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ROBERT WILLIAM STRINGALL

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# ENDOMORPHISM RINGS OF PRIMARY ABELIAN GROUPS

ROBERT W. STRINGALL

**This paper is concerned with the study of certain homomorphic images of the endomorphism rings of primary abelian groups. Let $E(G)$ denote the endomorphism ring of the abelian $p$-group $G$, and define $H(G) = \{\alpha \in E(G) \mid x \in G, px = 0$ and height $x < \infty$ imply height $\alpha(x) >$ height $x\}$. Then $H(G)$ is a two sided ideal in $E(G)$ which always contains the Jacobson radical. It is known that the factor ring $E(G)/H(G)$ is naturally isomorphic to a subring $R$ of a direct product $\Pi_{n=1}^{\infty} M_n$ with $\sum_{n=1}^{\infty} M_n$ contained in $R$ and where each $M_n$ is the ring of all linear transformations of a $Z_p$ space whose dimension is equal to the $n-1$ Ulm invarient of $G$. The main result of this paper provides a partial answer to the unsolved question of which rings $R$ can be realized as $E(G)/H(G)$.**

THEOREM. **Let $R$ be a countable subring of $\Pi_{\aleph_0} Z_p$ which contains the identity and $\sum_{\aleph_0} Z_p$. Then there exists a $p$-group G with a standard basic subgroup and containing no elements of infinite height such that $E(G)/H(G)$ is isomorphic to $R$. Moreover, $G$ can be chosen without proper isomorphic subgroups; in this case, $H(G)$ is the Jacobson radical of $E(G)$.**

## 1. Preliminaries.

(1.1) Throughout this paper $p$- represents a fixed prime number, $N$ the natural numbers, $Z$ the integers and $Z_{p^n}$ the ring of integers modulo $p^n$. All groups under consideration will be assumed to be $p$-primary and abelian. With few exceptions, the notation of [3], [5], and [8] will prevail.

Let $h_G(x)$ and $E(x)$ denote, respectively, the $p$-height of $x$ in $G$ and the exponential order of $x$. If $A$ is any subset of the group $G$, then $\{A\}$ will denote the subgroup of $G$ generated by $A$. Denote the $p^n$ layer of $G$ by $G[p^n]$. Finally, if $A$ is any set, let $|A|$ be the cardinal number of $A$.

(1.2) Let $G$ be a $p$-primary group and $B$ a basic subgroup of $G$. The group $B$ can be written as $B = \sum_{n \in N} B_n$ where each $B_n$ is a direct sum of, say $f(n)$, copies of $Z_{p^n}$. That is, $B_n = \sum_{f(n)} \{b_i\}$ where $E(b_i) = n$. Define $H_n = \{p^n G, B_{n+1}, B_{n+2}, \cdots\}$. It is readily verified that $G = B_1 \oplus \cdots \oplus B_n \oplus H_n$ for each $n \in N$. Thus, it is possible to define the projections $\pi_n$ $(n = 1, 2, \cdots)$ of $G$ onto $H_n$ corresponding to the decomposition $G = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus H_n$. Define $\rho_1 = 1 - \pi_1$ and

$\rho_n = \pi_{n-1} - \pi_n$ for $n > 1$. It follows that $\rho_n(G) = B_n$ and that $\rho_n$ is the projection of $G$ onto $B_n$.

**2. Endomorphism rings.** A few preliminary notions are needed before the main results can be presented. Although given in a different context, many of the results of this section are patterned after those of R. S. Pierce in his work [8].

LEMMA 2.1. *Let $G$ be a $p$-group and $B = \sum_{n \in N} B_n$ a basic subgroup of $G$. If $\alpha$ is an endomorphism of $B_n[p]$, then $\alpha$ can be extended to an endomorphism $\beta$ of $G$ such that $j \neq n$ implies $\beta(B_j) = 0$.*

*Proof.* Since $G = B_1 \oplus B_2 \oplus \cdots \oplus B_m \oplus H_m$ for each $m \in N$, for each $m \in N$, it is enough to show that $\alpha$ can be extended to $B_n$. Let

$$B_n = \sum_{i=1}^{f(n)} \{b_i\}$$

where, for each $i$, $E(b_i) = n$. For $b_i \in B_n$, write

$$\begin{aligned} \alpha(p^{n-1}b_i) &= a_1 p^{n-1}b_1 + \cdots + a_k p^{n-1}b_k \\ \beta(b_i) &= a_1 b_1 + \cdots + a_k b_k \end{aligned}$$

where $k$ and the integers $a_j$ $(0 \leqq a_j < p)$ are determined by $\alpha$. Compute $\beta(b_i)$ in this way for each $b_i \in B_n$, and extend $\beta$ linearly to $B_n$. It follows that $\beta$ is the desired extension of $\alpha$ to $B_n$.

LEMMA 2.2. *If $G$ is a $p$-group and $B$ a basic subgroup of $G$, then any bounded homomorphism of $B$ into $G$ can be extended to a bounded endomorphism of $G$.*

*Proof.* By definition, $G/B$ is divisible. Consequently,

$$G/B = p^n(G/B) = \frac{B + p^nG}{B}$$

for each positive integer $n$. It follows that $G = B + p^nG$ for each $n \in N$. Let $k \in N$ be such that $p^k\alpha = 0$, and write $x \in G$ as $x = b + p^ky$ where $b \in B$ and $y \in G$. It is easy to check that $x \to \alpha(b)$ defines a bounded extension of $\alpha$ to an endomorphism of $G$.

For proof of the following lemma see [8], Lemma 13.1.

LEMMA 2.3. *An endomorphism $\alpha$ of the $p$-group $G$ is an automorphism if and only if* $\ker \alpha \cap G[p] = 0$ *and* $\alpha(G[p] \cap p^nG) = G[p] \cap p^nG$ *for each integer* $n = 0, 1, 2, \cdots$.

For the $p$-group $G$, let $E(G)$ denote the ring of all endomorphisms of $G$. If $E_p(G)$ denotes the subcollection of $E(G)$ consisting of all bounded endomorphisms of $G$, then it is not difficult to show that $E_p(G)$ is a two sided ideal of $E(G)$.

LEMMA 2.4. *Let*

$$H_p(G) = \{\alpha \in E_p(G) \mid x \in G[p] \text{ and } h_G(x) \in N \text{ imply } h_G(\alpha(x)) > h_G(x)\} ,$$
$$K_p(G) = \{\alpha \in E_p(G) \mid \alpha(G[p]) = 0\} , \text{ and}$$
$$L_p(G) = \{\alpha \in E_p(G) \mid \alpha(G) \subseteqq pG\} .$$

*Then $H_p(G)$, $K_p(G)$ and $L_p(G)$ are two sided ideals of $E_p(G)$ contained in the Jacobson radical, $J(E_p(G))$, of $E_p(G)$. What is more, $K_p(G) + L_p(G) \subseteqq H_p(G)$.*

*Proof.* It is easy to check that $H_p(G)$, $K_p(G)$ and $L_p(G)$ are two sided ideals of both $E_p(G)$ and $E(G)$. It is also easy to verify that $K_p(G) \subseteqq H_p(G)$. It remains only to show that $L_p(G) \subseteqq H_p(G) \subseteqq J(E_p(G))$. To this end, suppose $\alpha \in L_p(G)$, $x \in G[p]$ and $h_G(x) = k \in N$. Since $h_G(x) = k$, it is possible to write $x = p^k y$ for some $y \in G$. It follows that

$$\alpha(x) = \alpha(p^k y) = p^k \alpha(y) \in p^k pG = p^{k+1}G$$

Hence, $h_G(\alpha(x)) \geqq k + 1 > h(x)$ and $\alpha \in H_p(G)$. Therefore, $L_p(G)$ is contained in $H_p(G)$. To show that $H_p(G)$ is contained in $J(E_p(G))$, let $\alpha \in H_p(G)$. Since $\alpha \in E_p(G)$, there exists a positive integer $k$ such that $p^k\alpha = 0$. Thus, if $x \in G[p]$ and $h_G(x) \geqq k$, then $\alpha(x) = 0$. Since $x \in G[p]$ implies $h_G(\alpha^k(x)) > k$, it follows that $\alpha^{k+1}(x) = 0$ for all $x \in G[p]$. If $x \in G[p]$ and $(1 - \alpha)(x) = 0$, then

$$x = \alpha(x) = \alpha^2(x) = \cdots = \alpha^{k+1}(x) = 0 .$$

Thus, $1 - \alpha$ is one-to-one on $G[p]$. Also, if $x \in G[p]$, then

$$(1 - \alpha)(x + \alpha(x) + \cdots + \alpha^k(x)) = x .$$

Therefore, $(1 - \alpha)(G[p] \cap p^nG) = G[p] \cap p^nG$ for each $n = 0, 1, 2, \cdots$. Applying 2.3, it is seen that $1 - \alpha$ has an inverse. Since $H_p(G)$ is an ideal of $E(G)$, $\alpha \in J(E(G)) \cap E_p(G) = J(E_p(G))$ (see [4], pp. 9 and 10).

It becomes necessary, at least for the remainder of this section, to fix the basic subgroup $B$ and a decomposition $B = \sum B_n$. This, naturally, determines the subgroup $H_n$, the cardinals $f(n)$ and the maps $\pi_n$ and $\beta_n$.

LEMMA 2.5. *There are group homomorphisms $\rho$ of $E_p(G)$ into $E_p(G)$, $\sigma$ of $E_p(G)$ into $K_p(G)$ and $\tau$ of $E_p(G)$ into $L_p(G)$ such that for $\alpha \in E_p(G)$*

$$(*)(\sigma\alpha)(b_n) = (1 - \pi_{n-1})(\alpha(b_n)),\ (\tau\alpha)(b_n) = \pi_n(\alpha(b_n))$$

*and* $(\rho\alpha)(b_n) = \rho_n(\alpha(b_n))$ *for* $b_n \in B_n$, $n = 1, 2, \cdots$. *Moreover,*

$$\rho^2 = \rho, \sigma^2 = \sigma, \tau^2 = \tau, \rho\sigma = \sigma\rho = \rho\tau = \tau\rho = \sigma\tau = \tau\sigma = 0, \rho + \sigma + \tau = 1\,,$$

*and* $\rho_n(\rho\alpha)\rho_n(b_n) = \rho\alpha(b_n)$ *for all* $b_n \in B_n$, $n = 1, 2, \cdots$.

