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RECIPROCITY AND JACOBI SUMS

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JOSEPH B. MUSKAT

Recently N. C. Ankeny derived a law of rth power reciprocity, where r is an odd prime:

q is an rth power residue, modulo $p \equiv 1 \pmod{r}$, if and only if the rth power of the Gaussian sum (or Lagrange resolvent) $\tau(\chi)$, which depends upon p and r, is an rth power in $GF(q^f)$, where q belongs to the exponent $f \pmod{r}$.

 $\tau(\chi)^r$ can be written as the product of algebraic integers known as Jacobi sums. Conditions in which the reciprocity criterion can be expressed in terms of a single Jacobi sum are presented in this paper.

That the law of prime power reciprocity is a generalization of the law of quadratic reciprocity is suggested by the following formulation of the latter:

If p and q are distinct odd primes, then q is a quadratic residue (mod p) if and only if $(-1)^{(p-1)/2}p = \tau(\psi)^2$ is a quadratic residue (mod q). Here ψ denotes the nonprincipal quadratic character modulo p (the Legendre symbol) and

$$au(\psi) = \sum\limits_{n=1}^{p-1} \psi(n) e^{2\pi i n/p}$$

is a Gaussian sum.

A complete statement of Ankeny's result is the following:

Let r be an odd prime. $Q(\zeta_r)$ will denote the cyclotomic field obtained by adjoining $\zeta_r = e^{2\pi i/r}$ to the field of rationals Q.

Let p be a prime $\equiv 1 \pmod{r}$. Let χ denote a fixed primitive rth power multiplicative character (mod p). Define the Gaussian sum

$$au(\chi^k) = \sum\limits_{n=1}^{p-1} \chi^k(n) e^{2\pi i n/p}, \qquad r
mid k \;.$$

Let q be a prime distinct from r, belonging to the exponent $f \pmod{r}$. Then

$$au(\chi)^{q^f-1} = [\tau(\chi)^r]^{(q^f-1)/r} \equiv \chi(q)^{-f} \pmod{q}$$
.

Consequently, if q is any one of the prime ideal divisors of the ideal (q) in $Q(\zeta_r)$, q is an rth power (mod p) if and only if $\tau(\chi)^r$ is an rth power in $Q(\zeta_r)/q$, a field of q^r elements; i.e.,

(1)
$$\chi(q) = 1$$
 if and only if $\tau(\chi)^r \equiv \beta^r \pmod{\mathfrak{q}}$

for some $\beta \in Q(\zeta_r)$ [1, Th. 2].

The following properties of the Gaussian sums are well known: Assume $k \not\equiv 0 \pmod{r}$.

(2)
$$au(\chi^k)\tau(\chi^{-k}) = p$$

 $au(\chi^k) \notin Q(\zeta_r), \text{ but } \tau(\chi^k)^t/\tau(\chi^{kt}) \in Q(\zeta_r).$

In particular,

 $au(\chi^k)^r \in Q(\zeta_r)$.

During the nineteenth century several people worked on special cases of the problem solved by Ankeny. C.G.J. Jacobi treated r = 3 in [3]. Using Cauchy's result that

$$\tau(\chi)^q/\tau(\chi^q) \equiv \chi(q)^{-q} (\text{mod } q), \qquad [6, \ p. 108]$$

T. Pepin showed that if $q \equiv \pm 1 \pmod{r}$, then $\chi(q) = 1$ if and only if $\tau(\chi)^r/\tau(\chi^2)^r$ is an *r*th power residue (mod q), ([6, pp. 117, 120]).

Define the Jacobi sums

$$\pi(\chi^a,\,\chi^b) = \sum\limits_{n=2}^{p-1} \chi^a(n) \chi^b (1-n) = \sum\limits_{j=0}^{r-1} c_j \zeta^j_r$$
 .

If r does not divide a, b, or a + b,

$$\pi(\chi^a, \chi^b) = \tau(\chi^a) \tau(\chi^b) / \tau(\chi^{a+b})$$
,

so by (2)

(3)
$$\pi(\chi^a, \chi^b)\pi(\chi^{-a}, \chi^{-b}) = p$$
.

