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ENDOMORPHISM RINGS OF PRIMARY ABELIAN GROUPS

ROBERT WILLIAM STRINGALL

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This paper is concerned with the study of certain homomorphic images of the endomorphism rings of primary abelian groups. Let $E(G)$ denote the endomorphism ring of the abelian p -group G , and define $H(G) = \{\alpha \in E(G) \mid x \in G, px = 0 \text{ and height } x < \infty \text{ imply height } \alpha(x) > \text{height } x\}$. Then $H(G)$ is a two sided ideal in $E(G)$ which always contains the Jacobson radical. It is known that the factor ring $E(G)/H(G)$ is naturally isomorphic to a subring R of a direct product $\prod_{n=1}^{\infty} M_n$ with $\sum_{n=1}^{\infty} M_n$ contained in R and where each M_n is the ring of all linear transformations of a Z_p space whose dimension is equal to the $n - 1$ Ulm invariant of G . The main result of this paper provides a partial answer to the unsolved question of which rings R can be realized as $E(G)/H(G)$.

THEOREM. Let R be a countable subring of $\prod_{\aleph_0} Z_p$ which contains the identity and $\sum_{\aleph_0} Z_p$. Then there exists a p -group G with a standard basic subgroup and containing no elements of infinite height such that $E(G)/H(G)$ is isomorphic to R . Moreover, G can be chosen without proper isomorphic subgroups; in this case, $H(G)$ is the Jacobson radical of $E(G)$.

1. Preliminaries.

(1.1) Throughout this paper p - represents a fixed prime number, N the natural numbers, Z the integers and Z_{p^n} the ring of integers modulo p^n . All groups under consideration will be assumed to be p -primary and abelian. With few exceptions, the notation of [3], [5], and [8] will prevail.

Let $h_G(x)$ and $E(x)$ denote, respectively, the p -height of x in G and the exponential order of x . If A is any subset of the group G , then $\langle A \rangle$ will denote the subgroup of G generated by A . Denote the p^n layer of G by $G[p^n]$. Finally, if A is any set, let $|A|$ be the cardinal number of A .

(1.2) Let G be a p -primary group and B a basic subgroup of G . The group B can be written as $B = \sum_{n \in N} B_n$ where each B_n is a direct sum of, say $f(n)$, copies of Z_{p^n} . That is, $B_n = \sum_{f(n)} \{b_i\}$ where $E(b_i) = n$. Define $H_n = \{p^n G, B_{n+1}, B_{n+2}, \dots\}$. It is readily verified that $G = B_1 \oplus \dots \oplus B_n \oplus H_n$ for each $n \in N$. Thus, it is possible to define the projections π_n ($n = 1, 2, \dots$) of G onto H_n corresponding to the decomposition $G = B_1 \oplus B_2 \oplus \dots \oplus B_n \oplus H_n$. Define $\rho_1 = 1 - \pi_1$ and

$\rho_n = \pi_{n-1} - \pi_n$ for $n > 1$. It follows that $\rho_n(G) = B_n$ and that ρ_n is the projection of G onto B_n .

2. **Endomorphism rings.** A few preliminary notions are needed before the main results can be presented. Although given in a different context, many of the results of this section are patterned after those of R. S. Pierce in his work [8].

LEMMA 2.1. *Let G be a p -group and $B = \sum_{n \in N} B_n$ a basic subgroup of G . If α is an endomorphism of $B_n[p]$, then α can be extended to an endomorphism β of G such that $j \neq n$ implies $\beta(B_j) = 0$.*

Proof. Since $G = B_1 \oplus B_2 \oplus \dots \oplus B_m \oplus H_m$ for each $m \in N$, for each $m \in N$, it is enough to show that α can be extended to B_n . Let

$$B_n = \sum_{i=1}^{f(n)} \{b_i\}$$

where, for each i , $E(b_i) = n$. For $b_i \in B_n$, write

$$\begin{aligned} \alpha(p^{n-1}b_i) &= a_1p^{n-1}b_1 + \dots + a_kp^{n-1}b_k \\ \beta(b_i) &= a_1b_1 + \dots + a_kb_k \end{aligned}$$

where k and the integers a_j ($0 \leq a_j < p$) are determined by α . Compute $\beta(b_i)$ in this way for each $b_i \in B_n$, and extend β linearly to B_n . It follows that β is the desired extension of α to B_n .

LEMMA 2.2. *If G is a p -group and B a basic subgroup of G , then any bounded homomorphism of B into G can be extended to a bounded endomorphism of G .*

Proof. By definition, G/B is divisible. Consequently,

$$G/B = p^n(G/B) = \frac{B + p^nG}{B}$$

for each positive integer n . It follows that $G = B + p^nG$ for each $n \in N$. Let $k \in N$ be such that $p^k\alpha = 0$, and write $x \in G$ as $x = b + p^ky$ where $b \in B$ and $y \in G$. It is easy to check that $x \rightarrow \alpha(b)$ defines a bounded extension of α to an endomorphism of G .

For proof of the following lemma see [8], Lemma 13.1.

LEMMA 2.3. *An endomorphism α of the p -group G is an automorphism if and only if $\ker \alpha \cap G[p] = 0$ and $\alpha(G[p] \cap p^nG) = G[p] \cap p^nG$ for each integer $n = 0, 1, 2, \dots$.*

For the p -group G , let $E(G)$ denote the ring of all endomorphisms of G . If $E_p(G)$ denotes the subcollection of $E(G)$ consisting of all bounded endomorphisms of G , then it is not difficult to show that $E_p(G)$ is a two sided ideal of $E(G)$.

LEMMA 2.4. *Let*

$$\begin{aligned} H_p(G) &= \{ \alpha \in E_p(G) \mid x \in G[p] \text{ and } h_{\alpha}(x) \in N \text{ imply } h_{\alpha}(\alpha(x)) > h_{\alpha}(x) \} , \\ K_p(G) &= \{ \alpha \in E_p(G) \mid \alpha(G[p]) = 0 \} , \text{ and} \\ L_p(G) &= \{ \alpha \in E_p(G) \mid \alpha(G) \subseteq pG \} . \end{aligned}$$

Then $H_p(G)$, $K_p(G)$ and $L_p(G)$ are two sided ideals of $E_p(G)$ contained in the Jacobson radical, $J(E_p(G))$, of $E_p(G)$. What is more, $K_p(G) + L_p(G) \subseteq H_p(G)$.

Proof. It is easy to check that $H_p(G)$, $K_p(G)$ and $L_p(G)$ are two sided ideals of both $E_p(G)$ and $E(G)$. It is also easy to verify that $K_p(G) \subseteq H_p(G)$. It remains only to show that $L_p(G) \subseteq H_p(G) \subseteq J(E_p(G))$. To this end, suppose $\alpha \in L_p(G)$, $x \in G[p]$ and $h_{\alpha}(x) = k \in N$. Since $h_{\alpha}(x) = k$, it is possible to write $x = p^k y$ for some $y \in G$. It follows that

$$\alpha(x) = \alpha(p^k y) = p^k \alpha(y) \in p^k pG = p^{k+1}G$$

Hence, $h_{\alpha}(\alpha(x)) \geq k + 1 > h(x)$ and $\alpha \in H_p(G)$. Therefore, $L_p(G)$ is contained in $H_p(G)$. To show that $H_p(G)$ is contained in $J(E_p(G))$, let $\alpha \in H_p(G)$. Since $\alpha \in E_p(G)$, there exists a positive integer k such that $p^k \alpha = 0$. Thus, if $x \in G[p]$ and $h_{\alpha}(x) \geq k$, then $\alpha(x) = 0$. Since $x \in G[p]$ implies $h_{\alpha}(\alpha^k(x)) > k$, it follows that $\alpha^{k+1}(x) = 0$ for all $x \in G[p]$. If $x \in G[p]$ and $(1 - \alpha)(x) = 0$, then

$$x = \alpha(x) = \alpha^2(x) = \dots = \alpha^{k+1}(x) = 0 .$$

Thus, $1 - \alpha$ is one-to-one on $G[p]$. Also, if $x \in G[p]$, then

$$(1 - \alpha)(x + \alpha(x) + \dots + \alpha^k(x)) = x .$$

Therefore, $(1 - \alpha)(G[p] \cap p^n G) = G[p] \cap p^n G$ for each $n = 0, 1, 2, \dots$. Applying 2.3, it is seen that $1 - \alpha$ has an inverse. Since $H_p(G)$ is an ideal of $E(G)$, $\alpha \in J(E(G)) \cap E_p(G) = J(E_p(G))$ (see [4], pp. 9 and 10).

It becomes necessary, at least for the remainder of this section, to fix the basic subgroup B and a decomposition $B = \sum B_n$. This, naturally, determines the subgroup H_n , the cardinals $f(n)$ and the maps π_n and β_n .

LEMMA 2.5. *There are group homomorphisms ρ of $E_p(G)$ into $E_p(G)$, σ of $E_p(G)$ into $K_p(G)$ and τ of $E_p(G)$ into $L_p(G)$ such that for $\alpha \in E_p(G)$*

$$(*) (\sigma\alpha)(b_n) = (1 - \pi_{n-1})(\alpha(b_n)), (\tau\alpha)(b_n) = \pi_n(\alpha(b_n))$$

and $(\rho\alpha)(b_n) = \rho_n(\alpha(b_n))$ for $b_n \in B_n$, $n = 1, 2, \dots$. Moreover,

$$\rho^2 = \rho, \sigma^2 = \sigma, \tau^2 = \tau, \rho\sigma = \sigma\rho = \rho\tau = \tau\rho = \sigma\tau = \tau\sigma = 0, \rho + \sigma + \tau = 1,$$

and $\rho_n(\rho\alpha)\rho_n(b_n) = \rho\alpha(b_n)$ for all $b_n \in B_n$, $n = 1, 2, \dots$.