*Proof.* It is clear that conditions (*) determine bounded homomorphisms of $B$ into $G$, which by 2.2 extend to $G$ as bounded endomorphisms. The remainder of the proof is similar to that of 13.4 in [8] and will not be given.

(2.6) LEMMA. *The mapping*

$$\lambda\colon \alpha \to ((\rho\alpha) \mid B_1[p], (\rho\alpha) \mid B_2[p], \cdots)$$

*is a ring homomorphism of* $E_p(G)$ *onto the ring direct sum*

$$\sum_{n=1}^{\infty} E(B_n[p])\,.$$

*The kernel of* $\lambda$ *is* $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$.

*Proof.* It is clear that $\lambda$ maps onto $\sum_{n\in N} E(B_n[p])$. In fact, if $(\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots) \in \sum_{n\in N} E(B_n[p])$ where $\alpha_k \in E(B_k[p])$ for $k = 1, 2, \cdots, n$, then by 2.1, each of the $\alpha_k$ have extensions $\beta_k$ to $G$ such that $j \neq k$ implies $\beta_k(B_j) = 0$. Obviously,

$$\lambda\Big(\sum_{i=1}^{n} \beta_i\Big) = (\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots)$$

and $p^n \sum_{i=1}^{n} \beta_i = 0$. Thus, $\lambda$ is onto $\sum_{n\in N} E(B_n[p])$. Clearly, $\lambda$ is additive. To show that $\lambda$ preserves products, let $b \in B_n[p]$. Then $h(b) = n - 1$, so that for some $c \in B_n$, $b = p^{n-1}c$. Also,

$$\rho(\alpha\beta(b)) = \rho_n(\alpha\beta(b)) = \rho_n(\alpha((\sigma\beta)(b) + (\rho\beta)(b) + (\tau\beta)(b)))\,.$$

Now, $\sigma\beta \in K_p(G)$ and $b \in G[p]$. Thus, $\sigma\beta(b) = 0$. Also, $\tau\beta \in L_p(G)$ implies that $\tau\beta(b) = \tau\beta(p^{n-1}c) = p^{n-1}\tau\beta(c) \in p^nG$, so that

$$\rho_n\alpha(\tau\beta(b)) \in p^nG \cap B_n = p^nB_n = 0\,.$$

Finally, $\rho\beta(b) = \rho_n\rho\beta(b)$. Thus,

$$\rho(\alpha\beta(b)) = \rho_n(\alpha\beta(b)) = \rho_n\alpha((\rho\beta)(b)) = (\rho\alpha)((\rho\beta)(b))\,.$$

Consequently, $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$. To show that the kernel of $\lambda$ is $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$, observe that $\lambda(\alpha) = 0$ if and only if $\rho\alpha \mid B_n[p] = 0$ for all $n \in N$. This condition is equivalent to $\rho\alpha(B[p]) =$

0 which, since $\rho\alpha$ is bounded, is equivalent to $\rho\alpha(G[p]) = 0$. Therefore, $\operatorname{Ker}(\lambda) = \{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$.

THEOREM 2.7. *The Jacobson radical of $E_p(G)$ is $H_p(G)$, and $K_p(G) + L_p(G) = H_p(G)$. Also, $E_p(G)/H_p(G)$ is ring isomorphic to the ring direct sum $\sum_{n \in N} M_n$ where each $M_n$ is the ring of all linear transformations of a $Z_p$-space of dimension $f(n)$.*

*Proof.* By 2.6 there is a ring homomorphism $\lambda$ of $E_p(G)$ onto a ring isomorphic to $\sum_{n \in N} M_n$. Moreover, the kernel, of $\lambda$ is $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$. The rings $M_n$ are surely primitive. Thus, by [4], proposition 1, p. 10, the Jacobson radical of $E_p(G)$ is contained in $\bigcap_{n \in N} \operatorname{Ker}(\delta_n\lambda) = \operatorname{Ker}\lambda$ where $\delta_n$ $(n = 1, 2, \cdots)$ is, temporarily, the projection map of $\sum_{n \in N} M_n$ onto $M_n$. Hence by 2.4,

$$K_p(G) + L_p(G) \subseteqq H_p(G) \subseteqq J(E_p(G)) \subseteqq \operatorname{Ker}\lambda .$$

To show that the kernel of $\lambda$ is contained in $K_p(G) + L_p(G)$, let $\alpha \in E_p(G)$ be such that $\rho\alpha \in K_p(G)$. By 2.5, $\rho\alpha + \sigma\alpha + \tau\alpha = \alpha$. It follows that $\alpha \in K_p(G) + L_p(G)$. Thus,

$$\operatorname{Ker}\lambda = \{\alpha \in E_p(G) \mid \rho\alpha \in K(G)\} \subseteqq K_p(G) + L_p(G) .$$

Hence,

$$\operatorname{Ker}\lambda = J(E_p(G)) = K_p(G) + L_p(G) = H_p(G) .$$

For proof of the following lemma, the reader is directed to R. S. Pierce's work [8], p. 284.

LEMMA 2.8. *Suppose $R$ is an associative ring and $S$ any two-sided ideal of $R$. Let $J(S)$ be the Jacobson radical of $S$ and*

$$J(R, S) = \{x \in R \mid xz \in J(S) \text{ for all } z \in S\} .$$

*Then the following statements are valid:*

(a) *$J(R, S)$ is a two-sided ideal of $R$ containing $J(R)$ the Jacobson radical of $R$;*

(b) *$J(R, S) = \{x \in R \mid wxz$ is quasi-regular for all $z, w$ in $S\}$;*

(c) *$J(R, S) = \{x \in R \mid zx \in J(S)$ for all $z \in S\}$;*

(d) *$J(R, S) \cap S = J(S)$;*

(e) *the image of $S$ under the natural projection of $R$ onto $R/J(R, S)$ is an ideal which isomorphic to $S/J(S)$.*

Recall that $M_n$ $(n = 1, 2, \cdots)$ is defined to be the ring of all linear transformations of a $Z_p$-space of dimension $f(n)$.

If $\xi$ is the natural map of $E(G)$ onto $E(G)/J(E(G), E_p(G))$, then, by 2.8 (e), $\xi(E_p(G))$ is isomorphic to $E_p(G)/J(E_p(G))$. By 2.7, there is an isomorphism $\lambda$ of $E_p(G)/J(E_p(G))$ onto the ring direct sum $\sum_{n\in N} M_n$. Let $\delta_n$ be the ring homomorphism of $E_p(G)$ onto $M_n$ obtained by composing $\lambda\xi$ with the projection of $\sum_{n\in N} M_n$ onto $M_n$. That is, for $\alpha \in E_p(G)$

$$\lambda\xi(\alpha) = (\delta_1\alpha, \delta_2\alpha, \cdots) .$$

It is easy to see that if $\rho_n$ $(n = 1, 2, \cdots)$ are as defined in 1.2, then

$$\delta_n(\rho_m) = 0 \quad \text{for } m \neq n \quad \text{and} \quad \delta_n(\rho_n) = 1 .$$

For $\alpha \in E(G)$, set $\mu(\alpha) = (\delta_1(\alpha\rho_1), \delta_2(\alpha\rho_2), \delta_3(\alpha\rho_3), \cdots)$.

THEOREM 2.9. *The correspondence*

$$\alpha \xrightarrow{\mu} (\delta_1(\alpha\rho_1), \delta_2(\alpha\rho_2), \delta_3(\alpha\rho_3), \cdots)$$

*is a ring homomorphism of $E(G)$ onto a subring $R$ of the ring direct product $\prod_{n\in N} M_n$ with kernel $J(E(G), E_p(G))$. Moreover, $R$ contains both the identity of $\prod_{n\in N} M_n$ and the ring direct sum $\sum_{n\in N} M_n$.*

*Proof.* See the proof of Theorem 14.3 in [8].

The following lemma gives an interesting characterization of $J(E(G), E_p(G))$.

LEMMA 2.10. $J(E(G), E_p(G)) = \{\alpha \in E(G) \mid x \in G[p]$ *and* $h_G(x) \in N$ *imply* $h_G(\alpha(x)) > h_G(x)\}$.

*Proof.* Suppose $\alpha \in E(G)$ and $h_G(\alpha(x)) > h(x)$ for all $x \in G[p]$ such that $h(x)$ is finite. Then if $\beta \in E(G)$, the product $\alpha\beta$ satisfies this same condition. That is, for elements $x$ in $G[p]$ of finite height, $h_G(\alpha\beta(x)) > h_G(x)$. In particular, if $\beta \in E_p(G)$, then $\alpha\beta$ is bounded and satisfies the foregoing condition. Thus, for $\beta \in E_p(G)$, $\alpha\beta \in H_p(G)$ which by 2.7 is $J(E_p(G))$. Consequently, $\alpha \in J(E(G), E_p(G))$ by definition. Conversely, suppose $\alpha \in J(E(G), E_p(G))$, $x \in G[p]$ and $h_G(x) < \infty$. The existence of a bounded endomorphism $\beta$ such that $\beta(x) = x$ is easy to verify (see, for example, [3], Theorem 24.7). By definition, $\alpha\beta \in J(E_p(G))$. Consequently, $h_G(\alpha(x)) = h_G(\alpha\beta(x)) > h_G(x)$.

The following two results will be needed later.

LEMMA 2.11. *Let $\alpha$ be any automorphism of the $p$-group $G$ without* *elements of infinite height. If* $\beta \in J(E(G), E_p(G))$, *then* $\alpha - \beta$ *is one-to-one.*

*Proof.* Suppose $0 \neq x \in G[p]$ and $(\alpha - \beta)(x) = 0$. Then by 2.10,

$$h_G(x) < h_G(\beta(x)) = h_G(\alpha(x)) \leqq h_G(\alpha^{-1}(\alpha(x))) = h_G(x) ,$$

a contradiction. Thus, $\ker(\alpha - \beta) \cap G[p] = 0$. This is enough to ensure that $\alpha - \beta$ is one-to-one.

