ we have

$$(6.1) \quad (1 - x^2)p_n^2(x) = O(n^{-1}).$$

Proof of this lemma follows at once from (4.1) using (5.4).

LEMMA 6.2. For the roots $x_{n\nu}$ ($\nu = 1, 2, \dots, \left[\frac{n}{2}\right]$, $n = 1, 2, \dots$) of the polynomial $p_n(x)$, we have

$$(6.2) \quad x_{n\nu}^2(1 - x_{n\nu}^2) \geq \frac{\nu^2}{4n^2}.$$

Proof. Let $2x_{n\nu}^2 - 1 = \cos \theta_{n\nu}$, then $(1 - x_{n\nu}^2) = \sin^2 \theta_{n\nu}/2$, and $x_{n\nu}^2 = \cos^2 \theta_{n\nu}/2$. Hence

$$x_{n\nu}^2(1 - x_{n\nu}^2) = \frac{4}{4} \cos^2 \frac{\theta_{n\nu}}{2} \sin^2 \frac{\theta_{n\nu}}{2} = \frac{1}{4} \sin^2 \theta_{n\nu}.$$

But from (5.7) we have

$$\theta_{n\nu} \geq \frac{\nu + \frac{1}{2}}{n + \frac{1}{2}} \pi > \frac{\nu\pi}{2n}$$

which gives

$$|\sin \theta_{n\nu}| > \left| \sin \frac{\nu\pi}{2n} \right| > \frac{\nu}{n}.$$

Therefore

$$x_{n\nu}^2(1 - x_{n\nu}^2) = \frac{1}{4} \sin^2 \theta_{n\nu} > \frac{\nu^2}{4n^2}.$$

7. We shall need the following lemmas for the estimation of the fundamental polynomials of the first kind.

LEMMA 7.1. *Let $x_{n\nu}$ be a root of $p_n(x)$, then*

$$(i) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}} \left[(p+1) \frac{x_{n\nu}^2}{(1-x_{n\nu}^2)} - \frac{q}{2} \right]$$

except when $n = 2m + 1$, and $\nu = m + 1$. In this case we have

$$(ii) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 0 .$$

Proof. It follows from (4.1) by differentiating with respect to x , for $n = 2m$

$$(7.1) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 4x_{n\nu} \left\{ \frac{\frac{d^2}{dt^2} P_m^{(p, (q-1/2))}(t)}{\frac{d}{dt} P_m^{(p, (q-1/2))}(t)} \right\}_{t=2x_{n\nu}^2-1} + \frac{1}{x_{n\nu}} .$$

By the substitution $t = 2x^2 - 1$, $\alpha = p$, $\beta = q - 1/2$, and $n = m$, the differential equation (4.2) gives

$$(7.2) \quad \left\{ \frac{\frac{d^2}{dt^2} P_m^{(p, (q-1/2))}(t)}{\frac{d}{dt} P_m^{(p, (q-1/2))}(t)} \right\}_{t=2x_{n\nu}^2-1} = \frac{1}{4x_{n\nu}^2(1-x_{n\nu}^2)} [-2(p+1) + (2p+q+3)(1-x_{n\nu}^2)] .$$

Substituting (7.2) in (7.1) we get

$$(7.3) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}} \left[(p+1) \frac{x_{n\nu}^2}{1-x_{n\nu}^2} - \frac{q}{2} \right] .$$

If however, $n = 2m + 1$ and $\nu \neq m + 1$, then it follows on account of (4.1) and (4.4) that

$$(7.4) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 4x_{n\nu} \left\{ \frac{\frac{d^2}{dt^2} P_m^{(p, (q+1/2))}(t)}{\frac{d}{dt} P_m^{(p, (q+1/2))}(t)} \right\}_{t=2x_{n\nu}^2-1} + \frac{3}{x_{n\nu}} .$$

But from (4.2) by putting $t = 2x^2 - 1$, $\alpha = p$, $\beta = q + 1/2$ and $n = m$ we get

$$(7.5) \quad \left\{ \frac{\frac{d^2}{dt^2} P_m^{(p, (q+1/2))}(t)}{\frac{d}{dt} P_m^{(p, (q+1/2))}(t)} \right\}_{t=2x_{n\nu}^2-1} = - \frac{1}{4x_{n\nu}^2(1-x_{n\nu}^2)} [-2(p+1) + (2p+q+5)(1-x_{n\nu}^2)]$$

substituting (7.5) in (7.4) we get

$$(7.6) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = \frac{2}{x_{n\nu}} \left[(p+1) \frac{x_{n\nu}^2}{1-x_{n\nu}^2} - \frac{q}{2} \right].$$