Proof. It is clear that conditions (*) determine bounded homomorphisms of B into G , which by 2.2 extend to G as bounded endomorphisms. The remainder of the proof is similar to that of 13.4 in [8] and will not be given.

(2.6) LEMMA. *The mapping*

$$\lambda: \alpha \rightarrow ((\rho\alpha) | B_1[p], (\rho\alpha) | B_2[p], \dots)$$

is a ring homomorphism of $E_p(G)$ onto the ring direct sum

$$\sum_{n=1}^{\infty} E(B_n[p]).$$

The kernel of λ is $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$.

Proof. It is clear that λ maps onto $\sum_{n \in N} E(B_n[p])$. In fact, if $(\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots) \in \sum_{n \in N} E(B_n[p])$ where $\alpha_k \in E(B_k[p])$ for $k = 1, 2, \dots, n$, then by 2.1, each of the α_k have extensions β_k to G such that $j \neq k$ implies $\beta_k(B_j) = 0$. Obviously,

$$\lambda\left(\sum_{i=1}^n \beta_i\right) = (\alpha_1, \alpha_2, \dots, \alpha_n, 0, 0, \dots)$$

and $p^n \sum_{i=1}^n \beta_i = 0$. Thus, λ is onto $\sum_{n \in N} E(B_n[p])$. Clearly, λ is additive. To show that λ preserves products, let $b \in B_n[p]$. Then $h(b) = n - 1$, so that for some $c \in B_n$, $b = p^{n-1}c$. Also,

$$\rho(\alpha\beta(b)) = \rho_n(\alpha\beta(b)) = \rho_n(\alpha((\sigma\beta)(b) + (\rho\beta)(b) + (\tau\beta)(b))).$$

Now, $\sigma\beta \in K_p(G)$ and $b \in G[p]$. Thus, $\sigma\beta(b) = 0$. Also, $\tau\beta \in L_p(G)$ implies that $\tau\beta(b) = \tau\beta(p^{n-1}c) = p^{n-1}\tau\beta(c) \in p^n G$, so that

$$\rho_n \alpha(\tau\beta(b)) \in p^n G \cap B_n = p^n B_n = 0.$$

Finally, $\rho\beta(b) = \rho_n \rho\beta(b)$. Thus,

$$\rho(\alpha\beta(b)) = \rho_n(\alpha\beta(b)) = \rho_n \alpha((\rho\beta)(b)) = (\rho\alpha)((\rho\beta)(b)).$$

Consequently, $\lambda(\alpha\beta) = \lambda(\alpha)\lambda(\beta)$. To show that the kernel of λ is $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$, observe that $\lambda(\alpha) = 0$ if and only if $\rho\alpha | B_n[p] = 0$ for all $n \in N$. This condition is equivalent to $\rho\alpha(B[p]) =$

0 which, since $\rho\alpha$ is bounded, is equivalent to $\rho\alpha(G[p]) = 0$. Therefore, $\text{Ker } (\lambda) = \{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$.

THEOREM 2.7. *The Jacobson radical of $E_p(G)$ is $H_p(G)$, and $K_p(G) + L_p(G) = H_p(G)$. Also, $E_p(G)/H_p(G)$ is ring isomorphic to the ring direct sum $\sum_{n \in \mathbb{N}} M_n$ where each M_n is the ring of all linear transformations of a Z_p -space of dimension $f(n)$.*

Proof. By 2.6 there is a ring homomorphism λ of $E_p(G)$ onto a ring isomorphic to $\sum_{n \in \mathbb{N}} M_n$. Moreover, the kernel, of λ is $\{\alpha \in E_p(G) \mid \rho\alpha \in K_p(G)\}$. The rings M_n are surely primitive. Thus, by [4], proposition 1, p. 10, the Jacobson radical of $E_p(G)$ is contained in $\bigcap_{n \in \mathbb{N}} \text{Ker}(\delta_n \lambda) = \text{Ker } \lambda$ where δ_n ($n = 1, 2, \dots$) is, temporarily, the projection map of $\sum_{n \in \mathbb{N}} M_n$ onto M_n . Hence by 2.4,

$$K_p(G) + L_p(G) \subseteq H_p(G) \subseteq J(E_p(G)) \subseteq \text{Ker } \lambda .$$

To show that the kernel of λ is contained in $K_p(G) + L_p(G)$, let $\alpha \in E_p(G)$ be such that $\rho\alpha \in K_p(G)$. By 2.5, $\rho\alpha + \sigma\alpha + \tau\alpha = \alpha$. It follows that $\alpha \in K_p(G) + L_p(G)$. Thus,

$$\text{Ker } \lambda = \{\alpha \in E_p(G) \mid \rho\alpha \in K(G)\} \subseteq K_p(G) + L_p(G) .$$

Hence,

$$\text{Ker } \lambda = J(E_p(G)) = K_p(G) + L_p(G) = H_p(G) .$$

For proof of the following lemma, the reader is directed to R. S. Pierce's work [8], p. 284.

LEMMA 2.8. *Suppose R is an associative ring and S any two-sided ideal of R . Let $J(S)$ be the Jacobson radical of S and*

$$J(R, S) = \{x \in R \mid xz \in J(S) \text{ for all } z \in S\} .$$

Then the following statements are valid:

- (a) $J(R, S)$ is a two-sided ideal of R containing $J(R)$ the Jacobson radical of R ;
- (b) $J(R, S) = \{x \in R \mid wxz \text{ is quasi-regular for all } z, w \text{ in } S\}$;
- (c) $J(R, S) = \{x \in R \mid zx \in J(S) \text{ for all } z \in S\}$;
- (d) $J(R, S) \cap S = J(S)$;
- (e) *the image of S under the natural projection of R onto $R/J(R, S)$ is an ideal which isomorphic to $S/J(S)$.*

Recall that M_n ($n = 1, 2, \dots$) is defined to be the ring of all linear

transformations of a Z_p -space of dimension $f(n)$.

If ξ is the natural map of $E(G)$ onto $E(G)/J(E(G), E_p(G))$, then, by 2.8 (e), $\xi(E_p(G))$ is isomorphic to $E_p(G)/J(E_p(G))$. By 2.7, there is an isomorphism λ of $E_p(G)/J(E_p(G))$ onto the ring direct sum $\sum_{n \in N} M_n$. Let δ_n be the ring homomorphism of $E_p(G)$ onto M_n obtained by composing $\lambda\xi$ with the projection of $\sum_{n \in N} M_n$ onto M_n . That is, for $\alpha \in E_p(G)$

$$\lambda\xi(\alpha) = (\delta_1\alpha, \delta_2\alpha, \dots).$$

It is easy to see that if ρ_n ($n = 1, 2, \dots$) are as defined in 1.2, then

$$\delta_n(\rho_m) = 0 \quad \text{for } m \neq n \quad \text{and} \quad \delta_n(\rho_n) = 1.$$

For $\alpha \in E(G)$, set $\mu(\alpha) = (\delta_1(\alpha\rho_1), \delta_2(\alpha\rho_2), \delta_3(\alpha\rho_3), \dots)$.

THEOREM 2.9. *The correspondence*

$$\alpha \xrightarrow{\mu} (\delta_1(\alpha\rho_1), \delta_2(\alpha\rho_2), \delta_3(\alpha\rho_3), \dots)$$

is a ring homomorphism of $E(G)$ onto a subring R of the ring direct product $\prod_{n \in N} M_n$ with kernel $J(E(G), E_p(G))$. Moreover, R contains both the identity of $\prod_{n \in N} M_n$ and the ring direct sum $\sum_{n \in N} M_n$.

Proof. See the proof of Theorem 14.3 in [8].

The following lemma gives an interesting characterization of $J(E(G), E_p(G))$.

LEMMA 2.10. $J(E(G), E_p(G)) = \{\alpha \in E(G) \mid x \in G[p] \text{ and } h_{\mathcal{G}}(x) \in N \text{ imply } h_{\mathcal{G}}(\alpha(x)) > h_{\mathcal{G}}(x)\}$.

Proof. Suppose $\alpha \in E(G)$ and $h_{\mathcal{G}}(\alpha(x)) > h(x)$ for all $x \in G[p]$ such that $h(x)$ is finite. Then if $\beta \in E(G)$, the product $\alpha\beta$ satisfies this same condition. That is, for elements x in $G[p]$ of finite height, $h_{\mathcal{G}}(\alpha\beta(x)) > h_{\mathcal{G}}(x)$. In particular, if $\beta \in E_p(G)$, then $\alpha\beta$ is bounded and satisfies the foregoing condition. Thus, for $\beta \in E_p(G)$, $\alpha\beta \in H_p(G)$ which by 2.7 is $J(E_p(G))$. Consequently, $\alpha \in J(E(G), E_p(G))$ by definition. Conversely, suppose $\alpha \in J(E(G), E_p(G))$, $x \in G[p]$ and $h_{\mathcal{G}}(x) < \infty$. The existence of a bounded endomorphism β such that $\beta(x) = x$ is easy to verify (see, for example, [3], Theorem 24.7). By definition, $\alpha\beta \in J(E_p(G))$. Consequently, $h_{\mathcal{G}}(\alpha(x)) = h_{\mathcal{G}}(\alpha\beta(x)) > h_{\mathcal{G}}(x)$.