THEOREM 2.12. *If* $G$ *is without elements of infinite height and has no proper isomorphic subgroups, then* $J(E(G), E_p(G)) = J(E(G))$.

*Proof.* If $\alpha \in J(E(G), E_p(G))$, then $1 - \alpha$ is an isomorphism by Lemma 2.11. Since $G$ has no proper isomorphic subgroups, $1 - \alpha$ is an automorphism. Therefore, $\alpha$ is quasi-regular for each $\alpha \in J(E(G), E_p(G))$ (see [4], p. 7). Since $J(E(G), E_p(G))$ is a right ideal, it follows that $J(E(G), E_p(G)) \subseteqq J(E(G))$ ([4], Theorem 1, p. 9). Finally, $J(E(G)) \subseteqq J(E(G), E_p(G))$ by 2.8 (a).

3. **Realizations of** $E(G)$. The primary concern of this paper is with the endomorphism rings of $p$-primary groups without elements of infinite height. The study of such rings can be greatly eased with the employment of some fairly simple notions.

Let $G$ be a $p$-group without elements of infinite height and $B = \sum_{n \in N} B_n$ a basic subgroup of $G$. Let $\bar{B}$ denote the closure (or torsion completion) of $B$. The group $\bar{B}$ can be defined as the torsion subgroup of the direct product $\prod_{n \in N} B_n$. That is,

$$\bar{B} = \{x \in \prod_{n \in N} B_n \mid p^k x = 0 \text{ for some } k \in N\} .$$

Naturally, $B$ is identified with the subgroup of $\bar{B}$ consisting of those elements which have at most a finite number of nonzero components. Thus, $B$ is a pure subgroup of $\bar{B}$. It is well known that there is a $B$-isomorphism of $G$ onto a pure subgroup of $\bar{B}$ (see [3], § 33). Thus, in a sense, the study of $p$-groups without elements of infinite height can be reduced to the study of pure subgroups of suitable closed groups $\bar{B}$.

It has already been asserted that $G$ should be a $p$-group with fixed basic subgroup $B$. In order that the above remarks will apply to $G$, require, in addition, that $G$ be without elements of infinite height. That is, both $B$ and $\bar{B}$ are fixed and $G$ is a pure subgroup of $\bar{B}$ which contains $B$.

If $\alpha, \beta$ are endomorphisms of $G$ which agree on $B$, then $B$ is contained in the kernel of the difference $\gamma = \alpha - \beta$. Thus, $\gamma(G)$ is a homomorphic image of the divisible group $G/B$, and, for this reason, is divisible. Since $G$ is reduced and since $\gamma(G) \subseteqq G$, it follows that $\gamma(G) = (\alpha - \beta)(G) = 0$. Thus, $\alpha = \beta$. Consequently, if $G$ is a reduced $p$-group, then every endomorphism of $G$ is completely determined by its effect on the elements of any basic subgroup.

By 2.2 and the above remarks, it follows that each bounded endomorphism of $B$ has a unique extension to an endomorphism of $G$. Because of this, it may be assumed that $E(G)$, the endomorphism ring of $G$, contains an embedded copy, denoted by $E_p(B)$, of the ring of all bounded endomorphisms of $B$. Thus, identify $E_p(B)$ with

$$\{\alpha \in E_p(G) \mid \alpha(B) \subseteqq B\} .$$

Suppose that $B \subseteqq G \subseteqq \bar{B}$ where $G$ is a pure subgroup of $\bar{B}$. It has been shown that every endomorphism of $G$ has a unique extension to $\bar{B}$ (see, for example, [6], pp. 84–85). Thus, it is possible to adopt the very useful convention of identifying the endomorphism ring of $G$ with the subring of the endomorphism ring of $\bar{B}$ consisting of endomorphisms of $\bar{B}$ which map $G$ into itself. That is,

$$E(G) = \{\alpha \in E(\bar{B}) \mid \alpha(G) \subseteqq G\} .$$

With this identification, $E_p(G)$ (the torsion subring of $E(G)$) becomes a subring of $E_p(\bar{B})$; namely,

$$E_p(G) = \{\alpha \in E_p(\bar{B}) \mid \alpha(G) \subseteqq G\} .$$

It is reasonable to expect the above identifications to carry over in some way to the images $\mu(E(G))$ where $\mu$ is the map defined in Theorem 2.9. The following results show that this is indeed the case.

Let $\xi$ be the map of Theorem 2.9 developed for $E(\bar{B})$. Then by using the definition of $\xi$ and the above convention, it is not hard to show, for pure subgroups $G$ of $\bar{B}$ containing $B$, that $\xi \mid E(G)$ and the map $\mu$, defined in 2.9 for $E(G)$, are identical. Because of this, it is possible to confine the investigation of all such maps $\mu$ to the map $\xi$ and its restrictions to subrings of $E(\bar{B})$.

By way of summation, the following is given.

LEMMA 3.1. *Let $G$ be pure subgroup of $\bar{B}$ which contains $B$. Let $\xi$ be the map of Theorem 2.9 defined for the $p$-group $\bar{B}$. The restriction of $\xi$ to $E(G)$ and the map of 2.9 developed for $G$ agree. Moreover, $J(E(G), E_p(G)) = J(E(\bar{B}), E_p(\bar{B})) \cap E(G)$.*

LEMMA 3.2. *If $G = B$ or $G = \bar{B}$, then $\xi(E(G)) = \prod M_n$.*

*Proof.* Suppose $(\alpha_1, \alpha_2, \cdots)$ is an arbitrary element of $\prod M_n$.

Each $\alpha_i$ ($i = 1, 2, \cdots$) may be considered as an endomorphism of $B_i[p]$. By 2.1, each $\alpha_i$ has an extension to an endomorphism $\beta_i$ of $B$ such that $\beta_i(B_j) = 0$ if $i \neq j$. Let $\alpha$ be the endomorphism of $B$ determined by the conditions:

$$\alpha(b_i) = \beta_i(b_i) \text{ for } b_i \in B_i \ \ i = 1, 2, \cdots .$$

By Lemma 2.2, $\alpha$ can be extended to $\bar{B}$. In either case, $\xi(\alpha) = (\alpha_1, \alpha_2, \cdots)$.

Up to this point it has been shown that $\prod M_n$ can be realized as a homomorphic image of $E(B)$ and $E(\bar{B})$. Using an example of R. S. Pierce, it can be shown that not every pure subgroup $G$ of $\bar{B}$ which contains $B$ can be so classified.

First, consider the ring of $p$-adic integers, $R_p$ (see [3], § 6). This ring can be thought of as the collection of all infinite sums of the form

$$r = r_0 + r_1 p + r_2 p^2 + \cdots$$

where $0 \leqq r_i < p$. Suppose $x \in G$, and $r \in R_p$ where

$$r = r_0 + r_1 p + r_2 p^2 + \cdots$$

and $0 \leqq r_i < p$. It is possible to assign a meaning to the product $rx$, namely,

$$rx = r_0 x + r_1 px + r_2 p^2 x + \cdots + r_n p^n x$$

where $n$ is any integer greater than $E(x)$. Clearly, this definition is independent of the integer $n$. It is easy to check that with this definition, $G$ becomes an $R_p$-module. Consequently, every element $r$ of $R_p$ induces an endomorphism of $G$, $x \rightarrow rx$, which will also be labeled $r$. What is more important, it is not difficult to show that this correspondence, between the elements of $R_p$ and the elements of $E(G)$, is a ring isomorphism. With this in mind, it is possible to assume that $R_p$ is a subring of the ring of all endomorphisms of $G$.

DEFINITION 3.3. An endomorphism $\alpha$ of the $p$-group $G$ is said to be a *small endomorphism* of $G$ provided the following condition is satisfied:

(*) for all $k \geqq 0$ there exists an integer $n$ such that $0(x) \leqq k$ and $h_G(x) \geqq n$ imply $\alpha(x) = 0$.

REMARK. The concept of small endomorphism is due to R. S. Pierce and can be found in his paper [8]. The equivalence of the above definition and that appearing in [8] can be shown using 3.1 and 2.10 in the above mentioned paper.

It is an easy consequence of the above definition that the collection of all small endomorphisms forms a subring $E_s(G)$ of the ring $E(G)$. Moreover, $E_s(G)$ is an ideal of $E(G)$.

R. S. Pierce has shown that there exists a $p$-group $H$ without elements of infinite height such that $E(H) = E_s(H) + R_p$ ([8], p. 297). The following results demonstrate a few of the many curious properties of such groups.

LEMMA 3.4. *If* $E(H) = E_s(H) + R_p$, *then* $E_s(H)$ *and* $R_p$ *are disjoint.*

*Proof.* Let $r \in R_p$ and $r = \sum_{i=0}^{\infty} r_i p^i$ where $0 \leqq r_i < p$. By definition, $r$ is a small endomorphism if and only if for all $k \geqq 0$ there exists an integer $n$ such that $x \in H$, $E(x) \leqq k$ and $h_H(x) \geqq n$ collectively imply $r(x) = 0$. Let $h$ be the least index such that $r_h \neq 0$. Let $k > h$, and for $l > k$ let $x_l = p^{l-k} b_l$ where $b_l \in B_l$, $E(b_l) = l$ and $h_H(b_l) = 0$. (Recall that $B = \sum_{n \in N} B_n$ is a basic subgroup of $H$). Then $x_l \in B \subseteqq H$, $E(x_l) = k$, $r(x_l) \neq 0$, and $h_H(x_l)$ increases indefinitely as $l$ increases. Thus $r$ is not a small endomorphism, and $E_s(H) \cap R_p = 0$.