In case $n = 2m + 1$ and $\nu = m + 1$, $x_{n\nu} = 0$ on account of (4.4). But the polynomial $p_n(x)$ is an odd function of x , therefore $p_n''(x_{n\nu}) = 0$ and in this case

$$(7.7) \quad \frac{p_n''(x_{n\nu})}{p_n'(x_{n\nu})} = 0.$$

8. Estimation of the fundamental polynomials of the first kind.

LEMMA 8.1. For $-1 \leq x \leq 1$, we have

$$(8.1) \quad \sum_{\nu=0}^{n+1} |r_{n\nu}(x)| = O(1).$$

Proof. From (1.7), (1.8) and Lemma 7.1 we get for $1 \leq \nu \leq n$

$$(8.2) \quad v_{n\nu}(x) = 1 - \frac{2}{x_{n\nu}} \left\{ \frac{px_{n\nu}^2}{(1-x_{n\nu}^2)} - \frac{q}{2} \right\} (x - x_{n\nu}).$$

From the representation (4.4) of $x_{n\nu}$'s it is clear that for $n = 2m + 1$, and $\nu = m + 1$, $x_{nm+1} = 0$. Whence from Lemma 7.1 (ii) and (1.7) it follows that

$$(8.3) \quad v_{nm+1}(x) \equiv 1.$$

For $x = 0$ it follows from (8.2) on account of $0 < q < 1$ and $0 < p \leq \frac{1}{2}$ that

$$(8.4) \quad v_{n\nu}(0) = 1 + \frac{2px_{n\nu}^2}{(1-x_{n\nu}^2)} - q \geq 1 - q > 0.$$

This inequality is also applicable on account of (8.3) when $n = 2m + 1$, and $\nu = m + 1$. For $-1 < x \leq 0$ and $x_{n\nu} \leq 0$ we have on

account of $v_{n\nu}(x_{n\nu}) = 1$ and (8.4)

$$(8.5) \quad v_{n\nu}(x) \geq 1 - q > 0 \quad (0 < q < 1).$$

Since $v_{n\nu}(x)$ is a linear function in the interval $0 \leq x < 1$ it follows from $v_{n\nu}(x_{n\nu}) = 1$ and $x_{n\nu} \geq 0$ that

$$(8.6) \quad v_{n\nu}(x) \geq 1 - q > 0 \quad \text{since } 0 < q < 1.$$

We shall now prove the inequality (8.1) in the interval $-1 < x \leq 0$. In this interval $r_{n\nu}(x) \geq 0$ for $x_{n\nu} \leq 0$. Also $r_{n0}(x)$ and $r_{n,n+1}(x)$ are positive. Hence from (1.12)

$$(8.7) \quad \begin{aligned} \sum_{\nu=0}^{n+1} |r_{n\nu}(x)| &= \sum_{x_{n\nu} \leq 0} |r_{n\nu}(x)| + \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| \\ &= \sum_{x_{n\nu} \leq 0} r_{n\nu}(x) + \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| \\ &= 1 - \sum_{x_{n\nu} > 0} r_{n\nu}(x) + \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| \\ &\leq 1 + 2 \sum_{x_{n\nu} > 0} |r_{n\nu}(x)|. \end{aligned}$$

On account of (8.2), (1.6) and (1.4) we obtain

$$(8.8) \quad \begin{aligned} \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| &= \sum_{x_{n\nu} > 0} \frac{1 - x^2}{1 - x_{n\nu}^2} \left| 1 - \frac{2}{x_{n\nu}} \left\{ \frac{px_{n\nu}^2}{1 - x_{n\nu}^2} - \frac{q}{2} \right\} (x - x_{n\nu}) \right| \\ &\quad \times \frac{p_n^2(x)}{p_n'^2(x_{n\nu})(x - x_{n\nu})^2} \\ &\leq \sum_{x_{n\nu} > 0} \frac{1 - x^2}{1 - x_{n\nu}^2} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})(x - x_{n\nu})^2} \\ &\quad + 2p \sum_{x_{n\nu} > 0} \frac{(1 - x^2)p_n^2(x)}{|x_{n\nu}|(1 - x_{n\nu}^2)^2 p_n'^2(x_{n\nu})(x - x_{n\nu})} \\ &\quad + (2p + q) \sum_{x_{n\nu} > 0} \frac{1 - x^2}{(1 - x_{n\nu}^2)} \frac{1}{|x_{n\nu}|} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})|x - x_{n\nu}|}. \end{aligned}$$