The following two results will be needed later.

LEMMA 2.11. *Let α be any automorphism of the p -group G without*

elements of infinite height. If $\beta \in J(E(G), E_p(G))$, then $\alpha - \beta$ is one-to-one.

Proof. Suppose $0 \neq x \in G[p]$ and $(\alpha - \beta)(x) = 0$. Then by 2.10,

$$h_{\mathfrak{a}}(x) < h_{\mathfrak{a}}(\beta(x)) = h_{\mathfrak{a}}(\alpha(x)) \leq h_{\mathfrak{a}}(\alpha^{-1}(\alpha(x))) = h_{\mathfrak{a}}(x),$$

a contradiction. Thus, $\ker(\alpha - \beta) \cap G[p] = 0$. This is enough to ensure that $\alpha - \beta$ is one-to-one.

THEOREM 2.12. *If G is without elements of infinite height and has no proper isomorphic subgroups, then $J(E(G), E_p(G)) = J(E(G))$.*

Proof. If $\alpha \in J(E(G), E_p(G))$, then $1 - \alpha$ is an isomorphism by Lemma 2.11. Since G has no proper isomorphic subgroups, $1 - \alpha$ is an automorphism. Therefore, α is quasi-regular for each $\alpha \in J(E(G), E_p(G))$ (see [4], p. 7). Since $J(E(G), E_p(G))$ is a right ideal, it follows that $J(E(G), E_p(G)) \subseteq J(E(G))$ ([4], Theorem 1, p. 9). Finally, $J(E(G)) \subseteq J(E(G), E_p(G))$ by 2.8 (a).

3. Realizations of $E(G)$. The primary concern of this paper is with the endomorphism rings of p -primary groups without elements of infinite height. The study of such rings can be greatly eased with the employment of some fairly simple notions.

Let G be a p -group without elements of infinite height and $B = \sum_{n \in \mathbb{N}} B_n$ a basic subgroup of G . Let \bar{B} denote the closure (or torsion completion) of B . The group \bar{B} can be defined as the torsion subgroup of the direct product $\prod_{n \in \mathbb{N}} B_n$. That is,

$$\bar{B} = \{x \in \prod_{n \in \mathbb{N}} B_n \mid p^k x = 0 \text{ for some } k \in \mathbb{N}\}.$$

Naturally, B is identified with the subgroup of \bar{B} consisting of those elements which have at most a finite number of nonzero components. Thus, B is a pure subgroup of \bar{B} . It is well known that there is a B -isomorphism of G onto a pure subgroup of \bar{B} (see [3], § 33). Thus, in a sense, the study of p -groups without elements of infinite height can be reduced to the study of pure subgroups of suitable closed groups \bar{B} .

It has already been asserted that G should be a p -group with fixed basic subgroup B . In order that the above remarks will apply to G , require, in addition, that G be without elements of infinite height. That is, both B and \bar{B} are fixed and G is a pure subgroup of \bar{B} which contains B .

If α, β are endomorphisms of G which agree on B , then B is contained in the kernel of the difference $\gamma = \alpha - \beta$. Thus, $\gamma(G)$ is a homomorphic image of the divisible group G/B , and, for this reason,

is divisible. Since G is reduced and since $\gamma(G) \subseteq G$, it follows that $\gamma(G) = (\alpha - \beta)(G) = 0$. Thus, $\alpha = \beta$. Consequently, if G is a reduced p -group, then every endomorphism of G is completely determined by its effect on the elements of any basic subgroup.

By 2.2 and the above remarks, it follows that each bounded endomorphism of B has a unique extension to an endomorphism of G . Because of this, it may be assumed that $E(G)$, the endomorphism ring of G , contains an embedded copy, denoted by $E_p(B)$, of the ring of all bounded endomorphisms of B . Thus, identify $E_p(B)$ with

$$\{\alpha \in E_p(G) \mid \alpha(B) \subseteq B\}.$$

Suppose that $B \subseteq G \subseteq \bar{B}$ where G is a pure subgroup of \bar{B} . It has been shown that every endomorphism of G has a unique extension to \bar{B} (see, for example, [6], pp. 84–85). Thus, it is possible to adopt the very useful convention of identifying the endomorphism ring of G with the subring of the endomorphism ring of \bar{B} consisting of endomorphisms of \bar{B} which map G into itself. That is,

$$E(G) = \{\alpha \in E(\bar{B}) \mid \alpha(G) \subseteq G\}.$$

With this identification, $E_p(G)$ (the torsion subring of $E(G)$) becomes a subring of $E_p(\bar{B})$; namely,

$$E_p(G) = \{\alpha \in E_p(\bar{B}) \mid \alpha(G) \subseteq G\}.$$

It is reasonable to expect the above identifications to carry over in some way to the images $\mu(E(G))$ where μ is the map defined in Theorem 2.9. The following results show that this is indeed the case.

Let ξ be the map of Theorem 2.9 developed for $E(\bar{B})$. Then by using the definition of ξ and the above convention, it is not hard to show, for pure subgroups G of \bar{B} containing B , that $\xi|E(G)$ and the map μ , defined in 2.9 for $E(G)$, are identical. Because of this, it is possible to confine the investigation of all such maps μ to the map ξ and its restrictions to subrings of $E(\bar{B})$.

By way of summation, the following is given.

LEMMA 3.1. *Let G be pure subgroup of \bar{B} which contains B . Let ξ be the map of Theorem 2.9 defined for the p -group \bar{B} . The restriction of ξ to $E(G)$ and the map of 2.9 developed for G agree. Moreover, $J(E(G), E_p(G)) = J(E(\bar{B}), E_p(\bar{B})) \cap E(G)$.*

LEMMA 3.2. *If $G = B$ or $G = \bar{B}$, then $\xi(E(G)) = \prod M_n$.*

Proof. Suppose $(\alpha_1, \alpha_2, \dots)$ is an arbitrary element of $\prod M_n$.

Each α_i ($i = 1, 2, \dots$) may be considered as an endomorphism of $B_i[p]$. By 2.1, each α_i has an extension to an endomorphism β_i of B such that $\beta_i(B_j) = 0$ if $i \neq j$. Let α be the endomorphism of B determined by the conditions:

$$\alpha(b_i) = \beta_i(b_i) \text{ for } b_i \in B_i \ i = 1, 2, \dots .$$

By Lemma 2.2, α can be extended to \bar{B} . In either case, $\xi(\alpha) = (\alpha_1, \alpha_2, \dots)$.

Up to this point it has been shown that $\prod M_n$ can be realized as a homomorphic image of $E(B)$ and $E(\bar{B})$. Using an example of R. S. Pierce, it can be shown that not every pure subgroup G of \bar{B} which contains B can be so classified.

First, consider the ring of p -adic integers, R_p (see [3], § 6). This ring can be thought of as the collection of all infinite sums of the form

$$r = r_0 + r_1p + r_2p^2 + \dots$$

where $0 \leq r_i < p$. Suppose $x \in G$, and $r \in R_p$ where

$$r = r_0 + r_1p + r_2p^2 + \dots$$

and $0 \leq r_i < p$. It is possible to assign a meaning to the product rx , namely,

$$rx = r_0x + r_1px + r_2p^2x + \dots + r_np^n x$$

where n is any integer greater than $E(x)$. Clearly, this definition is independent of the integer n . It is easy to check that with this definition, G becomes an R_p -module. Consequently, every element r of R_p induces an endomorphism of G , $x \rightarrow rx$, which will also be labeled r . What is more important, it is not difficult to show that this correspondence, between the elements of R_p and the elements of $E(G)$, is a ring isomorphism. With this in mind, it is possible to assume that R_p is a subring of the ring of all endomorphisms of G .

DEFINITION 3.3. An endomorphism α of the p -group G is said to be a *small endomorphism* of G provided the following condition is satisfied:

(*) for all $k \geq 0$ there exists an integer n such that $0(x) \leq k$ and $h_G(x) \geq n$ imply $\alpha(x) = 0$.

REMARK. The concept of small endomorphism is due to R. S. Pierce and can be found in his paper [8]. The equivalence of the above definition and that appearing in [8] can be shown using 3.1 and 2.10 in the above mentioned paper.

It is an easy consequence of the above definition that the collection of all small endomorphisms forms a subring $E_s(G)$ of the ring $E(G)$. Moreover, $E_s(G)$ is an ideal of $E(G)$.

R. S. Pierce has shown that there exists a p -group H without elements of infinite height such that $E(H) = E_s(H) + R_p$ ([8], p. 297). The following results demonstrate a few of the many curious properties of such groups.

LEMMA 3.4. *If $E(H) = E_s(H) + R_p$, then $E_s(H)$ and R_p are disjoint.*

Proof. Let $r \in R_p$ and $r = \sum_{i=0}^{\infty} r_i p^i$ where $0 \leq r_i < p$. By definition, r is a small endomorphism if and only if for all $k \geq 0$ there exists an integer n such that $x \in H$, $E(x) \leq k$ and $h_H(x) \geq n$ collectively imply $r(x) = 0$. Let h be the least index such that $r_h \neq 0$. Let $k > h$, and for $l > k$ let $x_l = p^{l-k} b_l$ where $b_l \in B_l$, $E(b_l) = l$ and $h_H(b_l) = 0$. (Recall that $B = \sum_{n \in \mathbb{N}} B_n$ is a basic subgroup of H). Then $x_l \in B \subseteq H$, $E(x_l) = k$, $r(x_l) \neq 0$, and $h_H(x_l)$ increases indefinitely as l increases. Thus r is not a small endomorphism, and $E_s(H) \cap R_p = 0$.