LEMMA 3.5. $\xi(E_s(H)) = \sum M_n$ *and* $\xi(R_p) = \{1\}$ *where* 1 *is the identity of* $\sum^c M_n$.

*Proof.* It is easy to see from the definitions of $\xi$ and $E_s(H)$ that $\xi(E_s(H)) \subseteqq \sum M_n$. Since $E_p(B) \subseteqq E_p(H) \subseteqq E_s(H)$ and $\xi(E_p(B)) = \sum M_n$, it follows that $\xi(E_s(H)) = \sum M_n$. Suppose $r = \sum_{i=0}^{\infty} r_i p^i \in R_p$. Write $r = r_0 + ps$ where $s = \sum_{i \in N} r_i p^{i-1}$. Clearly,

$$\xi(r) = \xi(r_0 + ps) = \xi(r_0) + \xi(ps) = \xi(r_0) \in \{1\}.$$

LEMMA 3.6.

$$\operatorname{Ker}(\xi \mid E(H)) = J(E_s(H)) + J(R_p) = J(E(H), E_p(H)) .$$

*Proof.* By 3.1, $\operatorname{Ker}(\xi \mid E(H)) = J(E(H), E_p(H))$. To show that $\operatorname{Ker}(\xi \mid E(H)) = J(E_s(H)) + J(R_p)$, let $\alpha + r$ be an arbitrary element in $E(H)$ where $\alpha \in E_s(H)$ and $r \in R_p$. Suppose, in addition, that $\xi(\alpha + r) = 0$. Since $\sum M_n$ and $\{1\}$ are obviously independent and since $\xi(\alpha + r) = \xi(\alpha) + \xi(r) \in \sum M_n + \{1\}$ by the foregoing lemma, $\xi(\alpha + r) = 0$ if and only if both $\xi(\alpha) = 0$ and $\xi(r) = 0$. Surely, $\xi(r) = 0$ if and only if $r \in pR_p$. Since $pR_p$ is the unique maximal ideal in $R_p$, $J(R_p) = pR_p$ (see [4], p. 9). Thus, the conditions $\xi(r) = 0$ and $r \in J(R_p)$ are equivalent. Moreover, $\xi(\alpha) = 0$ and $\alpha \in E_s(H)$ if and only if $\alpha \in J(E(H), E_p(H)) \cap E_s(H)$. By Lemmas 2.9 and 2.8 (d) of this paper and 14.4 of [8], $\xi(\alpha) = 0$ if and only if

$$\alpha \in J(E(H), E_s(H)) \cap E_s(H) = J(E_s(H)).$$

Thus,

$$\operatorname{Ker}(\xi \mid E(H)) = J(E_s(H)) + J(R_p) = J(E(H), E_p(H)) .$$

LEMMA 3.7. *If* $K(G) = \{\alpha \in E(G) \mid \alpha(G[p]) = 0\}$, *then* $K(G)$ *is a two sided ideal of* $E(G)$ *which is contained in the Jacobson radical of* $E(G)$.

*Proof.* It is obvious that $K(G)$ is an ideal of $E(G)$. Moreover, if $\alpha \in K(G)$, then $\ker(1-\alpha) \cap G[p] = 0$ and $(1-\alpha)(G[p] \cap p^n G) = G[p] \cap p^n G$. Thus, $1-\alpha$ is an automorphism by 2.3. It follows that $K(G)$ is a quasi regular ideal in $E(G)$; and is, therefore, contained in the Jacobson radical of $E(G)$ (see [4], p. 9, Theorem 1).

THEOREM 3.8. $E(H)/J(E(H)) = E(H)/J(E(H), E_p(H))$ *is ring isomorphic to* $\sum M_n + \{1\}$.

*Proof.* By 3.5 and 3.6, $\xi$ maps $E(H)$ onto $\sum M_n + \{1\}$ with kernel $J(E(H), E_p(H)) = J(E_s(H)) + J(R_p)$. Also, by 2.8 (a), $J(E(H)) \subseteqq J(E_s(H)) + J(R_p)$. Thus, it remains only to show that $J(R_p)$ and $J(E_s(H))$ are contained in $J(E(H))$. Since $E_s(H)$ is a two sided ideal of $E(H)$, $J(E_s(H)) = J(E(H)) \cap E_s(H)$ (see [4], p. 10). Thus, $J(E_s(H)) \subseteqq J(E(H))$. Since $J(R_p) = pR_p$ ($pR_p$ is the unique maximal ideal of $R_p$) and since $J(E(H))$ is an ideal, Lemma 3.7 is enough to insure that $J(R_p) \subseteqq J(E(H))$.

**4. An extension property.** In § 3, it was shown, using suitable pure subgroups of $\bar{B}$, that there are at least two distinct rings of the form $E(G)/J(E(G), E_p(G))$, namely, $\prod M_n$ and $\sum M_n + \{1\}$. It is the objective of the remainder of this paper to investigate some of the possible images $\xi(E(G))$ for $B \subseteqq G \subseteqq \bar{B}$.

For the duration, assume that $B = \sum_{i \in N} B_i$ where each $B_i = \{b_i\}$ is of rank one and of order $p^i$. In this case each $M_i$ automatically becomes fixed as a single copy of $Z_p$. That is, each $M_i$ will be the ring of all endomorphisms of a cyclic group, $\{c_i\}$, of order $p$.

For a subset $A$ of $N$, let $t(A)$ be the element of $\prod_{n \in N} M_n$ defined by the conditions

$$t(A)(c_j) = \begin{cases} c_j & \text{if } j \in A \\ 0 & \text{if } j \notin A . \end{cases}$$

It is obvious that if $r$ is any element of $\prod_{n \in N} M_n$ and if for each $i = 0, 1, \cdots, p-1$ $A_i(r) = \{j \in N \mid r(c_j) = ic_j\}$, then $r$ can be written in the form $r = \sum_{i=0}^{p-1} it(A_i(r))$.

LEMMA 4.1. *Let $R$ be any subring of $\prod_{n\in N} M_n$ with identity $e$. (The identity of $\prod_{n\in N} M_n$ and $e$ are not assumed to be identical.) Then $e = t(M)$ for some subset $M$ of $N$. Moreover, the collection $K(R) = \{A \subseteqq N \mid t(A) \in R\}$ forms a Boolean algebra of subsets of $M$.*

*Proof.* Using Fermat's theorem

$$e = e^{p-1} = \left(\sum_{i=0}^{p-1} it(A_i(e))\right)^{p-1} = \sum_{i=0}^{p=1} i^{p-1}t(A_i(e)) = \sum_{i=1}^{p=1} t(A_i(e)) = t(M)$$

where $M = \{i \in N \mid e(c_i) \neq 0\}$. If $t(A), t(B)$ are members of $R$, then $t(A \cap B) = t(A)t(B) \in R$ and $t(A \cap B) = t(A) + t(B) - t(A \cap B) \in R$. Since $t(A) = e \cdot t(A) = t(M) \cdot t(A) = t(M \cap A)$, it follows that $A \subseteqq M$ for all $A \in K(R)$. Thus, $t(M - A) = t(M) - t(A) = e - t(A) \in R$ for all $A \in K(R)$. This shows that $K(R)$ does indeed form a subalgebra of $P(M) = \{A \mid A \subseteqq M\}$.

LEMMA 4.2. *Let $R$ be a subring of $\prod_{n\in N} M_n$ with identity $e = t(M)$. If $r \in R$, then $t(A_k(r)) \in R$ for each $k = 0, 1, \cdots, p-1$.*

*Proof.*

$$r = 0 \cdot t(A_0(r)) + t(A_1(r)) + 2t(A_2(r)) + \cdots + (p-1)t(A_{p-1}(r)) .$$

Consider the product

$$s = \prod_{i\neq k, i=0,1,\cdots,p-1} (ie - r) .$$

It follows that $s \in R$. Clearly, if $i \notin A_k(r)$, then $s(c_j) = 0$ since $j \in A_i(r)$ for some $i$ and

$$(ie - r)(c_j) = ic_j - r(c_j) = ic_j - ic_j = 0 .$$

Also, if $j \in A_k(r)$, then

$$\begin{aligned} s(c_j) = (0 - k)(1 - k)(2 - k) \cdots ((k - 1) - k)((k + 1) - k) \\ \cdots ((p - 1) - k)(c_j) = (p - 1)!\, c_j . \end{aligned}$$

By Wilson's theorem, $(p-1)! \equiv -1$ (modulo $p$); consequently, $t(A_k(r)) = -s \in R$.

Suppose $R$ is a subring of $\prod M_n$ which contains $\sum M_n + \{1\}$. For each $A \in K(R)$, let $\rho(A) = \sum_{i\in A} \rho_i$. Define $\Gamma(R)$ to be the subgroup of $E(\bar{B})$ generated by the collection $\{\rho(A) \mid A \in K(R)\}$. Using Lemma 4.1 and 4.2 some elementary properties of $\Gamma(R)$ can be stated.

LEMMA 4.3. *If $\alpha \in \Gamma(R)$, then there exists an integer $n \geqq 0$, integers $a_1, a_2, \cdots, a_n$ and disjoint elements $A_1, A_2, \cdots, A_n$ in $K(R)$ such that*

$$\alpha = a_1\rho(A_1) + a_2\rho(A_2) + \cdots + a_n\rho(A_n) .$$

*Moreover, the group $\Gamma(R)$ is a subring of $E(\bar{B})$.*

*Proof.* For the first statement, induction can be used. For the induction step, it is enough to show that if

$$\alpha = a_1\rho(A_1) + a_2\rho(A_2) + \cdots + a_{n-1}\rho(A_{n-1}) + a_n\rho(A_n)$$

where $A_1, \cdots, A_{n-1}$ are disjoint, then the result holds. Using 4.1, $A_1, \cdots, A_n \in K(R)$ imply that

$$A_1 \cap A_n, \cdots, A_{n-1} \cap A_n;\ A_1 - A_n, \cdots, A_{n-1} - A_n;$$

and $A_n - \bigcup_{i=1}^{n-1} A_i$ are members of $K(R)$. Moreover, these sets are disjoint. Thus, if $\alpha$ is written

$$\alpha = a_1\rho(A_1 - A_n) + \cdots + a_{n-1}\rho(A_{n-1} - A_n) + (a_1 + a_n)\rho(A_1 \cap A_n) + \cdots + (a_{n-1} + a_n)\rho(A_{n-1} - A_n) + a_n\rho\left(A_n - \bigcup_{i-1}^{n-1} A_i\right) ,$$

then it is easily checked that this is the desired decomposition. To show that $\Gamma(R)$ and the subring of $E(\bar{B})$ generated by $\Gamma(R)$ are identical, it is enough to show that $\Gamma(R)$ is closed under composition. It suffices to note that if $A_1, A_2 \in K(R)$, then $\rho(A_1)\rho(A_2) = \rho(A_1 \cap A_2) \in \Gamma(R)$. This is obvious by Lemma 4.1 and the definition of $\Gamma(R)$.