(For information on Jacobi sums see [2, Ch. 20])

 $\tau(\chi)^r$ can be expressed as a product of Jacobi sums, as follows:

$$\tau(\chi)^r = \tau(\chi)\tau(\chi^{r-1})\prod_{j=1}^{r-2}\tau(\chi)\tau(\chi^j)/\tau(\chi^{j+1}) = p\prod_{j=1}^{r-2}\pi(\chi,\chi^j), \text{ by } (2) .$$

For r = 3, $\tau(\chi)^r = p\pi(\chi, \chi)$, so that knowing $\pi(\chi, \chi)$ gives complete information about reciprocity. For r > 3, however, it is often necessary to consider products of Jacobi sums. Some cases where $\pi(\chi, \chi)$ itself gives complete information about reciprocity are described in the following two theorems:

Notation. For brevity, let $\pi[t] = \pi(\chi^t, \chi^t)$. Let $\pi[1] = \sum_{j=0}^{r-1} c_j \zeta_r^j$. Then

$$\pi[t] = \sum\limits_{j=0}^{r-1} c_j \zeta_r^{jt}$$
 .

Let 2 belong to the exponent $s \pmod{r}$.

LEMMA.
$$\pi[t]^{q^h} \equiv \pi[tq^h] \pmod{q}$$
.

Proof.

$$\pi[t]^{q^h} = \left[\sum_{j=0}^{r-1} c_j \zeta_r^{jt}\right]^{q^h} \equiv \sum_{j=0}^{r-1} c_j^{q^h} \zeta_r^{jtq^h} \equiv \sum_{j=0}^{r-1} c_j \zeta_r^{jtq^h} \equiv \pi[tq^h] (\text{mod } q) \ .$$

THEOREM 1. Assume $2^{r-1} \not\equiv 1 \pmod{r^2}$. If there exists an integer u such that $q^u \equiv 2 \pmod{r}$, then $\tau(\chi)^r$ is an rth power in $Q(\zeta_r)/q$ if and only if $\pi(\chi, \chi)$ is.

Proof. By an identity attributed to Cauchy, [6, p. 112]

$$\begin{aligned} \tau(\chi)^{2^{s-1}} &= \pi[1]^{2^{s-1}}\pi[2]^{2^{s-2}}\pi[4]^{2^{s-3}}\cdots\pi[2^{s-2}]^{2}\pi[2^{s-1}] \\ &= \prod_{j=0}^{s-1}\pi[2^{j}]^{2^{s-j-1}} = \prod_{j=0}^{s-1}\pi[q^{u_j}]^{2^{s-j-1}} \\ &= \beta^r \prod_{j=0}^{s-1}\pi[q^{u_j}]^{q^{u_j(s-j-1)}}, \quad \text{for some } \beta \in Q(\zeta_r) \;. \end{aligned}$$

To the *j*th factor of the product in (4) apply the lemma with t = 1 and h = uj:

$$\begin{split} \tau(\chi)^{2^{s}-1} &\equiv \beta^{r} \prod_{j=0}^{s-1} \pi[q^{0}]^{q^{u(s-1)}} \equiv \beta^{r} \pi[1]^{sq^{u(s-1)}} \\ &\equiv \gamma^{r} \pi[1] 2^{s-1} s \pmod{q}, \quad \text{for some } \gamma \in Q(\zeta_{r}) \text{.} \end{split}$$

Since $r^{2} \not\downarrow 2^{r-1} - 1$, $r \not\not\downarrow (2^{s} - 1)/r$. Also, $r \not\not\downarrow 2^{s-1}s$. It follows that $\tau(\chi)^{r}$ is an *r*th power in $Q(\zeta_{r})/q$ if and only if $\pi(\chi, \chi)$ is.

EXAMPLE.
$$r = 7, q = 3.$$
 $s = 3, u = 2.$
 $\tau(\chi)^7 = \pi[1]^4 \pi[2]^2 \pi[4] = \beta^7 \pi[1]^{s^4} \pi[3^2]^{s^2} \pi[3^4]^{s^0}$
 $\equiv \beta^7[\pi[1]^{s^4}]^s \equiv \beta^7 \pi[1]^{s^4 \cdot s} (\text{mod } 3) .$

(A different treatment of the example was given in [5, p. 351].)