LEMMA 3.5. $\xi(E_s(H)) = \sum M_n$ and $\xi(R_p) = \{1\}$ where 1 is the identity of $\sum^c M_n$.

Proof. It is easy to see from the definitions of ξ and $E_s(H)$ that $\xi(E_s(H)) \subseteq \sum M_n$. Since $E_p(B) \subseteq E_p(H) \subseteq E_s(H)$ and $\xi(E_p(B)) = \sum M_n$, it follows that $\xi(E_s(H)) = \sum M_n$. Suppose $r = \sum_{i=0}^{\infty} r_i p^i \in R_p$. Write $r = r_0 + ps$ where $s = \sum_{i \in \mathbb{N}} r_i p^{i-1}$. Clearly,

$$\xi(r) = \xi(r_0 + ps) = \xi(r_0) + \xi(ps) = \xi(r_0) \in \{1\}.$$

LEMMA 3.6.

$$\text{Ker}(\xi | E(H)) = J(E_s(H)) + J(R_p) = J(E(H), E_p(H)).$$

Proof. By 3.1, $\text{Ker}(\xi | E(H)) = J(E(H), E_p(H))$. To show that $\text{Ker}(\xi | E(H)) = J(E_s(H)) + J(R_p)$, let $\alpha + r$ be an arbitrary element in $E(H)$ where $\alpha \in E_s(H)$ and $r \in R_p$. Suppose, in addition, that $\xi(\alpha + r) = 0$. Since $\sum M_n$ and $\{1\}$ are obviously independent and since $\xi(\alpha + r) = \xi(\alpha) + \xi(r) \in \sum M_n + \{1\}$ by the foregoing lemma, $\xi(\alpha + r) = 0$ if and only if both $\xi(\alpha) = 0$ and $\xi(r) = 0$. Surely, $\xi(r) = 0$ if and only if $r \in pR_p$. Since pR_p is the unique maximal ideal in R_p , $J(R_p) = pR_p$ (see [4], p. 9). Thus, the conditions $\xi(r) = 0$ and $r \in J(R_p)$ are equivalent. Moreover, $\xi(\alpha) = 0$ and $\alpha \in E_s(H)$ if and only if $\alpha \in J(E(H), E_p(H)) \cap E_s(H)$. By Lemmas 2.9 and 2.8 (d) of this paper and 14.4 of [8], $\xi(\alpha) = 0$ if and only if

$$\alpha \in J(E(H), E_s(H)) \cap E_s(H) = J(E_s(H)).$$

Thus,

$$\text{Ker}(\xi | E(H)) = J(E_s(H)) + J(R_p) = J(E(H), E_p(H)).$$

LEMMA 3.7. *If $K(G) = \{\alpha \in E(G) \mid \alpha(G[p]) = 0\}$, then $K(G)$ is a two sided ideal of $E(G)$ which is contained in the Jacobson radical of $E(G)$.*

Proof. It is obvious that $K(G)$ is an ideal of $E(G)$. Moreover, if $\alpha \in K(G)$, then $\ker(1 - \alpha) \cap G[p] = 0$ and $(1 - \alpha)(G[p] \cap p^n G) = G[p] \cap p^n G$. Thus, $1 - \alpha$ is an automorphism by 2.3. It follows that $K(G)$ is a quasi regular ideal in $E(G)$; and is, therefore, contained in the Jacobson radical of $E(G)$ (see [4], p. 9, Theorem 1).

THEOREM 3.8. $E(H)/J(E(H)) = E(H)/J(E(H), E_p(H))$ is ring isomorphic to $\sum M_n + \{1\}$.

Proof. By 3.5 and 3.6, ξ maps $E(H)$ onto $\sum M_n + \{1\}$ with kernel $J(E(H), E_p(H)) = J(E_s(H)) + J(R_p)$. Also, by 2.8 (a), $J(E(H)) \subseteq J(E_s(H)) + J(R_p)$. Thus, it remains only to show that $J(R_p)$ and $J(E_s(H))$ are contained in $J(E(H))$. Since $E_s(H)$ is a two sided ideal of $E(H)$, $J(E_s(H)) = J(E(H)) \cap E_s(H)$ (see [4], p. 10). Thus, $J(E_s(H)) \subseteq J(E(H))$. Since $J(R_p) = pR_p$ (pR_p is the unique maximal ideal of R_p) and since $J(E(H))$ is an ideal, Lemma 3.7 is enough to insure that $J(R_p) \subseteq J(E(H))$.

4. **An extension property.** In § 3, it was shown, using suitable pure subgroups of \bar{B} , that there are at least two distinct rings of the form $E(G)/J(E(G), E_p(G))$, namely, $\prod M_n$ and $\sum M_n + \{1\}$. It is the objective of the remainder of this paper to investigate some of the possible images $\xi(E(G))$ for $B \subseteq G \subseteq \bar{B}$.

For the duration, assume that $B = \sum_{i \in N} B_i$ where each $B_i = \{b_i\}$ is of rank one and of order p^i . In this case each M_i automatically becomes fixed as a single copy of Z_p . That is, each M_i will be the ring of all endomorphisms of a cyclic group, $\{c_i\}$, of order p .

For a subset A of N , let $t(A)$ be the element of $\prod_{n \in N} M_n$ defined by the conditions

$$t(A)(c_j) = \begin{cases} c_j & \text{if } j \in A \\ 0 & \text{if } j \notin A. \end{cases}$$

It is obvious that if r is any element of $\prod_{n \in N} M_n$ and if for each $i = 0, 1, \dots, p - 1$ $A_i(r) = \{j \in N \mid r(c_j) = ic_j\}$, then r can be written in the form $r = \sum_{i=0}^{p-1} it(A_i(r))$.

LEMMA 4.1. *Let R be any subring of $\prod_{n \in N} M_n$ with identity e . (The identity of $\prod_{n \in N} M_n$ and e are not assumed to be identical.) Then $e = t(M)$ for some subset M of N . Moreover, the collection $K(R) = \{A \subseteq N \mid t(A) \in R\}$ forms a Boolean algebra of subsets of M .*

Proof. Using Fermat's theorem

$$e = e^{p-1} = \left(\sum_{i=0}^{p-1} it(A_i(e)) \right)^{p-1} = \sum_{i=0}^{p-1} i^{p-1} t(A_i(e)) = \sum_{i=1}^{p-1} t(A_i(e)) = t(M)$$

where $M = \{i \in N \mid e(c_i) \neq 0\}$. If $t(A), t(B)$ are members of R , then $t(A \cap B) = t(A)t(B) \in R$ and $t(A \cup B) = t(A) + t(B) - t(A \cap B) \in R$. Since $t(A) = e \cdot t(A) = t(M) \cdot t(A) = t(M \cap A)$, it follows that $A \subseteq M$ for all $A \in K(R)$. Thus, $t(M - A) = t(M) - t(A) = e - t(A) \in R$ for all $A \in K(R)$. This shows that $K(R)$ does indeed form a subalgebra of $P(M) = \{A \mid A \subseteq M\}$.

LEMMA 4.2. *Let R be a subring of $\prod_{n \in N} M_n$ with identity $e = t(M)$. If $r \in R$, then $t(A_k(r)) \in R$ for each $k = 0, 1, \dots, p - 1$.*

Proof.

$$r = 0 \cdot t(A_0(r)) + t(A_1(r)) + 2t(A_2(r)) + \dots + (p - 1)t(A_{p-1}(r)) .$$

Consider the product

$$s = \prod_{i \neq k, i=0,1,\dots,p-1} (ie - r) .$$

It follows that $s \in R$. Clearly, if $i \notin A_k(r)$, then $s(c_j) = 0$ since $j \in A_i(r)$ for some i and

$$(ie - r)(c_j) = ic_j - r(c_j) = ic_j - ic_j = 0 .$$

Also, if $j \in A_k(r)$, then

$$\begin{aligned} s(c_j) &= (0 - k)(1 - k)(2 - k) \dots ((k - 1) - k)((k + 1) - k) \\ &\dots ((p - 1) - k)(c_j) = (p - 1)! c_j . \end{aligned}$$

By Wilson's theorem, $(p - 1)! \equiv -1 \pmod{p}$; consequently, $t(A_k(r)) = -s \in R$.

Suppose R is a subring of $\prod M_n$ which contains $\sum M_n + \{1\}$. For each $A \in K(R)$, let $\rho(A) = \sum_{i \in A} \rho_i$. Define $\Gamma(R)$ to be the subgroup of $E(\bar{B})$ generated by the collection $\{\rho(A) \mid A \in K(R)\}$. Using Lemma 4.1 and 4.2 some elementary properties of $\Gamma(R)$ can be stated.

LEMMA 4.3. *If $\alpha \in \Gamma(R)$, then there exists an integer $n \geq 0$, integers a_1, a_2, \dots, a_n and disjoint elements A_1, A_2, \dots, A_n in $K(R)$ such that*

$$\alpha = a_1\rho(A_1) + a_2\rho(A_2) + \cdots + a_n\rho(A_n) .$$

Moreover, the group $\Gamma(R)$ is a subring of $E(\bar{B})$.