LEMMA 4.4. $R = \xi(\Gamma(R))$.

*Proof.* If $r \in R$, then $r = \sum_{i=0}^{p-1} it(A_i(r))$ where $A_i(r) \in K(R)$ (see 4.2). Let $\alpha = \sum_{i=0}^{p=1} i\rho(A_i(r))$. Then $\alpha \in \Gamma(R)$ and $\xi(\alpha) = r$. Thus, $R \subseteqq \xi(\Gamma(R))$. On the other hand, suppose

$$\alpha = a_1\rho(A_1) + \cdots + a_n\rho(A_n) \in \Gamma(R) ,$$

where $a_1, \cdots, a_n \in Z$ and $A_1, \cdots, A_n \in K(R)$. Applying $\xi$,

$$\xi(\alpha) = a_1\xi\rho(A_1) + \cdots + a_n\xi\rho(A_1) + \cdots + a_n\xi(\rho(A_n)) = a_1t(A_1) + \cdots + a_nt(A_n) \in R$$

(see the definition of $K(R)$ in Lemma 4.1).

The following lemma is needed before the main result of this section can be given.

LEMMA 4.5. *Let $y = h\sum_{j\geq k} a_jp^{j-k}b_j$ where $h \in Z$, $k \in N$ and each $a_j(j \geqq k)$ is an integer such that $0 \leqq a_j < p$. If $A \subseteqq N$ and $i \in N$, then $p^{i-1}y \neq 0$ and $\rho(A)(p^{i-1}y) \in B$ imply that $\rho(A)(y) \in B$.*

*Proof*. Suppose $\rho(A)(y) \notin B$. Then if $A_0 = \{i \in A \mid a_i \neq 0\}$, $A_0$ is infinite. Since $\rho(A)(p^{i-1}y) \in B$, there is some $n \in A_0$ such that $\rho_n\rho(A)(p^{i-1}y) = 0$. Thus,

$$0 = \rho_n\rho(A)(p^{i-1}y) = \rho_n(p^{i-1}y) = p^{i-1}ha_np^{n-k}b_n = ha_np^{n+i-k-1}b_n ,$$

so that $p^{k+1-i}$ divides $h$. Since $p^{i-1}y \neq 0$, this cannot be the case.

THEOREM 4.6. *Let $G$ be a pure subgroup of $\bar{B}$ such that $B \subseteqq G$ and $\gamma(G) \subseteqq G$ for each $\gamma \in \Gamma(R)$. Suppose $x \in \bar{B}[p]$ is such that $\Gamma(R)(x) \cap G[p] \subseteqq B[p]$. Then there is a pure subgroup $H$ of $\bar{B}$ such that*

(i) $B \subseteqq G \subseteqq H$

(ii) $H[p] = G[p] + \Gamma(R)(x)$

(iii) $\gamma(H) \subseteqq H$ *for each* $\gamma \in \Gamma(R)$.

*Proof*. Write $x = \sum_{i \geqq k_0} a_ip^{i-1}b_i$ where $k_0 > 0$, $0 \leqq a_i < p$ for $i \geqq k_0$ and $a_{k_0} \neq 0$. Let $K$ be the subgroup of $\bar{B}$ generated by $B$ and the collection consisting of all sums of the form $\sum_{i \geqq k} a_ip^{i-k}b_i$ where $k \geqq k_0$. Consider the group $\bar{K}$ generated by all elements of the form $\gamma(z)$ for $z \in K$ and $\gamma(R)$. It is claimed that the group $H = \bar{K} + G$ has all the desired properties. First, note that $K$ is exactly the subset of $\bar{B}$ consisting of all elements which can be written as $b + h\sum_{j \geqq k} a_jp^{j-k}b_j$ for some $b \in B$, $h \in Z$ and $k \in N$ (the integers $a_j$ for $j \geqq k$ are determined by the element $x$). Also, if $y = b + h\sum_{j \geqq k} a_jp^{j-k}b_j \in K$, then $y$ may be written as $y = b' + p^nh\sum_{j \geqq k+n} a_jp^{j-(k+n)}b_j$, where $b' = b + h\sum_{j=k}^{k+n-1} a_jp^{j-k}b_j \in B$ and $\sum_{j \geqq k+n} a_jp^{j-(k+n)}b_j \in K$. Thus, $K/B$ is divisible. Suppose $n \in N, \gamma_1, \cdots, \gamma_k \in \Gamma(R)$ and $x_1, \cdots, x_k \in K$. Using the divisibility of $K/B$, choose $y_1, \cdots, y_k \in K$ such that $x_i - p^ny_i \in B$ for each $i = 1, \cdots, k$. Since $\gamma \in \Gamma(R)$ implies $\gamma(B) \subseteqq B$, it follows that

$$\begin{aligned}\gamma_1(x_1) + \cdots + \gamma_k(x_k) - p^n(\gamma_1(y_1) + \cdots + \gamma_k(y_k)) \\ = \gamma_1(x_1) - \gamma_1(p^ny_1) + \cdots + \gamma_k(x_k) - \gamma_k(p^ny_k) \\ = \gamma_1(x_1 - p^ny_1) + \cdots + \gamma_k(x_k - p^ny_k) \in B .\end{aligned}$$

This shows that $\bar{K}/B$ is divisible. (Note that $B \subseteqq \bar{K}$ since $1 \in \Gamma(R)$ and $B \subseteqq K$.) Now, both $\bar{K}/B$ and $G/B$ are divisible. Consequently, $H = \bar{K} + G$ is a pure subgroup of $\bar{B}$ since $(\bar{K} + G)/B = (\bar{K}/B) + (G/B)$ is a sum of divisible groups and hence divisible. Since, $\alpha, B \in \Gamma(R)$ imply that $\alpha\beta \in \Gamma(R)$ (see 4.3), it follows that $\gamma(\bar{K}) \subseteqq \bar{K}$ for all $\gamma \in \Gamma(R)$. Thus, $\gamma(H) \subseteqq H$ for each $\gamma \in \Gamma(R)$. It remains only to show that $H[p] = G[p] + \Gamma(R)(x)$. First, suppose that

$$y = h\sum_{j \geqq k} a_jp^{i-k}b_j \in K \quad \text{and} \quad A \in K(R)$$

Then $\rho(A)(y) \in G$ if and only if $\rho(A)(y) \in B$. To show that this assertion is correct, suppose that $\rho(A)(y)$ is a member of $G$. Then, if $i = E(y)$, $p^{i-1}y = h'x - b'$ for suitable $h' \in Z$ and $b' \in B$. Thus, since $\Gamma(R)(x) \cap G \subseteqq B$ and $B \subseteqq G$, it follows that $\rho(A)(p^{i-1}y + b') = \rho(A)(h'x) \in B$ and that $\rho(A)(p^{i-1}y) \in B$. But, $\rho(A)(p^{i-1}y) \in B$, $p^{i-1}y \neq 0$ and $y = h \sum_{j \geqq k} a_j p^{j-k} b_j$ imply, via 4.5 and the restriction on the $a_i (i \geqq k)$, that $\rho(A)(y) \in B$. The converse is trivial. Let

$$x_1, x_2, \cdots, x_n \in K, z \in G \quad \text{and} \quad \gamma_1, \gamma_2, \cdots, \gamma_n \in \Gamma .$$

Suppose

$$p(\gamma_1(x_1) + \gamma_2(x_2) + \cdots + \gamma_n(x_n) + z) = 0 .$$

For each $i = 1, 2, \cdots, n$, let $x_i = d_i + h_i \sum_{j \geqq k_i} a_j p^{j-k_i} b_j$ where $d_i \in B$, $h_i \in Z$ and $k_i \in N$. Let $k'$ be any positive integer greater than each of the integers $k_1, k_2, \cdots, k_n$. It is easily checked that there exist integers $m_1, m_2, \cdots, m_n$ and elements $d'_1, d'_2, \cdots, d'_n$ of $B$ such that for each $i = 1, \cdots, n$

$$x_i = d'_i + m_i \sum_{j \geqq k'} a_j p^{j-k'} b_j .$$

Thus, if $y = \sum_{j \geqq k'} a_j p^{j-k'} b_j$, then

$$\begin{aligned} \gamma_1(x_1) + \cdots + \gamma_n(x_n) &= \gamma_1(d'_1 + m_1 y) + \cdots + \gamma_n(d'_n + m_n y) \\ &= \gamma_1(d'_1) + \cdots + \gamma_n(d'_n) + (m_1\gamma_1 + \cdots + m_n\gamma_n)(y) \\ &= b + \gamma(y) \end{aligned}$$

where $b \in B$ and $\gamma \in \Gamma$. Since $\gamma \in \Gamma$, it is possible to write $\gamma = e_1\rho(A_1) + \cdots + e_m\rho(A_m)$ where $A_1, \cdots, A_m$ are disjoint members of $K(R)$ and where $e_1, \cdots, e_m \in Z$ (see 4.3). Now,

$$p(\gamma_1(x_1) + \gamma_2(x_2) + \cdots + \gamma_n(x_n) + z) = 0$$

implies $p(b + \gamma(y) + z) = 0$; and, therefore, $p\gamma(y) \in G + B = G$. Suppose that $e_i\rho(A_i)(y) \notin B$ for some $i = 1, \cdots, m$. Then since $b \in G$, $p\gamma(y) \in G$ and $\rho(A_i)(G) \subseteqq G$, it follows that

$$\rho(A_i)(p\gamma(y)) = pe_i\rho(A_i)(y) = \rho(A_i)(pe_i y) \in G .$$

Thus, as was noted, $\rho(A_i)(pe_i y) \in B$. Now,

$$\begin{aligned} \rho(A_i)(pe_i y) &= \rho(A_i)\Big(pe_i \sum_{j \geqq k'} a_j p^{j-k'} b_j\Big) \\ &= pe_i \sum_{\substack{j \geqq k' \\ j \in A_i}} a_j p^{j-k'} b_j \in B . \end{aligned}$$

Since, by assumption, $e_i\rho(A_i)(y) \notin B$, it follows that $\rho(A_i)(y) \notin B$. Thus, $p^{k'-1}$ divides $e_i$. Therefore, $e_i y = e'_i x - b'$ for suitable $e'_i \in Z$ and $b' \in B$.