Since $-1 < x \leq 0$ and $0 < x_{n\nu} < 1$, therefore $|x - x_{n\nu}| > |x_{n\nu}|$. Hence from (8.8),

$$(8.9) \quad \begin{aligned} \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| &\leq (1 + 2p + q) \sum_{x_{n\nu} > 0} \frac{1 - x^2}{1 - x_{n\nu}^2} \cdot \frac{1}{x_{n\nu}^2} \cdot \frac{p_n^2(x)}{p_n'^2(x_{n\nu})} \\ &\quad + 2|p| \sum_{x_{n\nu} > 0} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^2 x_{n\nu}^2} \frac{p_n^2(x)}{p_n'^2(x_{n\nu})}. \end{aligned}$$

Owing to (4.1) we have

$$(8.10) \quad p_n'(x_{n\nu}) = \begin{cases} 4x_{n\nu} P_m^{(p, 2(q-1/2))}(2x_{n\nu} - 1) & \text{for } n = 2m \\ 4x_{n\nu}^2 P_m^{(p, 2(q+1/2))}(2x_{n\nu} - 1) & \text{for } n = 2m + 1. \end{cases}$$

Thus for $n = 2m$, using (8.9) and (8.10); for n odd using (8.8), (8.10) and $x^2 < (x - x_{n\nu})^2$, we have

$$(8.11) \quad \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| \begin{cases} \frac{1}{16} (1 + 4p + q) \sum_{x_{n\nu} > 0} \frac{(1 - x^2) [P_m^{(p, (q-1/2))} (2x^2 - 1)]^2}{x_{n\nu}^4 (1 - x_{n\nu}^2)^2 \left[\frac{d}{dt} P_m^{(p, (q-1/2))} (t) \right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m \\ \leq \frac{1}{16} (1 + 4p + q) \sum_{x_{n\nu} > 0} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^2 x_{n\nu}^5} \cdot \frac{[P_m^{(p, (q+1/2))} (2x^2 - 1)]^2}{\left[\frac{d}{dt} P_m^{(p, (q+1/2))} (t) \right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m + 1. \end{cases}$$

Now Lemmas 6.1 and 6.2, with (5.8) give

$$(8.12) \quad \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| = \begin{cases} \left[\sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{q+2}}{n^{q+3}} \right] & \text{for } n = 2m \\ \left[\sum_{\nu=1}^m O(n^{-1}) \frac{n^5}{\nu^5} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^5}{\nu^5} \cdot \frac{\nu^{q+4}}{n^{q+5}} \right] & \text{for } n = 2m + 1 \end{cases}$$

and since $0 < p \leq \frac{1}{2}$, $0 < q < 1$, (8.12) gives

$$(8.13) \quad \sum_{x_{n\nu} > 0} |r_{n\nu}(x)| = O(1).$$

By a similar reasoning we can obtain for the interval $0 \leq x < 1$ and $x_{n\nu} \geq 0$, that

$$(8.14) \quad \sum_{x_{n\nu} < 0} |r_{n\nu}(x)| = O(1).$$

Hence from (8.13) and (8.14) we get the lemma for $1 \leq \nu \leq n$, and $-1 < x < 1$. For $\nu = 0$ and $n + 1$ it is easy to see from (1.6) with $\omega_n(x) = p_n(x)$ and (5.4) that

$$r_{n0}(x) = O(1) \quad \text{and} \quad r_{n, n+1}(x) = O(1).$$

At $x = \pm 1$, the lemma is trivial.