Proof. For the first statement, induction can be used. For the induction step, it is enough to show that if

$$\alpha = a_1\rho(A_1) + a_2\rho(A_2) + \cdots + a_{n-1}\rho(A_{n-1}) + a_n\rho(A_n)$$

where A_1, \dots, A_{n-1} are disjoint, then the result holds. Using 4.1, $A_1, \dots, A_n \in K(R)$ imply that

$$A_1 \cap A_n, \dots, A_{n-1} \cap A_n; A_1 - A_n, \dots, A_{n-1} - A_n;$$

and $A_n - \bigcup_{i=1}^{n-1} A_i$ are members of $K(R)$. Moreover, these sets are disjoint. Thus, if α is written

$$\begin{aligned} \alpha &= a_1\rho(A_1 - A_n) + \cdots + a_{n-1}\rho(A_{n-1} - A_n) + (a_1 + a_n)\rho(A_1 \cap A_n) \\ &+ \cdots + (a_{n-1} + a_n)\rho(A_{n-1} - A_n) + a_n\rho\left(A_n - \bigcup_{i=1}^{n-1} A_i\right), \end{aligned}$$

then it is easily checked that this is the desired decomposition. To show that $\Gamma(R)$ and the subring of $E(\bar{B})$ generated by $\Gamma(R)$ are identical, it is enough to show that $\Gamma(R)$ is closed under composition. It suffices to note that if $A_1, A_2 \in K(R)$, then $\rho(A_1)\rho(A_2) = \rho(A_1 \cap A_2) \in \Gamma(R)$. This is obvious by Lemma 4.1 and the definition of $\Gamma(R)$.

LEMMA 4.4. $R = \xi(\Gamma(R))$.

Proof. If $r \in R$, then $r = \sum_{i=0}^{p-1} it(A_i(r))$ where $A_i(r) \in K(R)$ (see 4.2). Let $\alpha = \sum_{i=0}^{p-1} i\rho(A_i(r))$. Then $\alpha \in \Gamma(R)$ and $\xi(\alpha) = r$. Thus, $R \subseteq \xi(\Gamma(R))$. On the other hand, suppose

$$\alpha = a_1\rho(A_1) + \cdots + a_n\rho(A_n) \in \Gamma(R) ,$$

where $a_1, \dots, a_n \in Z$ and $A_1, \dots, A_n \in K(R)$. Applying ξ ,

$$\begin{aligned} \xi(\alpha) &= a_1\xi\rho(A_1) + \cdots + a_n\xi\rho(A_1) + \cdots + a_n\xi(\rho(A_n)) = \\ &a_1t(A_1) + \cdots + a_nt(A_n) \in R \end{aligned}$$

(see the definition of $K(R)$ in Lemma 4.1).

The following lemma is needed before the main result of this section can be given.

LEMMA 4.5. Let $y = h \sum_{j \geq k} a_j p^{j-k} b_j$ where $h \in Z, k \in N$ and each $a_j (j \geq k)$ is an integer such that $0 \leq a_j < p$. If $A \subseteq N$ and $i \in N$, then $p^{i-1}y \neq 0$ and $\rho(A)(p^{i-1}y) \in B$ imply that $\rho(A)(y) \in B$.

Proof. Suppose $\rho(A)(y) \notin B$. Then if $A_0 = \{i \in A \mid a_i \neq 0\}$, A_0 is infinite. Since $\rho(A)(p^{i-1}y) \in B$, there is some $n \in A_0$ such that $\rho_n \rho(A)(p^{i-1}y) = 0$. Thus,

$$0 = \rho_n \rho(A)(p^{i-1}y) = \rho_n(p^{i-1}y) = p^{i-1} h a_n p^{n-k} b_n = h a_n p^{n+i-k-1} b_n,$$

so that p^{k+i-1} divides h . Since $p^{i-1}y \neq 0$, this cannot be the case.

THEOREM 4.6. *Let G be a pure subgroup of \bar{B} such that $B \subseteq G$ and $\gamma(G) \subseteq G$ for each $\gamma \in \Gamma(R)$. Suppose $x \in \bar{B}[p]$ is such that $\Gamma(R)(x) \cap G[p] \subseteq B[p]$. Then there is a pure subgroup H of \bar{B} such that*

- (i) $B \subseteq G \subseteq H$
- (ii) $H[p] = G[p] + \Gamma(R)(x)$
- (iii) $\gamma(H) \subseteq H$ for each $\gamma \in \Gamma(R)$.

Proof. Write $x = \sum_{i \geq k_0} a_i p^{i-1} b_i$ where $k_0 > 0$, $0 \leq a_i < p$ for $i \geq k_0$ and $a_{k_0} \neq 0$. Let K be the subgroup of \bar{B} generated by B and the collection consisting of all sums of the form $\sum_{i \geq k} a_i p^{i-k} b_i$ where $k \geq k_0$. Consider the group \bar{K} generated by all elements of the form $\gamma(z)$ for $z \in K$ and $\gamma \in \Gamma(R)$. It is claimed that the group $H = \bar{K} + G$ has all the desired properties. First, note that K is exactly the subset of \bar{B} consisting of all elements which can be written as $b + h \sum_{j \geq k} a_j p^{j-k} b_j$ for some $b \in B$, $h \in Z$ and $k \in N$ (the integers a_j for $j \geq k$ are determined by the element x). Also, if $y = b + h \sum_{j \geq k} a_j p^{j-k} b_j \in K$, then y may be written as $y = b' + p^n h \sum_{j \geq k+n} a_j p^{j-(k+n)} b_j$, where $b' = b + h \sum_{j=k}^{k+n-1} a_j p^{j-k} b_j \in B$ and $\sum_{j \geq k+n} a_j p^{j-(k+n)} b_j \in K$. Thus, K/B is divisible. Suppose $n \in N$, $\gamma_1, \dots, \gamma_k \in \Gamma(R)$ and $x_1, \dots, x_k \in K$. Using the divisibility of K/B , choose $y_1, \dots, y_k \in K$ such that $x_i - p^n y_i \in B$ for each $i = 1, \dots, k$. Since $\gamma \in \Gamma(R)$ implies $\gamma(B) \subseteq B$, it follows that

$$\begin{aligned} & \gamma_1(x_1) + \dots + \gamma_k(x_k) - p^n(\gamma_1(y_1) + \dots + \gamma_k(y_k)) \\ &= \gamma_1(x_1) - \gamma_1(p^n y_1) + \dots + \gamma_k(x_k) - \gamma_k(p^n y_k) \\ &= \gamma_1(x_1 - p^n y_1) + \dots + \gamma_k(x_k - p^n y_k) \in B. \end{aligned}$$

This shows that \bar{K}/B is divisible. (Note that $B \subseteq \bar{K}$ since $1 \in \Gamma(R)$ and $B \subseteq K$.) Now, both \bar{K}/B and G/B are divisible. Consequently, $H = \bar{K} + G$ is a pure subgroup of \bar{B} since $(\bar{K} + G)/B = (\bar{K}/B) + (G/B)$ is a sum of divisible groups and hence divisible. Since, $\alpha, B \in \Gamma(R)$ imply that $\alpha B \in \Gamma(R)$ (see 4.3), it follows that $\gamma(\bar{K}) \subseteq \bar{K}$ for all $\gamma \in \Gamma(R)$. Thus, $\gamma(H) \subseteq H$ for each $\gamma \in \Gamma(R)$. It remains only to show that $H[p] = G[p] + \Gamma(R)(x)$. First, suppose that

$$y = h \sum_{j \geq k} a_j p^{j-k} b_j \in K \quad \text{and} \quad A \in K(R)$$

Then $\rho(A)(y) \in G$ if and only if $\rho(A)(y) \in B$. To show that this assertion is correct, suppose that $\rho(A)(y)$ is a member of G . Then, if $i = E(y)$, $p^{i-1}y = h'x - b'$ for suitable $h' \in Z$ and $b' \in B$. Thus, since $\Gamma(R)(x) \cap G \subseteq B$ and $B \subseteq G$, it follows that $\rho(A)(p^{i-1}y + b') = \rho(A)(h'x) \in B$ and that $\rho(A)(p^{i-1}y) \in B$. But, $\rho(A)(p^{i-1}y) \in B$, $p^{i-1}y \neq 0$ and $y = h \sum_{j \geq k} a_j p^{j-k} b_j$ imply, via 4.5 and the restriction on the $a_i (i \geq k)$, that $\rho(A)(y) \in B$. The converse is trivial. Let

$$x_1, x_2, \dots, x_n \in K, z \in G \quad \text{and} \quad \gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma.$$