Consequently, $e_i\rho(A_i)(y) = \rho(A_i)(e_iy) \in \Gamma(R)(x) + B$. It follows that $\gamma_1(x_1) + \cdots + \gamma_n(x_n) \in \Gamma(R)(x) + B$ and that

$$\gamma_1(x_1) + \cdots + \gamma_n(x_n) + z \in \Gamma(R)(x) + G .$$

Thus, $\gamma_1(x_1) + \cdots + \gamma_n(x_n) + z = y + w$ where $y \in \Gamma(R)(x)$ and $w \in G$. Also,

$$0 = p(\gamma_1(x_1) + \cdots + \gamma_n(x_n) + z) = p(y + w) = pw$$

and $w \in G[p]$. This shows that $H[p] \subseteqq G[p] + \Gamma(R)(x)$. The opposite inclusion is obvious.

**5. The image.** This section is devoted to the construction of a class of pure subgroups of $\bar{B}$ having suitably restricted endomorphism rings. The methods used here are similar to those employed by P. Crawley in [2] and R. S. Pierce in [7].

DEFINITION 5.1. (*R. S. Pierce*) A family $\mathscr{F}$ of subsets of a set $F$ is called *weakly independent* if whenever $A_0, A_1, \cdots, A_n$ are distinct elements of $\mathscr{F}$, then $A_0$ is not contained in the union of the remaining sets $A_1, A_2, \cdots, A_n$.

THEOREM 5.2. (*R. S. Pierce*) *Let $F$ be a set of infinite cardinality $\varphi$. If $\psi$ is a cardinal number such that $0 < \psi \leqq \varphi$, then there is a family $\mathscr{F}$ of subsets of $F$ such that*

(a) *$\mathscr{F}$ is weakly independent,*

(b) *$|A| = \psi$ for all $A \in \mathscr{F}$,*

(c) *$|\mathscr{F}| = \varphi^\psi$.*

*Proof.* (See [8], p. 261.)

At this point it is convenient to set $\Theta = \{\alpha \mid \bar{B}[p] \mid \alpha \in E(\bar{B})\}$. It is clear that $\Theta$ is a ring with identity. For the moment, only the additive group structure of $\Theta$ will be considered.

LEMMA 5.3. *Let $\Gamma = \{\alpha_0 = 0, \alpha_1, \alpha_2, \cdots\}$ be any countable subgroup of $\Theta$ satisfying the following condition:*

(*) *for all nonzero $\alpha \in \Gamma$, $\alpha(c_j) \neq 0$ for an infinite number of indices $j \in N$.*

*There is a collection $T(\Gamma)$ of element in $\bar{B}[p]$ such that*

(i) $|T(\Gamma)| = 2^{\aleph_0}$,

(ii) $\sum_{x \in T(\Gamma)} \Gamma(x)$ *is direct* ($\Gamma(x) = \{\alpha(x) \mid \alpha \in \Gamma\}$).

(iii) $\alpha_i(x) \neq \alpha_j(x)$ *for all* $x \in T(\Gamma)$ *and for all* $i \neq j$,

(iv) $\alpha_i(x) = 0$ *for some* $x \in T(\Gamma)$ *implies* $\alpha_i = 0$.

*Proof.* Let $K = N \times N$. Well order $K$ in the following way: $(i, j) < (k, h)$ if $i + j < k + h$ or if $i + j = k + h$ and $i < k$. Now, each element of $\Gamma$ satisfies (*). Thus, since the set

$$\{(i, j) \in K \mid (i, j) < (k, h)\}$$

is finite for all elements $(k, h) \in K$, it is possible to define, inductively, an order preserving one-to-one map $f$ of $K$ into $N$ such that $h_{\bar{B}}(\alpha_i(c_{f(i,j)}))$ is finite (i.e., $\alpha_i(c_{f(i,j)}) \neq 0$) and is greater than the height or every nonzero element in the finite subgroup of $\bar{B}[p]$ generated by the collection $\{\alpha_k(c_{f(m,n)}) \mid k \leqq i \text{ and } (m, n) < (i, j)\}$. Let $\mathscr{F}$ be any weakly independent collection of subsets of $N$ such that $|\mathscr{F}| = 2^{\aleph_0}$. If $S \in \mathscr{F}$, let $x(S) \in \bar{B}[p]$ be defined by the expression:

$$x(S) = \sum_{j \in S} c_{f(i,j)} .$$

Let $T(\Gamma) = \{x(S) \mid S \in \mathscr{F}\}$. Suppose $S_1, S_2, \cdots, S_{n_0} \in \mathscr{F}$ are distinct, $x_i = x(S_i)$ for $i = 1, 2, \cdots, n_0$ and

$$\sum_{i=1}^{n_0} \alpha_{k_i}(x_i) = 0$$

for positive integers $k_1, k_2, \cdots, k_{n_0}$. Since $\mathscr{F}$ is weakly independent, there exists for each $i = 1, 2, \cdots, n_0$ an integer

$$m_i \in S_i - \bigcup_{\substack{j \neq i \\ j \leqq n_0}} S_j .$$

Let $k_i$ be the largest integer in the collection $\{k_1, \cdots, k_{n_0}\}$. Let $h_i = h_{\bar{B}}(\alpha_{k_i}(c_{f(k_i,m_i)})) + 1$. It follows that

$$(1) \qquad (1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i,m_i)}) \neq 0$$

and

$$(2) \qquad \begin{aligned} &(1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i,m_i)}) + (1 - \pi_{h_i})\alpha_{k_i}(x - c_{f(k_i,m_i)}) \\ &\quad + (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1,2,\cdots,n_0}} \alpha_{k_j}(x_j) = 0 . \end{aligned}$$

Now,

$$\begin{aligned} &(1 - \pi_{h_i})\alpha_{k_i}(x - c_{f(k_i,m_i)}) + (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1,2,\cdots,n_0}} \alpha_{k_j}(x_j) \\ &\quad = (1 - \pi_{h_i})\alpha_{k_i}(1 - \pi_{h_i})(x_i - c_{f(k_i,m_i)}) \\ &\quad + (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1,2,\cdots,n_0}} \alpha_{k_j}(1 - \pi_{h_i})(x_j) . \end{aligned}$$

Since $m_i \in S_i$, it follows from the definition of $x_i = x(S)$ and the order preserving property of the mapping $f$ that

$$(1 - \pi_{h_i})(x_i - c_{f(k_i, m_i)}) = \sum_{\substack{(m,n) < (k_i, m_i) \\ n \in S_i}} c_{f(m,n)} .$$

Hence, $\alpha_{k_i}(1 - \pi_{h_i})(x_i - c_{f(k_i, m_i)})$ belongs to the subgroup $S$ of $\bar{B}[p]$ generated by the collection

$$\{\alpha_k(c_{f(m,n)}) \mid k \leqq k_i, (m, n) < (k_i, m_i)\} .$$

Also, if $j \neq i$, then $m_i \notin S_j$ and $k_j \leqq k_i$. Therefore,

$$(1 - \pi_{h_i})(x_j) = \sum_{\substack{(m,n) < (k_i, m_i) \\ n \in S_j}} c_{f(m,n)} ;$$

and because of this, $\alpha_{k_j}((1 - \pi_{h_i})(x_j)) \in S$. Thus, from (1), (2) and the above, $(1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i, m_i)}) = (1 - \pi_{h_i})(z) \neq 0$ for some $z$ in $S$. It follows that $h_{\bar{B}}(\alpha_{k_i}(c_{f(k_i, m_i)}) = h_{\bar{B}}(z)$, a contradiction of the definition of the map $f$. Thus, $\sum_{x \in T(\Gamma)} \Gamma(x)$ is direct. Condition (i) is clear from the definition of $T(\Gamma)$. Condition (iv) follows from the preceding argument with $n = 1$. Since $\Gamma$ is a group, condition (iii) follows easily from (iv).

DEFINITION 5.4. Let $\Gamma$ be a subgroup of $\Theta$. An element $\alpha$ in $\Theta$ will be called $\Gamma$-exceptional provided there exists a collection $T(\Gamma, \alpha)$ of elements in $\bar{B}[p]$ such that

( i ) $|T(\Gamma, \alpha)| = 2^{\aleph_0}$,

(ii) $\Gamma(x), \Gamma(y), \{\alpha(x)\}, \{\alpha(y)\}$ are independent for all distinct $x, y \in T(\Gamma, \alpha)$,

(iii) $\alpha(x) \neq 0$ for all $x \in T(\Gamma, \alpha)$.

An endomorphism $\alpha \in E(\bar{B})$ will be called $\Gamma$-exceptional if $\alpha \mid \bar{B}[p]$ is $\Gamma$-exceptional.

