

# Pacific Journal of Mathematics

**ON A THEOREM BY HOFFMAN AND RAMSAY**

STELIOS A. NEGREPONTIS

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STELIOS NÉGREPONTIS

**Hoffman and Ramsay recently constructed, using the continuum hypothesis, a proper subalgebra of the algebra  $C(\beta N)$  of all complex-valued continuous functions on  $\beta N$  which is uniformly closed and separates points of  $\beta N$ . (This example settles in the negative a conjecture by Badé and Curtis). The purpose of this note is to indicate that this result does not need the continuum hypothesis.**

Their construction, in [3], naturally falls into two parts, the first of which is devoted to the proof that the space  $X$  of maximal ideals of the algebra  $L^\infty$  of all essentially bounded measurable complex-valued functions on the unit circle is homeomorphic to a subset of  $\beta N - N$ . It is in this part that they are forced to use the continuum hypothesis.

1. We have a simple proof of this fact without appeal to the continuum hypothesis.

**THEOREM.** *The space  $X$  is homeomorphic to a subset of  $\beta N - N$ .*

*Proof.* We first show that there is a compact separable space  $S$  in which  $X$  is embedded. This can be seen in at least two distinct ways. By a theorem in [2, p. 174] we can regard  $X$  as a subset of  $\mathcal{M}(H^\infty)$ , the space of maximal ideals of  $H^\infty$ . (For a definition of  $H^\infty$ , the reader should consult [2]). According to the "Corona" conjecture proved by Carleson, the open unit disc is dense in  $\mathcal{M}(H^\infty)$ , and thus,  $\mathcal{M}(H^\infty)$  is a separable space. Thus, we may take  $S = \mathcal{M}(H^\infty)$ . An alternative, and more elementary, way is to notice that  $L^\infty = C(X)$ , a fact proved in [2, p. 170], and hence that  $C(X)$  has cardinality  $\mathfrak{C} = 2^{\aleph_0}$ ; but this clearly implies that we can embed  $X$  in the compact "cube"  $[0, 1]^{\mathfrak{C}}$ , which according to [4, 3N] is separable. Thus, we may take  $S = [0, 1]^{\mathfrak{C}}$ .

In any case, let  $\varphi : N \rightarrow S$  be any mapping onto a dense subset of  $S$ . The Stone-Cech extension of  $\varphi$ , denoted by  $\bar{\varphi}$ , provides a mapping of  $\beta N$  onto  $S$ . As it is well-known, using Zorn's lemma, we can find a closed subset  $A$  of  $\bar{\varphi}^{-1}(X)$  such that  $\bar{\varphi} : A \rightarrow X$  is irreducible, i.e.  $\bar{\varphi}$  is onto, and  $\bar{\varphi}(B) \neq X$  for every proper closed subset  $B$  of  $A$ . Gleason's lemma [1, 2.3], which applies since  $X$  is extremally disconnected, states that  $\bar{\varphi}$  is a homeomorphism. We can embed  $\beta N$  into  $\beta N - N$ , and hence, the theorem is proved.

2. REMARKS. (i) The initial attempt to prove the above theorem, which failed, was the following: Let  $S^1$  be the unit circle in the plane, and let  $\sigma : X \rightarrow S^1$  be the natural projection (as described in [2, p. 171]). A quick proof of the theorem would follow, if it could be shown that  $\sigma$  were irreducible; but this is not the case.

(ii) The space  $X$  is the only example known to the author of an extremally disconnected subspace of  $\beta N$  which is not separable. Surely  $X$  is not a retract of  $\beta N$ . It follows that although the projective objects (in the category of compact spaces and continuous maps) are "the retracts of the free objects" (in the terminology of Rainwater [5]), it is not true that a projective object is a retract of every free object in which it is embedded.

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