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A HELLINGER INTEGRAL REPRESENTATION FOR BOUNDED LINEAR FUNCTIONALS

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The function space considered is that consisting of the complex-valued, quasicontinuous functions on a real interval $[a, b]$, anchored at a , and having the LUB norm. It is shown that each bounded linear functional on this Banach space has a Hellinger integral representation. A formula for the norm of the functional is given in terms of the integrating functions involved in its representation. A new existence criterion for the Hellinger integral is uncovered on the way to the representation theorem.

2. Definitions. In this section certain definitions and notational conventions are adopted for use in the succeeding sections. Throughout the paper, $[a, b]$ will denote a given interval and the word function will mean map from $[a, b]$ into the complex numbers.

DEFINITION 2.1. If c is any number in $(a, b]$, then R_c denotes a function such that $R_c(t) = 0$ if t is in $[a, c)$ and $R_c(t) = 1$ if $c \leq t \leq b$. If c is in $[a, b)$, then L_c denotes a function such that $L_c(t) = 0$ if $a \leq t \leq c$ and $L_c(t) = 1$ if t is in $(c, b]$. The functions L_c and R_c are called unit step functions. A linear combination of unit step functions is called a step function. Notice that each step function vanishes at a .

DEFINITION 2.2. We now specify the function space, $Q_0[a, b]$, which plays the central role. Its elements are the quasicontinuous functions anchored at a and they may be defined in two ways. First, $Q_0[a, b]$ is the set of all functions which vanish at a and which have a limit from the right at each t in $[a, b)$ and a limit from the left at each t in $(a, b]$. Second, let $B[a, b]$ be the Banach space of bounded functions, with LUB norm. Then $Q_0[a, b]$ is the closure, in $B[a, b]$, of the linear space of all step functions. So $Q_0[a, b]$ is a Banach space with norm $\|x\| = LUB |x(t)|$ for all t in $[a, b]$. Also, each bounded linear functional on $Q_0[a, b]$ is determined by its values on the step functions, since the latter form a dense linear subspace.

For proof of the equivalence of these two formulations of $Q_0[a, b]$, see [1, Lemma 4.16].

DEFINITION 2.3. Suppose g is any subset of $[a, b]$. If x is a function, then x_g denotes a function such that $x_g(t) = x(t)$ if t is in g

and $x_g(t) = 0$ if t is in $[a, b]$ but not in g . If F is a linear functional defined on $Q_0[a, b]$ and it is true that x_g is in $Q_0[a, b]$ for each x in $Q_0[a, b]$, then F'_g denotes a linear functional such that $F'_g(x) = F(x_g)$ for each x in $Q_0[a, b]$.

DEFINITION 2.4. “ v has bounded slope variation with respect to u ” means that v is a function, u is a real-valued, increasing function, and there exists a nonnegative number B such that if $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ with $n > 1$, then

$$\sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right| \leq B .$$

The least such number B is denoted by $V_a^b(dv/du)$ and is called the slope variation of v with respect to u over $[a, b]$.

DEFINITION 2.5. Suppose each of u, v , and w is a function and u is increasing. “The Hellinger integral $\int_a^b dw dv/du$ exists” means that $\int_a^b dw dv/du$ is a number and for each positive number ϵ there exists a subdivision D of $[a, b]$ such that if $\{t_p\}_{p=0}^n$ is any refinement of D then

$$\left| \int_a^b \frac{dw dv}{du} - \sum_{p=1}^n \frac{[w(t_p) - w(t_{p-1})] \cdot [v(t_p) - v(t_{p-1})]}{u(t_p) - u(t_{p-1})} \right| < \epsilon$$

Clearly, this integral has a unique value.

DEFINITION 2.6. If u is an increasing function and v is a function and c is in $[a, b)$ then “ $D_u^+v(c)$ exists” means that

$$\lim_{t \rightarrow c^+} \frac{v(t) - v(c)}{u(t) - u(c)}$$

exists and equals $D_u^+v(c)$. The notation $D_u^-v(c)$ is used in a corresponding manner for numbers c in $(a, b]$.