REMARK. If $\alpha \in E(\bar{B})$ and $\alpha \mid \bar{B}[p] = 0$, then by 3.7 and 2.8 (a) $\alpha \in J(E(\bar{B}), E_p(\bar{B}))$. Thus, the kernel of the map $\alpha \rightarrow \alpha \mid B[p]$ is contained in $J(E(\bar{B}), E_p(\bar{B}))$. It follows that $\xi$ can be considered as a map from $\Theta$ to $\prod M_n$ by defining for each $\alpha \in \Theta$, $\xi(\alpha) = \xi(\beta)$ where $\beta \in E(B)$ and $\alpha = \beta \mid \bar{B}[p]$. Extensive use will be made of this convention in what follows.

LEMMA 5.5. *Let $\Gamma$ be any countable subgroup of $\Theta$. Suppose $\alpha \in \Theta$ is such that $\alpha \notin \Gamma$ and $\Delta = \{\Gamma, \alpha\}$ satisfies the following condition:*

(*) *for all nonzero $\beta \in \Delta$, $\beta(c_j) \neq 0$ for an infinite number of indices $j \in N$. Then $\alpha$ is $\Gamma$-exceptional.*

*Proof.* Since $\Delta = \{\Gamma, \alpha\}$ is obviously countable and satisfies (*), Lemma 5.3 can be applied to conclude that there exists a collection $T(\Delta)$ with the properties:

(i) $|T(\Delta)| = 2^{\aleph_0}$,

(ii) $\sum_{x \in T(\Delta)} \Delta(x)$ is direct.

(iii) $\gamma(x) \neq \beta(x)$ for all $x \in T(\Delta)$ and distinct $\beta, \gamma \in \Delta$,

(iv) $\beta \in \Delta$ and $\beta(x) = 0$ for some $x \in T(\Delta)$ implies $\beta = 0$.

Set $T(\Gamma, \alpha) = T(\Delta)$. Clearly, conditions (i) and (iii) of 5.4 are satisfied. Let $x, y \in T(\Gamma, \alpha)$ be distinct, and suppose there is a relation of the form $\beta(x) + k\alpha(x) + \gamma(y) + h\alpha(y) = 0$ where $\beta, \gamma \in \Gamma$ and $h, k \in Z$. By (ii), it is clear that both $\beta(x) + k\alpha(x) = 0$ and $\gamma(y) + h\alpha(y) = 0$. It follows by (iv) that $\beta + k\alpha = 0$ and $\gamma + h\alpha = 0$. Since $E(\alpha) = 1$ and $\alpha \notin \Gamma$, this last condition implies that both $h\alpha = 0$ and $k\alpha = 0$. Thus $\beta(x) = k\alpha(x) = \gamma(y) = h\alpha(y) = 0$, and condition (ii) of 5.4 is also satisfied. This completes the proof.

COROLLARY 5.6. *Let $\Gamma$ by any countable subgroup of $\Theta$ satisfying*

(*) *for all nonzero $\gamma \in \Gamma, \gamma(c_j) \neq 0$ for an infinite number of indices $j \in N$.*

*Suppose $\alpha \in \Theta$ is such that $\xi(\alpha)$ is not a member of $\xi(\Gamma) + (\sum M_n)$. Then $\alpha$ is $\Gamma$-exceptional.*

*Proof.* Clearly, $\alpha \notin \Gamma$ since $\xi(\alpha) \notin \xi(\Gamma)$. Consequently, it is enough to show that $\Delta = \{\Gamma, \alpha\}$ satisfies condition (*). Suppose, to the contrary, that there exist $n \in N$ and $\beta \in \Delta$ such that $\beta \neq 0$ and $\beta(c_j) = 0$ for all $j > n$. It is possible to write $\beta = \gamma + k\alpha$ where $\gamma \in \Gamma$ and $k \in Z$. Since $\Gamma$ satisfies (*) and $E(\alpha) = 1$, it can be assumed that $k \not\equiv 0$ (modulo $p$). Now, $\beta = \gamma + k\alpha$ and

$$\xi(k\alpha) = \xi(\beta - \gamma) = \xi(\beta) - \xi(\gamma) \in \sum M_n + \xi(\Gamma) .$$

Since $k$ is relatively prime to $p$, it follows that $\xi(\alpha) \in \sum M_n + \xi(\Gamma)$, a contradiction.

COROLLARY 5.7. *Let $\Gamma$ be any countable subgroup of $\Theta$ satisfying the following condition:*

(**) *for all nonzero $\gamma \in \Gamma$ there exists a sequence of integers $\{a_i\}_{i \in N}$ such that $\gamma(c_i) = a_i c_i$ for each $i \in N$, and $a_i c_i \neq 0$ for an infinite number of indices $i \in N$.*

*Let $\alpha \in \Theta$ be such that $\alpha(c_i) - \rho_i \alpha(c_i) \neq 0$ for an infinite number of indices $i \in N$. Then $\alpha$ is $\Gamma$-exceptional.*

*Proof.* If $\gamma \in \Gamma$, then $\gamma(c_i) - \rho_i \gamma(c_i) = a_i c_i - a_i c_i = 0$ for all $i \in N$. Thus, $\alpha \notin \Gamma$. As before, let $\Delta = \{\Gamma, \alpha\}$, suppose $\gamma + k\alpha \in \Delta$. If $k \equiv 0$ (modulo $p$), then either $\gamma = 0$ or $(\gamma + k\alpha)(c_i) = \gamma(c_i) \neq 0$ for an infinite number of indices $i \in N$. If $\gamma = 0$, then $\gamma + k\alpha = 0$; and there is nothing to show. Suppose $k \not\equiv 0$ (modulo $p$). It follows that

$$\begin{aligned}(1-\rho_i)(\gamma+k\alpha)(c_i) &= (\gamma+k\alpha)(c_i)-\rho_i(\gamma+k\alpha)(c_i)\\ &= (\gamma-\rho_i\gamma)(c_i)+k(\alpha-\rho_i\alpha)(c_i)\\ &= k(\alpha-\rho_i\alpha)(c_i)\neq 0\end{aligned}$$

for an infinite number of indices $i \in N$. Consequently, $\gamma + k\alpha$ must have this same property, and by 5.5, $\alpha$ is $\Gamma$-exceptional.

Let $R$ be any countable subring of $\prod M_n$ which contains $\sum M_n + \{1\}$. Let $\Gamma(R)$ be as defined in § 4. That is, $\Gamma(R)$ is the subgroup of $E(\bar{B})$ generated by the collection $\{\rho(A) \mid A \in K(R)\}$. Define $\Gamma$ to be the subgroup of $\Theta$ defined by $\Gamma = \{\gamma \mid \bar{B}[p] \mid \gamma \in \Gamma(R)\}$. Note that $\Gamma$ is a $p$-group in which every element has order $p$. By Zorn's lemma, it is possible to choose a subgroup $\Delta$ of $\Gamma$ which contains the identity and which is maximal with respect to having only the zero element in common with the subgroup $\{\gamma \in \Gamma \mid \xi(\gamma) \in \sum M_n\}$. Obviously, $\Delta$ is a countable subgroup of $\Theta$ which satisfies condition (*) of 5.3. Let $\mathscr{E}$ be the collection of all those elements in $\Theta$ which are $\Delta$-exceptional. By 5.6, if $\alpha \in \Theta$ and $\xi(\alpha) \notin \xi(\Delta) + \sum M_n$, then $\alpha$ is $\Delta$-exceptional. Since $\xi(\Delta) + (\sum M_n)$ is countable and since $\xi$ maps onto $\prod M_n$ by Lemma 3.2, it follows that $|\mathscr{E}| = 2^{\aleph_0}$. Let $\Omega$ be the first ordinal of cardinality $2^{\aleph_0}$, and let $\varphi \leftrightarrow \alpha_\varphi$ be a one-to-one correspondence between the elements of $\mathscr{E}$ and the ordinals $\varphi < \Omega$.

LEMMA 5.8. *There exist collections* $\{G_\varphi \mid \varphi < \Omega\}$, $\{P_\varphi \mid \varphi < \Omega\}$ *and* $\{U_\varphi \mid \varphi < \Omega\}$ *such that*

(i) *for all* $\varphi < \Omega$, $G_\varphi$ *is a pure subgroup of* $\bar{B}$ *containing* $B$, $P_\varphi = G_\varphi[p]$ *and* $U_\varphi$ *is a subset of* $\bar{B}[p]$,

(ii) $G_\varphi \subseteqq G_\chi$ *and* $U_\varphi \subseteqq U_\chi$ *whenever* $\varphi \leqq \chi < \Omega$,

(iii) $|P_\varphi| \leqq (|\varphi| + 1)\aleph_0$ *and* $|U_\varphi| \leqq (|\varphi| + 1)\aleph_0$,

(iv) $\gamma(G_\varphi) \subseteqq G_\varphi$ *for all* $\gamma \in \Gamma(R)$ *and each* $\varphi < \Omega$,

(v) $P_\varphi \cap U_\varphi = \varnothing$ *for all* $\varphi < \Omega$,

(vi) *for each* $\varphi < \Omega$ *there exists* $z_\varphi \in P_\varphi$ *such that* $\alpha_\varphi(z_\varphi) \in U_\varphi$.