Suppose

$$p(\gamma_1(x_1) + \gamma_2(x_2) + \dots + \gamma_n(x_n) + z) = 0.$$

For each $i = 1, 2, \dots, n$, let $x_i = d_i + h_i \sum_{j \geq k_i} a_j p^{j-k_i} b_j$ where $d_i \in B$, $h_i \in Z$ and $k_i \in N$. Let k' be any positive integer greater than each of the integers k_1, k_2, \dots, k_n . It is easily checked that there exist integers m_1, m_2, \dots, m_n and elements d'_1, d'_2, \dots, d'_n of B such that for each $i = 1, \dots, n$

$$x_i = d'_i + m_i \sum_{j \geq k'} a_j p^{j-k'} b_j.$$

Thus, if $y = \sum_{j \geq k'} a_j p^{j-k'} b_j$, then

$$\begin{aligned} \gamma_1(x_1) + \dots + \gamma_n(x_n) &= \gamma_1(d'_1 + m_1 y) + \dots + \gamma_n(d'_n + m_n y) \\ &= \gamma_1(d'_1) + \dots + \gamma_n(d'_n) + (m_1 \gamma_1 + \dots + m_n \gamma_n)(y) \\ &= b + \gamma(y) \end{aligned}$$

where $b \in B$ and $\gamma \in \Gamma$. Since $\gamma \in \Gamma$, it is possible to write $\gamma = e_1 \rho(A_1) + \dots + e_m \rho(A_m)$ where A_1, \dots, A_m are disjoint members of $K(R)$ and where $e_1, \dots, e_m \in Z$ (see 4.3). Now,

$$p(\gamma_1(x_1) + \gamma_2(x_2) + \dots + \gamma_n(x_n) + z) = 0$$

implies $p(b + \gamma(y) + z) = 0$; and, therefore, $p\gamma(y) \in G + B = G$. Suppose that $e_i \rho(A_i)(y) \notin B$ for some $i = 1, \dots, m$. Then since $b \in G$, $p\gamma(y) \in G$ and $\rho(A_i)(G) \subseteq G$, it follows that

$$\rho(A_i)(p\gamma(y)) = p e_i \rho(A_i)(y) = \rho(A_i)(p e_i y) \in G.$$

Thus, as was noted, $\rho(A_i)(p e_i y) \in B$. Now,

$$\begin{aligned} \rho(A_i)(p e_i y) &= \rho(A_i) \left(p e_i \sum_{j \geq k'} a_j p^{j-k'} b_j \right) \\ &= p e_i \sum_{\substack{j \geq k' \\ j \in A_i}} a_j p^{j-k'} b_j \in B. \end{aligned}$$

Since, by assumption, $e_i \rho(A_i)(y) \notin B$, it follows that $\rho(A_i)(y) \notin B$. Thus, $p^{k'-1}$ divides e_i . Therefore, $e_i y = e'_i x - b'$ for suitable $e'_i \in Z$ and $b' \in B$.

Consequently, $e_i \rho(A_i)(y) = \rho(A_i)(e_i y) \in \Gamma(R)(x) + B$. It follows that $\gamma_1(x_1) + \dots + \gamma_n(x_n) \in \Gamma(R)(x) + B$ and that

$$\gamma_1(x_1) + \dots + \gamma_n(x_n) + z \in \Gamma(R)(x) + G.$$

Thus, $\gamma_1(x_1) + \dots + \gamma_n(x_n) + z = y + w$ where $y \in \Gamma(R)(x)$ and $w \in G$. Also,

$$0 = p(\gamma_1(x_1) + \dots + \gamma_n(x_n) + z) = p(y + w) = pw$$

and $w \in G[p]$. This shows that $H[p] \subseteq G[p] + \Gamma(R)(x)$. The opposite inclusion is obvious.

5. The image. This section is devoted to the construction of a class of pure subgroups of \bar{B} having suitably restricted endomorphism rings. The methods used here are similar to those employed by P. Crawley in [2] and R. S. Pierce in [7].

DEFINITION 5.1. (*R. S. Pierce*) A family \mathcal{F} of subsets of a set F is called *weakly independent* if whenever A_0, A_1, \dots, A_n are distinct elements of \mathcal{F} , then A_0 is not contained in the union of the remaining sets A_1, A_2, \dots, A_n .

THEOREM 5.2. (*R. S. Pierce*) Let F be a set of infinite cardinality φ . If ψ is a cardinal number such that $0 < \psi \leq \varphi$, then there is a family \mathcal{F} of subsets of F such that

- (a) \mathcal{F} is weakly independent,
- (b) $|A| = \psi$ for all $A \in \mathcal{F}$,
- (c) $|\mathcal{F}| = \varphi^\psi$.

Proof. (See [8], p. 261.)

At this point it is convenient to set $\theta = \{\alpha | \bar{B}[p] | \alpha \in E(\bar{B})\}$. It is clear that θ is a ring with identity. For the moment, only the additive group structure of θ will be considered.

LEMMA 5.3. Let $\Gamma = \{\alpha_0 = 0, \alpha_1, \alpha_2, \dots\}$ be any countable subgroup of θ satisfying the following condition:

(*) for all nonzero $\alpha \in \Gamma, \alpha(c_j) \neq 0$ for an infinite number of indices $j \in N$.

There is a collection $T(\Gamma)$ of element in $\bar{B}[p]$ such that

- (i) $|T(\Gamma)| = 2^{\aleph_0}$,
- (ii) $\sum_{x \in T(\Gamma)} \Gamma(x)$ is direct ($\Gamma(x) = \{\alpha(x) | \alpha \in \Gamma\}$).
- (iii) $\alpha_i(x) \neq \alpha_j(x)$ for all $x \in T(\Gamma)$ and for all $i \neq j$,
- (iv) $\alpha_i(x) = 0$ for some $x \in T(\Gamma)$ implies $\alpha_i = 0$.

Proof. Let $K = N \times N$. Well order K in the following way: $(i, j) < (k, h)$ if $i + j < k + h$ or if $i + j = k + h$ and $i < k$. Now, each element of Γ satisfies (*). Thus, since the set

$$\{(i, j) \in K \mid (i, j) < (k, h)\}$$

is finite for all elements $(k, h) \in K$, it is possible to define, inductively, an order preserving one-to-one map f of K into N such that $h_{\bar{B}}(\alpha_i(c_{f(i,j)}))$ is finite (i.e., $\alpha_i(c_{f(i,j)}) \neq 0$) and is greater than the height or every nonzero element in the finite subgroup of $\bar{B}[p]$ generated by the collection $\{\alpha_k(c_{f(m,n)}) \mid k \leq i \text{ and } (m, n) < (i, j)\}$. Let \mathcal{F} be any weakly independent collection of subsets of N such that $|\mathcal{F}| = 2^{n_0}$. If $S \in \mathcal{F}$, let $x(S) \in \bar{B}[p]$ be defined by the expression:

$$x(S) = \sum_{j \in S} c_{f(i,j)}.$$

Let $T(\Gamma) = \{x(S) \mid S \in \mathcal{F}\}$. Suppose $S_1, S_2, \dots, S_{n_0} \in \mathcal{F}$ are distinct, $x_i = x(S_i)$ for $i = 1, 2, \dots, n_0$ and

$$\sum_{i=1}^{n_0} \alpha_{k_i}(x_i) = 0$$

for positive integers k_1, k_2, \dots, k_{n_0} . Since \mathcal{F} is weakly independent, there exists for each $i = 1, 2, \dots, n_0$ an integer

$$m_i \in S_i - \bigcup_{\substack{j \neq i \\ j \leq n_0}} S_j.$$

Let k_i be the largest integer in the collection $\{k_1, \dots, k_{n_0}\}$. Let $h_i = h_{\bar{B}}(\alpha_{k_i}(c_{f(k_i, m_i)})) + 1$. It follows that

$$(1) \quad (1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i, m_i)}) \neq 0$$

and

$$(2) \quad (1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i, m_i)}) + (1 - \pi_{h_i})\alpha_{k_i}(x - c_{f(k_i, m_i)}) + (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1, 2, \dots, n_0}} \alpha_{k_j}(x_j) = 0.$$

Now,

$$\begin{aligned} & (1 - \pi_{h_i})\alpha_{k_i}(x - c_{f(k_i, m_i)}) + (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1, 2, \dots, n_0}} \alpha_{k_j}(x_j) \\ &= (1 - \pi_{h_i})\alpha_{k_i}(1 - \pi_{h_i})(x_i - c_{f(k_i, m_i)}) \\ &+ (1 - \pi_{h_i}) \sum_{\substack{j \neq i \\ j=1, 2, \dots, n_0}} \alpha_{k_j}(1 - \pi_{h_i})(x_j). \end{aligned}$$

Since $m_i \in S_i$, it follows from the definition of $x_i = x(S)$ and the order preserving property of the mapping f that

$$(1 - \pi_{h_i})(x_i - c_{f(k_i, m_i)}) = \sum_{\substack{(m, n) < (k_i, m_i) \\ n \in S_i}} c_{f(m, n)}.$$

Hence, $\alpha_{k_i}(1 - \pi_{h_i})(x_i - c_{f(k_i, m_i)})$ belongs to the subgroup S of $\bar{B}[p]$ generated by the collection

$$\{\alpha_k(c_{f(m, n)}) \mid k \leq k_i, (m, n) < (k_i, m_i)\}.$$

Also, if $j \neq i$, then $m_i \notin S_j$ and $k_j \leq k_i$. Therefore,

$$(1 - \pi_{h_i})(x_j) = \sum_{\substack{(m, n) < (k_i, m_i) \\ n \in S_j}} c_{f(m, n)};$$

and because of this, $\alpha_{k_j}((1 - \pi_{h_i})(x_j)) \in S$. Thus, from (1), (2) and the above, $(1 - \pi_{h_i})\alpha_{k_i}(c_{f(k_i, m_i)}) = (1 - \pi_{h_i})(z) \neq 0$ for some z in S . It follows that $h_{\bar{B}}(\alpha_{k_i}(c_{f(k_i, m_i)})) = h_{\bar{B}}(z)$, a contradiction of the definition of the map f . Thus, $\sum_{x \in T(\Gamma)} \Gamma(x)$ is direct. Condition (i) is clear from the definition of $T(\Gamma)$. Condition (iv) follows from the preceding argument with $n = 1$. Since Γ is a group, condition (iii) follows easily from (iv).