THEOREM 2. Assume $2^{r-1} \not\equiv 1 \pmod{r^2}$, r > 3, and $s \equiv 2 \pmod{4}$. If there exists an integer v such that $q^* \equiv 4 \pmod{r}$, then $\tau(\chi)^r$ is an rth power in $Q(\zeta_r)/q$ if and only if $\pi(\chi, \chi)$ is.

Proof.

$$\tau(\chi)^{2^{s}-1} = \prod_{j=0}^{s/2-1} \pi[2^{2j}]^{2^{s-1-2j}} \pi[2^{2j+1}]^{2^{s-2-2j}}$$

,

$$= \prod_{j=0}^{s/2-1} \pi [q^{v_j}]^{2^{s-1-2j}} \pi [2q^{v_j}]^{2^{s-2-2j}}$$

(5)
$$= \beta^r \prod_{j=0}^{s/2-1} \pi [q^{v_j}]^{2q^{v(s/2-1-j)}} \pi [2q^{v_j}]^{q^{v(s/2-1-j)}}$$

for some $\beta \in Q(\zeta_r)$,

$$(6) \qquad \equiv \beta^{r} [\pi[q^{0}]^{2q^{v(s/2-1)}} \pi[2q^{0}]^{q^{v(s/2-1)}}]^{s/2} (\mathrm{mod} \ q),$$

by applying the Lemma with h = vj and t = 1, then 2, to the *j*th factor of (5). Now apply the Lemma to the second factor of (6) with t = 2, h = v(s - 2)/4:

$$egin{aligned} & au(\chi)^{2^{s}-1} \equiv eta^{r}[\pi[1]^{2q^{v(s/2-1)}}\pi[2q^{v(s-2)/4}]^{q^{v(s-2)/4}}]^{s/2} \ & \equiv eta^{r}[\pi[1]^{2q^{v(s/2-1)}}\pi[2\cdot4^{(s-2)/4}]^{q^{v(s-2)/4}}]^{s/2} \ & \equiv \gamma^{r}[\pi[1]^{2^{s-1}}\pi[2^{s/2}]^{2^{s/2-1}}]^{s/2} \ , \end{aligned}$$

for some $\gamma \in Q(\zeta_r)$,

$$\equiv \gamma^r [\pi[1]^{2^{s-1}} \pi[-1]^{2^{s/2-1}}]^{s/2} (\mathrm{mod} \ q)$$
 .

By (3)

$$au(\chi)^{2^{s-1}} \equiv \gamma^{r} [p^{2^{s/2-1}} \pi [1]^{2^{s-1}} - 2^{2^{s/2-1}}]^{s/2} (\mathrm{mod} \ q)$$
.

Since r > 3, $q \not\equiv 1 \pmod{r}$, so p is an rth power in $Q(\zeta_r)/\mathfrak{q}$.

$$2^{s-1} - 2^{s/2-1} \equiv 1 \pmod{r}$$
, so
 $au(\chi)^{2^{s-1}} \equiv \delta^r \pi [1]^{s/2} \pmod{\mathfrak{q}}$,

for some $\delta \in Q(\zeta_r)$. $r \nmid (2^s - 1)/r$, $r \nmid s/2$, and the theorem follows.

In Theorem 3 of [5] the above results were proved for the following values of q, under the restriction $2^{r-1} \not\equiv 1 \pmod{r^2}$:

(a) $q \equiv 2 \pmod{r}$.

(b) $r > 3, q \equiv -2 \pmod{r}$.

Part (a) is included in Theorem 1. Part (b) has three cases:

If s is odd, $(-2)^{s+1} = 2^s \cdot 2 \equiv 2 \pmod{r}$. Theorem 1 applies, with u = s + 1.

If $s \equiv 2 \pmod{4}$, $(-2)^2 = 4$. Theorem 2 applies, with v = 2.

If $s \equiv 0 \pmod{4}$, $(-2)^{s/2+1} = -(2)^{s/2} (2) \equiv 2 \pmod{r}$. Theorem 1 applies, with u = s/2 + 1.

For certain small values of q and r it is possible to characterize when $\chi(q) = 1$ in terms of the coefficients of $\pi[1] \pmod{p}$. Pepin gave the following three (the first not quite correctly).