*Proof*. The proof is by transfinite induction. Suppose $G_\varphi$ and $U_\varphi$ exist for all $\varphi < \chi$. Let $G'_\chi = \bigcup_{\varphi<\chi} G_\varphi + B$. $P'_\chi = \bigcup_{\varphi<\chi} P_\varphi + B[p]$ and $U'_\chi = \bigcup_{\varphi<\chi} U_\varphi$. Note that $G'_\chi[p] = P'_\chi$, that $\gamma(G'_\chi) \subseteqq G'_\chi$ for each $\gamma \in \Gamma(R)$, and that $G'_\chi$ is a pure subgroup of $\bar{B}$. Suppose there is an element $z$ in $P'_\chi \cap U'_\chi$. The existence of $z$ implies the existence of ordinals $\psi < \chi$ and $\omega < \chi$ such that $z \in U_\psi$ and $z \in P_\omega + B[p] = P_\omega$. Let $\varphi$ be largest of $\psi$ and $\omega$. Then $z \in [P_\varphi + B[p]] \cap U_\varphi = P_\varphi \cap U_\varphi$, contrary to the induction hypothesis. Thus, $P'_\chi \cap U'_\chi = \varnothing$. Since $|P_\varphi| \leqq (|\varphi| + 1)\aleph_0$ and $|U_\varphi| \leqq (|\varphi| + 1)\aleph_0$ for each $\varphi < \chi$, it follows that $|P'_\chi| \leqq (|\chi| + 1)\aleph_0$ and $|U'_\chi| \leqq (|\chi| + 1)\aleph_0$. Thus,

$$|\{P'_\chi, U'_\chi, B[p]\}| \leqq |P'_\chi|\,|U'_\chi|\,\aleph_0 \leqq (|\chi| + 1)\aleph_0 < 2^{\aleph_0}$$

Since $\alpha_\chi$ is $\varDelta$-exceptional, there is a collection $T(\alpha_\chi) \subseteqq \bar{B}[p]$ such that
( a ) $|T(\alpha_\chi)| = 2^{\aleph_0}$
( b ) $y, z \in T(\alpha_\chi)$ imply that $\varDelta(y), \varDelta(z), \{\alpha_\chi(y)\}, \{\alpha_\chi(z)\}$
are independent and $\alpha_\chi(y), \alpha_\chi(z)$ are nonzero. Therefore, it is possible to find $z_\chi \in T(\alpha_\chi)$ such that $\alpha_\chi(z_\chi) \neq 0$ and

$$(\sharp) \qquad \{\varDelta(z_\chi), \alpha_\chi(z_\chi)\} \cap \{P'_\chi, U'_\chi, B[p]\} = 0 .$$

Now suppose $\gamma \in \Gamma(R)$. Since every element of $\Gamma$ has order $p$ and since $\varDelta$ is maximal with respect to having zero intersection with $\{\gamma \in \Gamma \mid \xi(\gamma) \in \sum M_n\}$, it is possible to write $\gamma \mid \bar{B}[p]$ as $\alpha + \beta$ where $\alpha \in \varDelta$, $\beta \in \Gamma$ and $\xi(\beta) \in \sum M_n$. Since $\xi(\beta) \in \sum M_n$, it follows from the definition of $\Gamma(R)$ that $\beta(\bar{B}[p]) \subseteqq B[p]$. Therefore,

$$\gamma(z_\chi) = \alpha(z_\chi) + \beta(z_\chi) \in \varDelta(z_\chi) + B[p] .$$

Consequently, if $\gamma(z_\chi) \in G'_\chi[p] = P'_\chi$, then (using ($\sharp$)) $\gamma(z_\chi) \in B[p]$. Thus, $G'_\chi$ and $z_\chi$ satisfy the hypothesis of 4.6. Let $G_\chi$ be the pure subgroup of $\bar{B}$ obtained by the application of 4.6. Then

$$P_\chi = G_\chi[p] = G'_\chi[p] + \Gamma(R)(z_\chi) = P'_\chi + \varDelta(z_\chi)$$

and $\gamma(G_\chi) \subseteqq G_\chi$ for each $\gamma \in \Gamma(R)$. Also, $|P_\chi| \leqq (|\chi| + 1)\aleph_0$. Let $U_\chi$ be the set obtained by adjoining $\alpha_\chi(z_\chi)$ to $U'_\chi$. Then $|U_\chi| \leqq (|\chi| + 1)\aleph_0$, and conditions (i), (ii), (iii), (iv) and (vi) obviously are satisfied. To show that (v) holds, suppose $z \in P_\chi \cap U_\chi$. There are two cases to consider:

*Case* 1. $z = \alpha_\chi(z_\chi)$ and $z = y + \beta(z_\chi)$ for $y \in P'_\chi$ and $\beta \in \varDelta$. By ($\sharp$), $\alpha_\chi(z_\chi) - \beta(z_\chi) = y = 0$. Thus, applying (b), it is clear that $\alpha_\chi(z_\chi) = 0$. This is a contradiction of the choice of $z_\chi$.

*Case* 2. $z \in U'_\chi$ and $z = y + \beta(z_\chi)$ for $y \in P'_\chi$ and $\beta \in \varDelta$. In this case, $0 = z - y = \beta(z_\chi)$ by ($\sharp$). Consequently, $y = z \in U'_\chi$. This is a contradiction since $U'_\chi \cap P'_\chi = \varnothing$.

LEMMA 5.9. *Let* $G(R) = \bigcup_{\chi<\Omega} G_\chi$, $P(R) = \bigcup_{\chi<\Omega} P_\chi$ *and* $U(R) = \bigcup_{\chi<\Omega} U_\chi$. *Then*
( i ) $G(R)$ *is a pure subgroup of* $\bar{B}$,
( ii ) $G(R)[p] = P(R)$,
(iii) $P(R) \cap U(R) = \varnothing$,
(iv) $\gamma(G(R)) \subseteqq G(R)$ *for each* $\gamma \in \Gamma(R)$,
( v ) *if* $\alpha \in E(\bar{B})$ *and if* $\alpha$ *is* $\varDelta$-*exceptional, then* $\alpha \notin E(G(R))$.

*Proof*. The arguments for (i), (ii), (ii), and (iv) are quite easy and can be found in the proof of 5.8. To show (v), suppose $\alpha$ is

$\Delta$-exceptional. Then there exist $\varphi < \Omega$ and $z_\varphi \in P(R)$ such that $\alpha(z_\varphi) \in U_\varphi$ (see (vi) of 5.8). Since $P(R) \cap U(R) = \emptyset$ and $G(R)[p] = P(R)$ by (iii) and (ii), it follows that $\alpha \notin E(G(R))$.

THEOREM 5.10. *Let $R$ be any countable subring of the ring direct product $\prod M_n$. Suppose that $R$ contains $\sum M_n + \{1\}$. There is a pure subgroup $G$ of $\bar{B}$, containing $B$, such that $\xi(E(G)) = R$. Moreover,*

$$\frac{E(G)}{J(E(G), E_p(G))} \cong R .$$

*Proof.* Let $G = G(R)$. By 4.4, $R = \xi(\Gamma(R))$. Thus, since $\Gamma(R) \subseteqq E(G)$ by (iv) of 5.9, $R \subseteqq \xi(E(G(R)))$. Suppose $\alpha \in E(G(R))$ and $\xi(\alpha) \notin R$. By 4.4, $\xi(\Delta) \subseteqq \xi(\Gamma) \subseteqq \xi(\Gamma(R)) = R$. Thus, $\xi(\Delta) + (\sum M_n) \subseteqq R$, and Lemma 5.6 may be applied to infer that $\alpha$ is $\Delta$-exceptional. This is contrary to (v) of 5.9. Therefore, $\xi(E(G(R))) = R$. It follows from 3.1 that

$$\frac{E(G)}{J(E(G), E_p(G))} \cong R .$$

LEMMA 5.12. *Let $U$ and $V$ be vector spaces over a field such that $V \subseteqq U$. Let $U/V$ be finite dimensional. Suppose $\alpha \in E(U)$, $\alpha$ is one-to-one and $\alpha(V) = V$. Then $\alpha$ is an automorphism of $U$.*

*Proof.* Since $\alpha(V) = V$, $\alpha$ induces an endomorphism $\alpha'$ of $U/V$ ($\alpha'(u + V) = \alpha'(u) + V$ for $u \in U$). Moreover, $\alpha'$ is one-to-one; and, consequently, the dimensions of $U/V$ and $\alpha'(U/V)$ are equal and finite. It follows that $\alpha'(U/V) = U/V$; and therefore, $\alpha(U) = U$ by a standard argument.

THEOREM 5.13. *The groups $G = G(R)$ have no proper isomorphic subgroups.*

*Proof.* Let $\alpha$ be an isomorphism of $G$ into $G$. By (v) of 5.9, $\alpha$ is not $\Delta$-exceptional. By 5.6, 5.7 and the definition of the map $\xi$, there must exist an integer $n$ and an element $\beta \in \Delta$ such that $\alpha(c_i) = \beta(c_i)$ for all $i > n$. Since $\alpha$ is an isomorphism, $0 \neq \alpha(c_i) = \beta(c_i)$ for all $i > n$. It follows that $\alpha$ and $\beta$ agree on $(\pi_n G)[p] = \pi_n(G[p])$ (see § I for the definition of $\pi_n$). Now, $\Delta \subset \Gamma = \{\gamma \mid \bar{B}[p] \mid \gamma \in \Gamma(R)\}$, $\beta \in \Delta$ and $\beta(c_i) \neq 0$ for $i > n$ imply, using Fermat's theorem, that $\beta^{p-1}$ acts as the identity on $\pi_n G[p]$. It follows that $\beta$ maps $(\pi_n G)[p] \cap p^k(\pi_n G)$ onto itself for each $k = 0, 1, \cdots$. Thus,

$$\alpha(G[p] \cap p^k G) = \alpha((\pi_n G)[p] \cap p^k(\pi_n G) = (\pi_n G)[p] \cap p^k(\pi_n G) = G[p] \cap p^k G$$

for each $k = n, n+1, \cdots$. Suppose $m \geqq 1$ is the largest integer such that $\alpha(G[p] \cap p^{m-1}G) \neq G[p] \cap p^{m-1}G$. It has been shown that if $m$ exists, then $m \leqq n$. An application of 5.12 to $U = G[p] \cap p^{m-1}G$ and $V = G[p] \cap p^m G$ shows that the existence of such an integer $m$ is impossible. Consequently, $\alpha(G[p] \cap p^k G) = G[p] \cap p^k G$ for all $k \geqq 0$. By Lemma 2.3, it follows that $\alpha$ is an automorphism of $G$.

COROLLARY 5.14. *Let $R$ be any countable subring of the ring direct product $\prod M_n$. Suppose that $R$ contains $\sum M_n + \{1\}$. There is a pure subgroup $G$ of $\bar{B}$ which contains $B$ such that*

$$\frac{E(G)}{J(E(G))} \cong R .$$

*Proof.* Let $G = G(R)$ and apply 5.10, 5.13 and 2.12.

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