DEFINITION 5.4. Let Γ be a subgroup of θ . An element α in θ will be called Γ -exceptional provided there exists a collection $T(\Gamma, \alpha)$ of elements in $\bar{B}[p]$ such that

(i) $|T(\Gamma, \alpha)| = 2^{\aleph_0}$,

(ii) $\Gamma(x), \Gamma(y), \{\alpha(x)\}, \{\alpha(y)\}$ are independent for all distinct $x, y \in T(\Gamma, \alpha)$,

(iii) $\alpha(x) \neq 0$ for all $x \in T(\Gamma, \alpha)$.

An endomorphism $\alpha \in E(\bar{B})$ will be called Γ -exceptional if $\alpha|_{\bar{B}[p]}$ is Γ -exceptional.

REMARK. If $\alpha \in E(\bar{B})$ and $\alpha|_{\bar{B}[p]} = 0$, then by 3.7 and 2.8 (a) $\alpha \in J(E(\bar{B}), E_p(\bar{B}))$. Thus, the kernel of the map $\alpha \rightarrow \alpha|_{B[p]}$ is contained in $J(E(\bar{B}), E_p(\bar{B}))$. It follows that ξ can be considered as a map from θ to $\prod M_n$ by defining for each $\alpha \in \theta$, $\xi(\alpha) = \xi(\beta)$ where $\beta \in E(B)$ and $\alpha = \beta|_{\bar{B}[p]}$. Extensive use will be made of this convention in what follows.

LEMMA 5.5. Let Γ be any countable subgroup of θ . Suppose $\alpha \in \theta$ is such that $\alpha \notin \Gamma$ and $\Delta = \{\Gamma, \alpha\}$ satisfies the following condition:

(*) for all nonzero $\beta \in \Delta$, $\beta(c_j) \neq 0$ for an infinite number of indices $j \in N$. Then α is Γ -exceptional.

Proof. Since $\Delta = \{\Gamma, \alpha\}$ is obviously countable and satisfies (*), Lemma 5.3 can be applied to conclude that there exists a collection $T(\Delta)$ with the properties:

- (i) $|T(\mathcal{A})| = 2^{\aleph_0}$,
- (ii) $\sum_{x \in T(\mathcal{A})} \mathcal{A}(x)$ is direct.
- (iii) $\gamma(x) \neq \beta(x)$ for all $x \in T(\mathcal{A})$ and distinct $\beta, \gamma \in \mathcal{A}$,
- (iv) $\beta \in \mathcal{A}$ and $\beta(x) = 0$ for some $x \in T(\mathcal{A})$ implies $\beta = 0$.

Set $T(\Gamma, \alpha) = T(\mathcal{A})$. Clearly, conditions (i) and (iii) of 5.4 are satisfied. Let $x, y \in T(\Gamma, \alpha)$ be distinct, and suppose there is a relation of the form $\beta(x) + k\alpha(x) + \gamma(y) + h\alpha(y) = 0$ where $\beta, \gamma \in \Gamma$ and $h, k \in Z$. By (ii), it is clear that both $\beta(x) + k\alpha(x) = 0$ and $\gamma(y) + h\alpha(y) = 0$. It follows by (iv) that $\beta + k\alpha = 0$ and $\gamma + h\alpha = 0$. Since $E(\alpha) = 1$ and $\alpha \notin \Gamma$, this last condition implies that both $h\alpha = 0$ and $k\alpha = 0$. Thus $\beta(x) = k\alpha(x) = \gamma(y) = h\alpha(y) = 0$, and condition (ii) of 5.4 is also satisfied. This completes the proof.

COROLLARY 5.6. *Let Γ be any countable subgroup of Θ satisfying (*) for all nonzero $\gamma \in \Gamma, \gamma(c_j) \neq 0$ for an infinite number of indices $j \in N$.*

Suppose $\alpha \in \Theta$ is such that $\xi(\alpha)$ is not a member of $\xi(\Gamma) + (\sum M_n)$. Then α is Γ -exceptional.

Proof. Clearly, $\alpha \notin \Gamma$ since $\xi(\alpha) \notin \xi(\Gamma)$. Consequently, it is enough to show that $\mathcal{A} = \{\Gamma, \alpha\}$ satisfies condition (*). Suppose, to the contrary, that there exist $n \in N$ and $\beta \in \mathcal{A}$ such that $\beta \neq 0$ and $\beta(c_j) = 0$ for all $j > n$. It is possible to write $\beta = \gamma + k\alpha$ where $\gamma \in \Gamma$ and $k \in Z$. Since Γ satisfies (*) and $E(\alpha) = 1$, it can be assumed that $k \not\equiv 0$ (modulo p). Now, $\beta = \gamma + k\alpha$ and

$$\xi(k\alpha) = \xi(\beta - \gamma) = \xi(\beta) - \xi(\gamma) \in \sum M_n + \xi(\Gamma).$$

Since k is relatively prime to p , it follows that $\xi(\alpha) \in \sum M_n + \xi(\Gamma)$, a contradiction.

COROLLARY 5.7. *Let Γ be any countable subgroup of Θ satisfying the following condition:*

(**) *for all nonzero $\gamma \in \Gamma$ there exists a sequence of integers $\{a_i\}_{i \in N}$ such that $\gamma(c_i) = a_i c_i$ for each $i \in N$, and $a_i c_i \neq 0$ for an infinite number of indices $i \in N$.*

Let $\alpha \in \Theta$ be such that $\alpha(c_i) - \rho_i \alpha(c_i) \neq 0$ for an infinite number of indices $i \in N$. Then α is Γ -exceptional.

Proof. If $\gamma \in \Gamma$, then $\gamma(c_i) - \rho_i \gamma(c_i) = a_i c_i - a_i c_i = 0$ for all $i \in N$. Thus, $\alpha \notin \Gamma$. As before, let $\mathcal{A} = \{\Gamma, \alpha\}$, suppose $\gamma + k\alpha \in \mathcal{A}$. If $k \equiv 0$ (modulo p), then either $\gamma = 0$ or $(\gamma + k\alpha)(c_i) = \gamma(c_i) \neq 0$ for an infinite number of indices $i \in N$. If $\gamma = 0$, then $\gamma + k\alpha = 0$; and there is nothing to show. Suppose $k \not\equiv 0$ (modulo p). It follows that

$$\begin{aligned}
 (1 - \rho_i)(\gamma + k\alpha)(c_i) &= (\gamma + k\alpha)(c_i) - \rho_i(\gamma + k\alpha)(c_i) \\
 &= (\gamma - \rho_i\gamma)(c_i) + k(\alpha - \rho_i\alpha)(c_i) \\
 &= k(\alpha - \rho_i\alpha)(c_i) \neq 0
 \end{aligned}$$

for an infinite number of indices $i \in N$. Consequently, $\gamma + k\alpha$ must have this same property, and by 5.5, α is Γ -exceptional.

Let R be any countable subring of $\prod M_n$ which contains $\sum M_n + \{1\}$. Let $\Gamma(R)$ be as defined in § 4. That is, $\Gamma(R)$ is the subgroup of $E(\bar{B})$ generated by the collection $\{\rho(A) \mid A \in K(R)\}$. Define Γ to be the subgroup of θ defined by $\Gamma = \{\gamma \mid \bar{B}[p] \mid \gamma \in \Gamma(R)\}$. Note that Γ is a p -group in which every element has order p . By Zorn's lemma, it is possible to choose a subgroup Δ of Γ which contains the identity and which is maximal with respect to having only the zero element in common with the subgroup $\{\gamma \in \Gamma \mid \xi(\gamma) \in \sum M_n\}$. Obviously, Δ is a countable subgroup of θ which satisfies condition (*) of 5.3. Let \mathcal{E} be the collection of all those elements in θ which are Δ -exceptional. By 5.6, if $\alpha \in \theta$ and $\xi(\alpha) \notin \xi(\Delta) + \sum M_n$, then α is Δ -exceptional. Since $\xi(\Delta) + (\sum M_n)$ is countable and since ξ maps onto $\prod M_n$ by Lemma 3.2, it follows that $|\mathcal{E}| = 2^{\aleph_0}$. Let Ω be the first ordinal of cardinality 2^{\aleph_0} , and let $\varphi \leftrightarrow \alpha_\varphi$ be a one-to-one correspondence between the elements of \mathcal{E} and the ordinals $\varphi < \Omega$.