3. Lemmas. This section contains results which are used in the proofs given for theorems in § 4.

LEMMA 3.1. If n is an integer greater than 2 and k_0, k_1, \dots, k_n is a sequence of complex numbers and e_1, e_2, \dots, e_n is a sequence of positive real numbers then

$$\begin{aligned} & \sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_p}{e_{p+1}} - \frac{k_p - k_{p-1}}{e_p} \right| \\ & \geq \frac{1}{e_n} \left(\sum_{q=1}^n e_q \right) \left| \frac{k_n - k_0}{\sum_{q=1}^n e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q} \right| + \sum_{p=1}^{n-2} \left| \frac{k_{p+1} - k_0}{\sum_{q=1}^{p+1} e_q} - \frac{k_p - k_0}{\sum_{q=1}^p e_q} \right| \end{aligned}$$

Proof by induction. For the case $n = 3$,

$$\begin{aligned} & \left| \frac{k_3 - k_2}{e_3} - \frac{k_2 - k_1}{e_2} \right| + \left| \frac{k_2 - k_1}{e_2} - \frac{k_1 - k_0}{e_1} \right| \\ &= \left| \frac{k_3 - k_2}{e_3} - \frac{k_2 - k_1}{e_2} \right| + \frac{e_1}{e_2} \left| \frac{k_2 - k_0}{e_1 + e_2} - \frac{k_1 - k_0}{e_1} \right| \\ & \quad + \left| \frac{k_2 - k_0}{e_1 + e_2} - \frac{k_1 - k_0}{e_1} \right|. \end{aligned}$$

But by the triangle inequality, the sum of the first two terms of the right-hand member is greater than or equal to

$$\begin{aligned} & \left| \frac{(k_3 - k_0) - (k_2 - k_0)}{e_3} - \frac{(k_2 - k_0) - (k_1 - k_0)}{e_2} \right| \\ & \quad + \left| \frac{(k_2 - k_0)e_1}{e_2(e_1 + e_2)} - \frac{k_1 - k_0}{e_2} \right| \\ &= \left| \frac{k_3 - k_0}{e_3} - \frac{(k_2 - k_0)(e_1 + e_2 + e_3)}{e_3(e_1 + e_2)} \right|. \end{aligned}$$

Thus it may be seen that the conclusion is true for this case.

For the final step in the induction we begin by noting that

$$\begin{aligned} \sum_{p=1}^n \left| \frac{k_{p+1} - k_p}{e_{p+1}} - \frac{k_p - k_{p-1}}{e_p} \right| &\geq \sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_0}{\sum_{q=1}^{p+1} e_q} - \frac{k_p - k_0}{\sum_{q=1}^p e_q} \right| \\ & \quad + \frac{1}{e_{n+1}} \cdot \left(\sum_{q=1}^{n+1} e_q \right) \left| \frac{k_{n+1} - k_0}{\sum_{q=1}^{n+1} e_q} - \frac{k_n - k_0}{\sum_{q=1}^n e_q} \right| \end{aligned}$$

is true provided the last term of the left-hand member is greater than or equal to the sum of the last term of the right-hand member and

$$\left(1 - \frac{1}{e_n} \cdot \sum_{q=1}^n e_q \right) \left| \frac{k_n - k_0}{\sum_{q=1}^n e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q} \right|.$$

But this is true provided the sum of the last term of the left-hand member and

$$\frac{1}{e_n} \cdot \left(\sum_{q=1}^{n-1} e_q \right) \left| \frac{k_n - k_0}{\sum_{q=1}^n e_q} - \frac{k_{n-1} - k_0}{\sum_{q=1}^{n-1} e_q} \right|$$

is greater than or equal to the last term of the right-hand member. This last sum, is, by the triangle inequality, greater than or equal to

$$\begin{aligned} & \left| \frac{(k_{n+1} - k_0) - (k_n - k_0)}{e_{n+1}} - \frac{(k_n - k_0) - (k_{n-1} - k_0)}{e_n} \right. \\ & \quad \left. + \frac{(k_n - k_0) \cdot \sum_{q=1}^{n-1} e_q}{e_n \cdot \sum_{q=1}^n e_q} - \frac{k_{n-1} - k_0}{e_n} \right| \\ & = \left| \frac{k_{n+1} - k_0}{e_{n+1}} - \frac{(k_n - k_0) \cdot \sum_{q=1}^{n+1} e_q}{e_{n+1} \cdot \sum_{q=1}^n e_q} \right|. \end{aligned}$$

Thus each of the inequalities is true. Hence Lemma 3.1.