9. Estimation of the fundamental polynomials of the second kind. In this section we shall estimate the quantity

$$\sum_{\nu=1}^n |\rho_{n\nu}(x)|.$$

We shall prove the following:

LEMMA 9.1. For $-1 \leq x \leq 1$ and $n = 1, 2, \dots$ we have

$$(9.1) \quad \sum_{\nu=1}^n |\rho_{n\nu}(x)| = O(n^{-\delta/2}), \quad \text{where } \delta = \min(2p, q) > 0.$$

Proof. From (1.9) and (1.4) with $\omega_n(x) = p_n(x)$

$$(9.2) \quad \sum_{\nu=1}^n \rho_{n\nu}(x) = \sum_{\nu=1}^n (x - x_{n\nu}) \frac{1 - x^2}{1 - x_{n\nu}^2} \frac{p_n^2(x)}{p_n^2(x_{n\nu})(x - x_{n\nu})^2}.$$

Now setting

$$(9.3) \quad \sum_{\nu=1}^n |\rho_{n\nu}(x)| = \sum_{x_{n\nu} \leq 0} |\rho_{n\nu}(x)| + \sum_{x_{n\nu} > 0} |\rho_{n\nu}(x)|$$

and considering the interval $-1 < x \leq 0$, we have for $x_{n\nu}$'s > 0 ,

$$|x - x_{n\nu}| > |x_{n\nu}|.$$

Thus from (9.2) and (8.10)

$$\sum_{x_{n\nu} > 0} |\rho_{n\nu}(x)| \leq \begin{cases} \frac{1}{16} \sum_{x_{n\nu} > 0} \frac{1}{|x_{n\nu}|^3} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^{3/2}} \frac{[P_m^{(p, (q-1/2))}(2x^2 - 1)]^2}{\left[\frac{d}{dt} P_m^{(p, (q-1/2))}(t)\right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m \\ \frac{1}{16} \sum_{x_{n\nu} > 0} \frac{1}{x_{n\nu}^4} \frac{(1 - x^2)}{(1 - x_{n\nu}^2)^2} \frac{[P_m^{(p, (q+1/2))}(2x^2 - 1)]^2}{\left[\frac{d}{dt} P_m^{(p, (q+1/2))}(t)\right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m + 1 \end{cases}$$

which on account of (5.8) and the Lemmas 6.1 and 6.2, gives,

$$(9.4) \quad \sum_{x_{n\nu} > 0} |\rho_{n\nu}(x)| \leq \begin{cases} \sum_{\nu=1}^m O(n^{-1}) \frac{n^3}{\nu^3} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^3}{\nu^3} \cdot \frac{\nu^{q+2}}{n^{q+3}} & \text{for } n = 2m \\ \sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \cdot \frac{\nu^{q+4}}{n^{q+5}} & \text{for } n = 2m + 1 \end{cases}$$

Since $0 < p \leq \frac{1}{2}$ and $0 < q < 1$, it follows from (9.4) that

$$(9.5) \quad \sum_{x_{n\nu} > 0} |\rho_{n\nu}(x)| \leq O(n^{-\delta}) \quad -1 < x \leq 0$$

where $\delta = \min(2p, q) > 0$.

Again let $x_{n\nu} \leq 0$, $-1 < x \leq 0$ and

$$(9.6) \quad \sum_{x_{n\nu} \leq 0} |\rho_{n\nu}(x)| = \sum_{\substack{x_{n\nu} \leq 0 \\ |x - x_{n\nu}| \leq n^{-\delta/2}} |\rho_{n\nu}(x)| + \sum_{\substack{x_{n\nu} \leq 0 \\ |x - x_{n\nu}| > n^{-\delta/2}} |\rho_{n\nu}(x)| = \Sigma' + \Sigma''.$$

On account of (9.2) the following holds in the interval $-1 < x \leq 0$.

$$\begin{aligned}
 (9.7) \quad \Sigma' |\rho_{n\nu}(x)| &\leq n^{-\delta/2} \Sigma' \frac{(1-x^2)p_n^2(x)}{(1-x_{n\nu}^2)p_n'^2(x_{n\nu})(x-x_{n\nu})^2} \\
 &\leq \frac{n^{-\delta/2}}{|1-q|} \Sigma' \frac{(1-x^2)}{(1-x_{n\nu}^2)} v_{n\nu}(x) \frac{p_n^2(x)}{p_n'^2(x_{n\nu})(x-x_{n\nu})^2} \\
 &\leq \frac{n^{-\delta/2}}{|1-q|} \Sigma' r_{n\nu}(x) \leq \frac{n^{-\delta/2}}{|1-q|}.
 \end{aligned}$$

From (9.6) we have

$$\Sigma'' |\rho_{n\nu}(x)| \leq n^{\delta/2} \Sigma'' \frac{(1-x^2)}{(1-x_{n\nu}^2)} \frac{p_n^2(x)}{p_n'^2(x_{n\nu})}.$$