Let r = 5. $\chi(3) = 1$ if and only if $c_1 \equiv c_4 \pmod{3}$ and

 $c_2 \equiv c_3 \pmod{3} [6, p.132].$

Let r = 7. $\chi(3) = 1$ if and only if $c_1 \equiv c_2 \equiv c_4 \pmod{3}$ and

$$c_3 \equiv c_5 \equiv c_6 \pmod{3}$$
 [6, pp. 145-146].

 $\chi(2) = 1$ if and only if c_0 is odd [6, p. 122].

Analogous criteria for r = 5, q = 7 and r = 7, q = 5 can be found in [5, p. 349].

A more general result, which yields only a sufficient condition, however, was suggested by Emma Lehmer [4], who proved it for r = 5.

THEOREM 3: Assume $2^{r-1} \not\equiv 1 \pmod{r^2}$, and r > 3. Let g be a primitive root, modulo r. If $c_g \equiv c_{g^3} \equiv c_{g^5} \equiv \cdots \equiv c_{g^{r-2}} \pmod{q}$ and $c_g^2 \equiv c_{g^4} \equiv c_{g^6} \equiv \cdots \equiv c_1 \pmod{q}$, then q is an rth power residue (mod p).

$$\begin{array}{ll} Proof. \quad \mathrm{Let} \ \lambda = \sum\limits_{j=0}^{\frac{r-3}{2}} \zeta_r^{g^2 j}, \ \mu = \sum\limits_{j=0}^{\frac{r-3}{2}} \zeta_r^{g^2 j+1} \\ \pi[1] = \sum\limits_{j=0}^{r-1} c_j \zeta_r^j = \sum\limits_{j=1}^{r-1} (c_j - c_0) \zeta_r^j \equiv (c_1 - c_0) \lambda + (c_g - c_0) \ \mu \ (\mathrm{mod} \ q). \end{array}$$

Similarly,

$$\pi[g] \equiv (c_{\scriptscriptstyle 1} - c_{\scriptscriptstyle 0}) \mu + (c_g - c_{\scriptscriptstyle 0}) \, \lambda \, ({
m mod} \, \, q) \; .$$

If 2 is a quadratic residue, modulo r,

$$\begin{split} \tau(\chi)^{2^{s-1}} &= \prod_{j=0}^{s-1} \pi[2^j]^{2^{s-j-1}} \equiv \prod_{j=0}^{s-1} \left[(c_1 - c_0) \,\lambda + (c_g - c_0) \mu \right]^{2^{s-j-1}} \\ &\equiv \left[(c_1 - c_0) \lambda + (c_g - c_0) \mu \right]^{2^{s-1}} (\text{mod } q) \text{.} \end{split}$$

If 2 is a quadratic nonresidue, modulo r,

$$egin{aligned} & au(\chi)^{2^{s}-1} = & \prod_{j=0}^{s/2-1} \pi [2^{2j}]^{2^{s-1-2j}} \pi [2^{2j+1}]^{2^{s-2-2j}} \ & \equiv [(c_1-c_0)\lambda + (c_g-c_0)\mu]^{2(2^{s}-1)/3} [(c_1-c_0)\mu + (c_g-c_0)\lambda]^{(2^{s}-1)/3} \ & (\mathrm{mod} \ q) \ . \end{aligned}$$

In both cases $\tau(\chi)^{z^{s-1}}$ has been shown to be an *r*th power in $Q(\zeta_r)/\mathfrak{q}$. Since $r \not\models (2^s - 1)/r$, $\tau(\chi)^r$ is an *r*th power in $Q(\zeta_r)/\mathfrak{q}$, and applying (1) yields the theorem.

COROLLARY. Assume $2^{r-1} \not\equiv 1 \pmod{r^2}$. If $c_1 \equiv c_2 \equiv \cdots \equiv c_{r-1} \pmod{q}$, then q is an rth power residue (mod p).

Proof. If r > 3, apply Theorem 3. If r = 3, $\tau(\chi)^3 \equiv (c_0 - c_1)^3$ (mod q).

A computation by John Brillhart shows that 1093 and 3511 are the only primes r less than 2^{24} for which $2^{r-1} \equiv 1 \pmod{r^2}$.

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