LEMMA 3.2. *If n is an integer greater than 2 and k_0, k_1, \dots, k_n is a number sequence and s_0, s_1, \dots, s_n is an increasing real number sequence, then*

$$\sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_p}{s_{p+1} - s_p} - \frac{k_p - k_{p-1}}{s_p - s_{p-1}} \right| \geq \sum_{p=1}^{n-1} \left| \frac{k_{p+1} - k_0}{s_{p+1} - s_0} - \frac{k_p - k_0}{s_p - s_0} \right|.$$

This result follows immediately from Lemma 3.1 by the transformation: $s_p - s_{p-1} = e_p$ for $p = 1, 2, \dots, n$.

LEMMA 3.3. *If v has bounded slope variation with respect to u then $D_u^- v(t)$ exists for each t in $(a, b]$ and $D_u^+ v(t)$ exists for each t in $[a, b)$.*

Proof. Suppose c is in $[a, b)$ and $\lim_{t \rightarrow c^+} (v(t) - v(c))/(u(t) - u(c))$ does not exist. Then there exists a positive number ε such that if r is in (c, b) then there exists a number s in (c, r) for which

$$\left| \frac{v(r) - v(c)}{u(r) - u(c)} - \frac{v(s) - v(c)}{u(s) - u(c)} \right| \geq \varepsilon.$$

It may be seen, then, that if n is an integer greater than 2 there exists an increasing number sequence s_0, s_1, \dots, s_n with $s_0 = c$ and each term in $[c, b]$ such that

$$\sum_{p=1}^{n-1} \left| \frac{v(s_{p+1}) - v(c)}{u(s_{p+1}) - u(c)} - \frac{v(s_p) - v(c)}{u(s_p) - u(c)} \right| \geq (n-1)\varepsilon.$$

But from this inequality and Lemma 3.2 it follows that

$$\sum_{p=1}^{n-1} \left| \frac{v(s_{p+1}) - v(s_p)}{u(s_{p+1}) - u(s_p)} - \frac{v(s_p) - v(s_{p-1})}{u(s_p) - u(s_{p-1})} \right| \geq (n-1)\varepsilon.$$

Since there exists an integer n for which $(n - 1)\varepsilon > V_a^b(dv/du)$, this is a contradiction. Hence $D_u^+v(c)$ exists for each c in $[a, b]$. An argument similar to that just given shows that $D_u^-v(c)$ exists for each c in $(a, b]$. Hence Lemma 3.3.

LEMMA 3.4. *Suppose v has bounded slope variation with respect to u . If t is in $(a, b]$, then $\int_a^b dR_t dv/du$ exists and is equal to $D_u^-v(t)$. If t is in $[a, b)$, then $\int_a^b dL_t dv/du$ exists and is equal to $D_u^+v(t)$.*

This lemma follows readily from Lemma 3.3 and the observation that, in each of the two equations implied by Lemma 3.4, each approximant for the right-hand member is an approximant for the left-hand member.

LEMMA 3.5. *If v has bounded slope variation with respect to u then the functional F , given by*

$$F(x) = \int_a^b \frac{dx dv}{du},$$

is linear on its domain, the dv/du -integrable functions x , and these form a linear space.

Proof of lemma is not given.

LEMMA 3.6. *If S is a step function and v has bounded slope variation with respect to u then*

$$\int_a^b \frac{dS dv}{du} \text{ exists.}$$

This lemma follows from Definition 2.5 and Lemmas 3.4 and 3.5.

LEMMA 3.7. *If a normed linear space A may be written as a direct sum $A = B \oplus C$ of two of its subspaces in such a way that*

$$\|a\| = \text{Max} \{ \|Pr_1(a)\|, \|Pr_2(a)\| \}$$

for each a in A , then

$$\|F\| = \|F \circ Pr_1\| + \|F \circ Pr_2\|,$$

for each bounded linear functional F on A .

Proof of this lemma is not given.

LEMMA 3.8. Suppose h is subset of $[a, b]$ and f and g are mutually exclusive subsets of h whose union is h . Suppose, moreover, that if x is any function in $Q_0[a, b]$, then each of x_f, x_g , and x_h is in $Q_0[a, b]$. If F is a bounded linear functional from $Q_0[a, b]$ then each of F_f, F_g , and F_h is a bounded linear functional and

$$\|F_f\| + \|F_g\| = \|F_h\| \leq \|F\|.$$

This lemma is a mere application of Lemma 3.7.