But owing to (8.10), we have

$$\Sigma'' |\rho_{n\nu}(x)| \leq \begin{cases} \frac{n^{\delta/2}}{16} \Sigma'' \frac{1}{x_{n\nu}^2} \frac{(1-x^2)}{(1-x_{n\nu}^2)} \frac{p_n^2(x)}{\left[\frac{d}{dt} P_{m(t)}^{(p, (q-1/2))} \right]_{t=2x_{n\nu}^2-1}} & \text{for } n = 2m \\ \frac{n^{\delta/2}}{16} \Sigma'' \frac{(1-x^2)}{x_{n\nu}^4(1-x_{n\nu}^2)^2} \frac{p_n^2(x)}{\left[\frac{d}{dt} P_m^{(p, (q+1/2))}(t) \right]_{t=2x_{n\nu}^2-1}^2} & \text{for } n = 2m + 1 \end{cases}$$

which by (5.8), and Lemmas 6.1 and 6.2 gives

$$(9.8) \quad \Sigma'' |\rho_{n\nu}(x)| \leq \begin{cases} n^{\delta/2} \left[\sum_{\nu=1}^m O(n^{-1}) \frac{n^2}{\nu^2} \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^2}{\nu^2} \frac{\nu^{q+2}}{n^{q+3}} \right] & \text{for } n = 2m \\ n^{\delta/2} \left[\frac{(1-x^2)p_n^2(x)}{p_n'^2(0)} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \frac{\nu^{2p+3}}{n^{2p+4}} + \sum_{\nu=1}^m O(n^{-1}) \frac{n^4}{\nu^4} \frac{\nu^{q+4}}{n^{q+5}} \right] & \text{for } n = 2m + 1. \end{cases}$$

For $n = 2m + 1$ we obtain by using (6.2)

$$\frac{(1-x^2)p_n^2(x)}{p_n'^2(0)} = \frac{(1-x^2)x^2 P_{m(2x^2-1)}^{2(p, (q+1/2))}}{[P_{m(-1)}^{(p, (q+1/2))}]^2} = \frac{O(n^{-1})}{\binom{m + \frac{q+1}{2}}{m}^2}.$$

From this as well as from (9.8) we see that in the interval $-1 < x \leq 0$

$$(9.9) \quad \sum_{\nu=1}^n |\rho_{n\nu}(x)| \leq O(n^{-\delta/2}).$$

Similarly it follows in the interval $0 \leq x < 1$ that

$$\sum_{\nu=1}^n |\rho_{n\nu}(x)| \leq O(n^{-\delta/2}).$$

At $x = \pm 1$, the lemma obviously holds.

10. The proof of the Theorem 1. We now apply the usual argument. We have $S_n(x, f)$ our interpolating polynomial and $\Pi(x)$ an arbitrary polynomial of degree $2n$ at most. Then there holds

$$(10.1) \quad S_n(x, f) - f(x) = S_n(x, f - \Pi) + (\Pi(x) - f(x)).$$

From (2.1) and (1.11) we get

$$(10.2) \quad S_n(x, f) - f(x) = \sum_{\nu=0}^{n+1} \{f(x_{n\nu}) - \Pi(x_{n\nu})\} r_{n\nu}(x) + \sum_{\nu=0}^n (y_{n\nu}^* - \Pi'(x_{n\nu})) \rho_{n\nu}(x).$$

Now by Weistrass approximation theorem for $-1 \leq x \leq 1$

$$(10.3) \quad \Pi(x) - f(x) = o(1).$$

Now

$$(10.4) \quad \left| \sum_{\nu=0}^{n+1} \{f(x_{n\nu}) - \Pi(x_{n\nu})\} r_{n\nu}(x) \right| \leq \max_{-1 \leq x \leq 1} |f(x) - \Pi(x)| \sum_{\nu=0}^{n+1} |r_{n\nu}(x)| = o(1)$$

owing to (10.3) and Lemma 8.1

If $M = \max_x |\Pi'(x)|$ then in the interval $-1 \leq x \leq 1$

$$(10.5) \quad \left| \sum_{\nu=1}^n (y_{n\nu}^* - \Pi'(x_{n\nu})) \rho_{n\nu}(x) \right| \leq (cn^\eta + M) \sum_{\nu=1}^n |\rho_{n\nu}(x)| = o(1)$$

in consequence of Lemma 9.1 and $|\beta_{n\nu}| \leq cn^\eta$, where $0 \leq \eta < \frac{\delta}{2} < 1$ and $\delta = (2p, q) > 0$.