4. Theorems. In this section a representation for the bounded linear functionals on $Q_0[a, b]$ in terms of the Hellinger integral is developed and a formula for their norms is given.

THEOREM 4.1. If x is in $Q_0[a, b]$ and v has bounded slope variation with respect to u , then $\int_a^b dx dv/du$ exists and

$$\left| \int_a^b \frac{dx dv}{du} \right| \leq \left\{ V_a^b \frac{dv}{du} + |D_u^- v(b)| \right\} \|x\|.$$

Proof. Let S_1, S_2, S_3, \dots be a sequence of step functions such that $\|S_p - x\| < 1/p$ if p is a positive integer. Suppose n is an integer greater than 1, $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ and q is a positive integer. Then, using summation by parts,

$$\begin{aligned} & \sum_{p=1}^n \frac{[S_q(t_p) - S_q(t_{p-1})][v(t_p) - v(t_{p-1})]}{u(t_p) - u(t_{p-1})} \\ &= - \sum_{p=1}^{n-1} S_q(t_p) \left\{ \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right\} \\ & \quad + S_q(t_n) \frac{v(b) - v(t_{n-1})}{u(b) - u(t_{n-1})}. \end{aligned}$$

It is thus evident that the left-hand member of this equation is, in absolute value, less than or equal to

$$\|S_q\| \left\{ V_a^b \frac{dv}{du} + \left| \frac{v(b) - v(t_{n-1})}{u(b) - u(t_{n-1})} \right| \right\}.$$

From this and Lemmas 3.3 and 3.6 one may conclude that

$$\left| \int_a^b \frac{dS_q dv}{du} \right| \leq \|S_q\| \left\{ V_a^b \frac{dv}{du} + |D_u^- v(b)| \right\}$$

(It is to be noted that this inequality holds true with S_q replaced by any other function in $Q_0[a, b]$ for which the integral exists). If m is an integer greater than q , then, since $\|S_q - S_m\| < 2/q$, it follows that

$$\left| \int_a^b \frac{d(S_q - S_m)dv}{du} \right| \leq \frac{2}{q} \left\{ V_a^b \frac{dv}{du} + |D_u^- v(b)| \right\}.$$

Consequently, the sequence

$$\left\{ \int_a^b \frac{dS_q dv}{du} \right\}_{q=1}^\infty$$

is a Cauchy sequence and so has a sequential limit. Call this limit I . We now show that the approximants to $\int_a^b dx dv/du$ tend, under refinement, to I .

There exists a number B such that

$$V_a^b \frac{dv}{du} + \left| \frac{v(b) - v(t)}{u(b) - u(t)} \right| < B$$

for each t in $[a, b]$. Since $\|x - S_p\| < 1/p$ for $p = 1, 2, \dots$, it follows that

$$\left| \sum_{i=1}^m \frac{\{x(s_i) - S_p(s_i) - [x(s_{i-1}) - S_p(s_{i-1})]\}[v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})} \right| < \frac{B}{p}$$

for any subdivision $\{s_i\}_{i=0}^m$ of $[a, b]$ and any positive integer p . For each positive integer p there exists a subdivision D_p of $[a, b]$ such that if $\{s_i\}_{i=0}^m$ is any refinement of D_p then

$$\sum_{i=1}^m \frac{[S_p(s_i) - S_p(s_{i-1})][v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})} - \int_a^b \frac{dS_p dv}{du} \left| < \frac{B}{p}.$$

Since

$$\left| \int_a^b \frac{dS_p dv}{du} - I \right| \leq \frac{2B}{p} \quad \text{for } p = 1, 2, \dots$$

it follows that, for each positive integer p ,

$$\left| \sum_{i=1}^m \frac{[x(s_i) - x(s_{i-1})][v(s_i) - v(s_{i-1})]}{u(s_i) - u(s_{i-1})} - I \right| < \frac{4B}{p}$$

provided $\{s_i\}_{i=0}^m$ is a refinement of D_p . Hence $\int_a^b dx dv/du$ exists and its value is I . That the integral satisfies the inequality of the conclusion may be seen from the parenthetical note above. Hence Theorem 4.1.

THEOREM 4.2. *Suppose v has bounded slope variation with respect to u and F is the functional defined by*

$$F(x) = \int_a^b \frac{dx dv}{du}$$

for each x in $Q_0[a, b]$. Then F is a bounded linear functional whose norm is $V_a^b(dv/du) + |D_u^-v(b)|$.

Proof. It is clear from Lemma 3.5 and Theorem 4.1 that F is linear and bounded and that the norm of F does not exceed $V_a^b(dv/du) + |D_u^-v(b)|$. We now construct a function z in $Q_0[a, b]$ such that $\|z\| = 1$ and $F(z)$ equals the sum of $|D_u^-v(b)|$ and the approximant for $V_a^b(dv/du)$ corresponding to a preassigned subdivision of $[a, b]$.

Suppose $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ with $n > 1$. Define d_p , for $p = 1, 2, \dots, (n-1)$, by

$$d_p = \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})}$$

if this expression is not zero and $d_p = 1$ if the expression is zero. For $p = 1, 2, \dots, (n-1)$, let z_p be a function such that

$$z_p(t) = \begin{cases} -\frac{u(t) - u(t_{p-1})}{u(t_p) - u(t_{p-1})} \cdot \frac{|d_p|}{d_p} & \text{for } t \text{ in } [t_{p-1}, t_p] \\ -\frac{u(t_{p+1}) - u(t)}{u(t_{p+1}) - u(t_p)} \cdot \frac{|d_p|}{d_p} & \text{for } t \text{ in } [t_p, t_{p+1}] \\ 0 & \text{for } t \text{ in } [a, b] \text{ but not in } [t_{p-1}, t_{p+1}]. \end{cases}$$

If $D_u^-v(b) = 0$, let $z_n = R_b$. If $D_u^-v(b) \neq 0$ let $z_n = (D_u^-v(b)/|D_u^-v(b)|)R_b$. Finally, let $z = \sum_{p=1}^n z_p$.

Each of z, z_1, z_2, \dots, z_n is in $Q_0[a, b]$ and it may be verified that

$$\int_a^b \frac{dz_p dv}{du} = \left[\frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right] \frac{|d_p|}{d_p}$$

for $p = 1, 2, \dots, (n-1)$ and $\int_a^b (dz_n dv/du) = |D_u^-v(b)|$. Hence,

$$\int_a^b \frac{dz dv}{du} = \sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right| + |D_u^-v(b)|$$

If t is in $[a, t_1]$, then

$$|z(t)| = \left| -\frac{u(t) - u(a)}{u(t_1) - u(a)} \cdot \frac{|d_1|}{d_1} \right| \leq 1.$$

If t is in $[t_{n-1}, b]$, then

$$|z(t)| = \left| -\frac{u(b) - u(t)}{u(b) - u(t_{n-1})} \cdot \frac{|d_{n-1}|}{d_{n-1}} \right| \leq 1.$$

If p is one of $1, 2, \dots, (n-2)$ and t is in $[t_p, t_{p+1}]$ then

$$|z(t)| = \left| -\frac{u(t) - u(t_p)}{u(t_{p+1}) - u(t_p)} \frac{|d_{p+1}|}{d_{p+1}} - \frac{u(t_{p+1}) - u(t)}{u(t_{p+1}) - u(t_p)} \frac{|d_p|}{d_p} \right| \leq 1.$$

And $|z(b)| = 1$. Hence $\|z\| = 1$.

It may be inferred from the foregoing that the norm of F is not less than $V_a^b(dv/du) + |D_u^-v(b)|$. Hence Theorem 4.2.

THEOREM 4.3. *If F is a bounded linear functional from $Q_0[a, b]$ then there exist two functions u and v , with v having bounded slope variation with respect to u , such that*

$$F(x) = \int_a^b \frac{dx dv}{du}$$

for each x in $Q_0[a, b]$.

Proof. Suppose c is in $(a, b]$. If r and s are numbers such that $a < r < s < c$, then, by Lemma 3.8, $\|F_{(r,c)}\| \geq \|F_{(s,c)}\| \geq 0$. Consequently, $\lim_{t \rightarrow c^-} \|F_{(t,c)}\|$ exists. Let λ denote the function such that $\lambda(c) = \lim_{t \rightarrow c^-} \|F_{(t,c)}\|$ for each number c in $(a, b]$ and $\lambda(a) = 0$. Similarly, let ρ denote the function such that $\rho(c) = \lim_{t \rightarrow c^+} \|F_{(c,t)}\|$ for each c in $[a, b)$ and $\rho(b) = 0$.