Thus (10.2), (10.3), (10.4) and (10.5) complete the proof of our Theorem 1.

11. Proof of Theorem 2. The conjugate points belonging to our point-system owing to (4.6), (1.8) and Lemma 7.1 (i) are given by

$$(11.1) \quad X_{n\nu} = x_{n\nu} + \frac{x_{n\nu}}{2 \left\{ \frac{px_{n\nu}^2}{1 - x_{n\nu}^2} - \frac{q}{2} \right\}} = x_{n\nu} \left[\frac{2p + (1 - 2p - q)(1 - x_{n\nu}^2)}{2p - (2p + q)(1 - x_{n\nu}^2)} \right] \quad x_{n\nu} \neq 0.$$

If however $x_{n\nu} = 0$ i.e., in the case when $n = 2m + 1$ and $\nu = m + 1$, then it follows from (4.6), (1.8) and Lemma 7.1(ii) that

$$X_{2m+1, m+1} = \infty .$$

Now we shall make use of the following statements in the proof of Theorem 2.

Let (α, β) be a fixed part of the interval $[-1, 1]$ but as small as we please. Consider the fundamental point system (4.3) or (4.4). We prove that for any value of n sufficiently large at least one member of the series of triangular matrix of the fundamental point-system lies within the interval (α, β) . Let

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x < \alpha \\ (x - \alpha)(\beta - x) & \text{for } \alpha \leq x \leq \beta \\ 0 & \text{for } \beta < x \leq 1 . \end{cases}$$

Then $f(x)$ is apparently continuous in the interval $-1 \leq x \leq 1$. Let us assume that it is not so then there is a series $n_1 < n_2 < n_3 \cdots < n_i \cdots$ such that no member of the point group belonging to these indices $x_{n_i, 1}, x_{n_i, 2}, \cdots, x_{n_i, n_i}$ ($i = 1, 2, \cdots$) lie in the interval (α, β) . Therefore in the interval $-1 \leq x \leq 1$ $\lim_{i \rightarrow \infty} S_{n_i}(f, x) = 0$ holds. On the other-hand according to Theorem 1 in place of $x = \alpha + \beta/2$

$$\lim_{i \rightarrow \infty} S_{n_i}(f, x) = f\left(\frac{\alpha + \beta}{2}\right) = \left(\frac{\alpha - \beta}{2}\right)^2 \neq 0$$

contradicts the foregoing inference, i.e., point-system (4.3) or (4.4) lie thickly in the interval $-1 \leq x \leq 1$. It can also be proved that the conjugate point-system belonging to (4.3) or (4.4) thickly cover the interval $-1 \leq x \leq 1$.

The conjugate points belonging to points $x_{n\nu} \neq 0$ can according to (11.1) be obtained from the function

$$g(x) = x \left[\frac{1 - q - (1 - 2p - q)x^2}{(2p + q)x^2 - q} \right]$$

in the places $x_{n\nu}$. In the interval $-1 \leq x \leq 1$, $g'(x) < 0$. Therefore the function $g(x)$ in the interval $(-\sqrt{q/2p + q}, \sqrt{q/2p + q})$ which on account of $0 < p \leq \frac{1}{2}$ and $0 < q < 1$ forms a part interval of $[-1, 1]$ diminishes continuously, is continuous and its value includes all values from $+\infty$ to $-\infty$. There must also be two points a_1 and b_1 different from each other within the interval $[-\sqrt{q/2p + q}, \sqrt{q/2p + q}]$ so that $g(a_1) = -1$ and $g(b_1) = 1$. Since $g'(x) < 0$ it follows that $-1 \leq g(x) \leq 1$ holds in the interval $b_1 \leq x \leq a_1$. Let a_2 and b_2 be again two different real values for which $-1 < a_2 < b_2 < 1$ holds.

Then there must obviously lie in the interval (a_1, b_1) two different points a_3 and b_3 such that $g(a_3) = a_2$ and $g(b_3) = b_2$. Since we have already proved that at least one point of each series of the point-system (4.3) or (4.4) must belong to the index n within the interval (a_3, b_3) . Therefore it follows that the conjugate points belonging to the fundamental points lying within the interval (α, β) must owing to monotony of $g(x)$ from this index onwards lie within the interval (a_2, b_2) , a_2 and b_2 can lie as near to each other as we please. Thus Theorem 2 is proved.

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