Now it may be seen from the definition of λ and Lemma 3.8 that if $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$, then

$$\sum_{p=0}^n \lambda(t_p) \leq \|F\|.$$

A similar statement is true of ρ . Thus there exists a countable subset M of $[a, b]$ such that if t is in $[a, b]$ but not in M then $\lambda(t) = \rho(t) = 0$.

Let u denote an increasing function such that (1) if t is in (a, b) and $\lambda(t) > 0$, then $u(t) - u(t-) > 0$, and (2) if t is in $[a, b)$ and $\rho(t) > 0$, then $u(t+) - u(t) > 0$. For each t in $[a, b]$ let u_t denote the function such that $u_t(s) = 0$ for $a \leq s \leq t$ and $u_t(s) = u(s) - u(t)$ for $t \leq s \leq b$. Let v denote the function such that $v(t) = -F(u_t)$ for each t in $[a, b]$.

Suppose $\{t_p\}_{p=0}^n$ is a subdivision of $[a, b]$ and $n > 1$. Then, by the definition of v and the linearity of F there exists a number sequence $\{d_p\}_{p=1}^{n-1}$, with $|d_p| = 1$ for $p = 1, 2, \dots, (n - 1)$, such that

$$\begin{aligned} & \sum_{p=1}^{n-1} \left| \frac{v(t_{p+1}) - v(t_p)}{u(t_{p+1}) - u(t_p)} - \frac{v(t_p) - v(t_{p-1})}{u(t_p) - u(t_{p-1})} \right| \\ &= F \left(\sum_{p=1}^{n-1} \left[\frac{u_{t_{p+1}} - u_{t_p}}{u(t_{p+1}) - u(t_p)} - \frac{u_{t_p} - u_{t_{p-1}}}{u(t_p) - u(t_{p-1})} \right] d_p \right). \end{aligned}$$

It may be verified that the norm of the function which is the argument of F in the right-hand member of the equation is 1. Consequently the left-hand member is less than or equal to $\|F\|$. Thus it may be inferred that v has bounded slope variation with respect to u .

Let G denote the bounded linear functional such that

$$G(x) = \int_a^b \frac{dx dv}{du}$$

for each x in $Q_0[a, b]$. Suppose c is in (a, b) . By Lemma 3.4

$$\begin{aligned} G(R_c) &= D_u^- v(c) \\ G(R_c) &= \lim_{t \rightarrow c^-} \frac{v(c) - v(t)}{u(c) - u(t)} \\ &= \lim_{t \rightarrow c^-} F\left(\frac{u_t - u_c}{u(c) - u(t)}\right). \end{aligned}$$

For t in (a, c) , one has

$$\left| \frac{u_t(s) - u_c(s)}{u(c) - u(t)} - R_c(s) \right| \leq \begin{cases} 0 & \text{if } s \text{ is in } [a, t] \\ \frac{u(c-) - u(t)}{u(c) - u(t)} & \text{if } s \text{ is in } (t, c) \\ 0 & \text{if } c \leq s \leq b \end{cases}$$

so that

$$\begin{aligned} \left| \frac{v(c) - v(t)}{u(c) - u(t)} - F(R_c) \right| &= \left| F_{(t,c)}\left(\frac{u_t - u_c}{u(c) - u(t)} - R_c\right) \right| \\ &\leq \|F_{(t,c)}\| \cdot \frac{u(c-) - u(t)}{u(c) - u(t)} \leq \|F_{(t,c)}\|. \end{aligned}$$

Now $\lim_{t \rightarrow c^-} \|F_{(t,c)}\| = \lambda(c)$. But if $\lambda(c) > 0$, then $u(c) - u(c-) > 0$ so that

$$\lim_{t \rightarrow c^-} \frac{u(c-) - u(t)}{u(c) - u(t)} = 0.$$

So, whether $\lambda(c)$ is positive or zero, one has that

$$\lim_{t \rightarrow c^-} \left| \frac{v(c) - v(t)}{u(c) - u(t)} - F(R_c) \right| = 0.$$

Hence $F(R_c) = G(R_c)$ for each c in (a, b) . A similar argument shows that $F(L_c) = G(L_c)$ for each c in $[a, b)$. Therefore $F(S) = G(S)$ for every step function S . Thus, $F = G$. Hence Theorem 4.3. Clearly, the norm of F is given by the expression appearing in Theorem 4.2.

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