LEMMA 5.8. *There exist collections $\{G_\varphi \mid \varphi < \Omega\}$, $\{P_\varphi \mid \varphi < \Omega\}$ and $\{U_\varphi \mid \varphi < \Omega\}$ such that*

- (i) *for all $\varphi < \Omega$, G_φ is a pure subgroup of \bar{B} containing B , $P_\varphi = G_\varphi[p]$ and U_φ is a subset of $\bar{B}[p]$,*
- (ii) *$G_\varphi \subseteq G_\chi$ and $U_\varphi \subseteq U_\chi$ whenever $\varphi \leq \chi < \Omega$,*
- (iii) *$|P_\varphi| \leq (|\varphi| + 1)\aleph_0$ and $|U_\varphi| \leq (|\varphi| + 1)\aleph_0$,*
- (iv) *$\gamma(G_\varphi) \subseteq G_\varphi$ for all $\gamma \in \Gamma(R)$ and each $\varphi < \Omega$,*
- (v) *$P_\varphi \cap U_\varphi = \emptyset$ for all $\varphi < \Omega$,*
- (vi) *for each $\varphi < \Omega$ there exists $z_\varphi \in P_\varphi$ such that $\alpha_\varphi(z_\varphi) \in U_\varphi$.*

Proof. The proof is by transfinite induction. Suppose G_φ and U_φ exist for all $\varphi < \chi$. Let $G'_\chi = \bigcup_{\varphi < \chi} G_\varphi + B$. $P'_\chi = \bigcup_{\varphi < \chi} P_\varphi + B[p]$ and $U'_\chi = \bigcup_{\varphi < \chi} U_\varphi$. Note that $G'_\chi[p] = P'_\chi$, that $\gamma(G'_\chi) \subseteq G'_\chi$ for each $\gamma \in \Gamma(R)$, and that G'_χ is a pure subgroup of \bar{B} . Suppose there is an element z in $P'_\chi \cap U'_\chi$. The existence of z implies the existence of ordinals $\psi < \chi$ and $\omega < \chi$ such that $z \in U_\psi$ and $z \in P_\omega + B[p] = P_\omega$. Let φ be largest of ψ and ω . Then $z \in [P_\varphi + B[p]] \cap U_\varphi = P_\varphi \cap U_\varphi$, contrary to the induction hypothesis. Thus, $P'_\chi \cap U'_\chi = \emptyset$. Since $|P_\varphi| \leq (|\varphi| + 1)\aleph_0$ and $|U_\varphi| \leq (|\varphi| + 1)\aleph_0$ for each $\varphi < \chi$, it follows that $|P'_\chi| \leq (|\chi| + 1)\aleph_0$ and $|U'_\chi| \leq (|\chi| + 1)\aleph_0$. Thus,

$$|\{P'_\chi, U'_\chi, B[p]\}| \leq |P'_\chi| |U'_\chi| \aleph_0 \leq (|\chi| + 1)\aleph_0 < 2^{\aleph_0}$$

Since α_x is Δ -exceptional, there is a collection $T(\alpha_x) \subseteq \bar{B}[p]$ such that

- (a) $|T(\alpha_x)| = 2^{\aleph_0}$
- (b) $y, z \in T(\alpha_x)$ imply that $\Delta(y), \Delta(z), \{\alpha_x(y)\}, \{\alpha_x(z)\}$ are independent and $\alpha_x(y), \alpha_x(z)$ are nonzero. Therefore, it is possible to find $z_x \in T(\alpha_x)$ such that $\alpha_x(z_x) \neq 0$ and

$$(\#) \quad \{\Delta(z_x), \alpha_x(z_x)\} \cap \{P'_x, U'_x, B[p]\} = 0.$$

Now suppose $\gamma \in \Gamma(R)$. Since every element of Γ has order p and since Δ is maximal with respect to having zero intersection with $\{\gamma \in \Gamma \mid \xi(\gamma) \in \sum M_n\}$, it is possible to write $\gamma \mid \bar{B}[p]$ as $\alpha + \beta$ where $\alpha \in \Delta, \beta \in \Gamma$ and $\xi(\beta) \in \sum M_n$. Since $\xi(\beta) \in \sum M_n$, it follows from the definition of $\Gamma(R)$ that $\beta(\bar{B}[p]) \subseteq B[p]$. Therefore,

$$\gamma(z_x) = \alpha(z_x) + \beta(z_x) \in \Delta(z_x) + B[p].$$

Consequently, if $\gamma(z_x) \in G'_x[p] = P'_x$, then (using $\#$) $\gamma(z_x) \in B[p]$. Thus, G'_x and z_x satisfy the hypothesis of 4.6. Let G_x be the pure subgroup of \bar{B} obtained by the application of 4.6. Then

$$P_x = G_x[p] = G'_x[p] + \Gamma(R)(z_x) = P'_x + \Delta(z_x)$$

and $\gamma(G_x) \subseteq G_x$ for each $\gamma \in \Gamma(R)$. Also, $|P_x| \leq (|\chi| + 1)\aleph_0$. Let U_x be the set obtained by adjoining $\alpha_x(z_x)$ to U'_x . Then $|U_x| \leq (|\chi| + 1)\aleph_0$, and conditions (i), (ii), (iii), (iv) and (vi) obviously are satisfied. To show that (v) holds, suppose $z \in P_x \cap U_x$. There are two cases to consider:

Case 1. $z = \alpha_x(z_x)$ and $z = y + \beta(z_x)$ for $y \in P'_x$ and $\beta \in \Delta$. By $\#$, $\alpha_x(z_x) - \beta(z_x) = y = 0$. Thus, applying (b), it is clear that $\alpha_x(z_x) = 0$. This is a contradiction of the choice of z_x .

Case 2. $z \in U'_x$ and $z = y + \beta(z_x)$ for $y \in P'_x$ and $\beta \in \Delta$. In this case, $0 = z - y = \beta(z_x)$ by $\#$. Consequently, $y = z \in U'_x$. This is a contradiction since $U'_x \cap P'_x = \emptyset$.

LEMMA 5.9. Let $G(R) = \bigcup_{x < \alpha} G_x, P(R) = \bigcup_{x < \alpha} P_x$ and $U(R) = \bigcup_{x < \alpha} U_x$. Then

- (i) $G(R)$ is a pure subgroup of \bar{B} ,
- (ii) $G(R)[p] = P(R)$,
- (iii) $P(R) \cap U(R) = \emptyset$,
- (iv) $\gamma(G(R)) \subseteq G(R)$ for each $\gamma \in \Gamma(R)$,
- (v) if $\alpha \in E(\bar{B})$ and if α is Δ -exceptional, then $\alpha \notin E(G(R))$.

Proof. The arguments for (i), (ii), (ii), and (iv) are quite easy and can be found in the proof of 5.8. To show (v), suppose α is

Δ -exceptional. Then there exist $\varphi < \Omega$ and $z_\varphi \in P(R)$ such that $\alpha(z_\varphi) \in U_\varphi$ (see (vi) of 5.8). Since $P(R) \cap U(R) = \emptyset$ and $G(R)[p] = P(R)$ by (iii) and (ii), it follows that $\alpha \notin E(G(R))$.

THEOREM 5.10. *Let R be any countable subring of the ring direct product $\prod M_n$. Suppose that R contains $\sum M_n + \{1\}$. There is a pure subgroup G of \bar{B} , containing B , such that $\xi(E(G)) = R$. Moreover,*

$$\frac{E(G)}{J(E(G), E_p(G))} \cong R.$$

Proof. Let $G = G(R)$. By 4.4, $R = \xi(\Gamma(R))$. Thus, since $\Gamma(R) \subseteq E(G)$ by (iv) of 5.9, $R \subseteq \xi(E(G(R)))$. Suppose $\alpha \in E(G(R))$ and $\xi(\alpha) \notin R$. By 4.4, $\xi(\Delta) \subseteq \xi(\Gamma) \subseteq \xi(\Gamma(R)) = R$. Thus, $\xi(\Delta) + (\sum M_n) \subseteq R$, and Lemma 5.6 may be applied to infer that α is Δ -exceptional. This is contrary to (v) of 5.9. Therefore, $\xi(E(G(R))) = R$. It follows from 3.1 that

$$\frac{E(G)}{J(E(G), E_p(G))} \cong R.$$

LEMMA 5.12. *Let U and V be vector spaces over a field such that $V \subseteq U$. Let U/V be finite dimensional. Suppose $\alpha \in E(U)$, α is one-to-one and $\alpha(V) = V$. Then α is an automorphism of U .*

Proof. Since $\alpha(V) = V$, α induces an endomorphism α' of U/V ($\alpha'(u + V) = \alpha(u) + V$ for $u \in U$). Moreover, α' is one-to-one; and, consequently, the dimensions of U/V and $\alpha'(U/V)$ are equal and finite. It follows that $\alpha'(U/V) = U/V$; and therefore, $\alpha(U) = U$ by a standard argument.

THEOREM 5.13. *The groups $G = G(R)$ have no proper isomorphic subgroups.*

Proof. Let α be an isomorphism of G into G . By (v) of 5.9, α is not Δ -exceptional. By 5.6, 5.7 and the definition of the map ξ , there must exist an integer n and an element $\beta \in \Delta$ such that $\alpha(c_i) = \beta(c_i)$ for all $i > n$. Since α is an isomorphism, $0 \neq \alpha(c_i) = \beta(c_i)$ for all $i > n$. It follows that α and β agree on $(\pi_n G)[p] = \pi_n(G[p])$ (see §I for the definition of π_n). Now, $\Delta \subset \Gamma = \{\gamma | \bar{B}[p] | \gamma \in \Gamma(R)\}$, $\beta \in \Delta$ and $\beta(c_i) \neq 0$ for $i > n$ imply, using Fermat's theorem, that β^{p-1} acts as the identity on $\pi_n G[p]$. It follows that β maps $(\pi_n G)[p] \cap p^k(\pi_n G)$ onto itself for each $k = 0, 1, \dots$. Thus,

$$\alpha(G[p] \cap p^k G) = \alpha((\pi_n G)[p] \cap p^k(\pi_n G)) = (\pi_n G)[p] \cap p^k(\pi_n G) = G[p] \cap p^k G$$

for each $k = n, n + 1, \dots$. Suppose $m \geq 1$ is the largest integer such that $\alpha(G[p] \cap p^{m-1}G) \neq G[p] \cap p^{m-1}G$. It has been shown that if m exists, then $m \leq n$. An application of 5.12 to $U = G[p] \cap p^{m-1}G$ and $V = G[p] \cap p^mG$ shows that the existence of such an integer m is impossible. Consequently, $\alpha(G[p] \cap p^kG) = G[p] \cap p^kG$ for all $k \geq 0$. By Lemma 2.3, it follows that α is an automorphism of G .

COROLLARY 5.14. *Let R be any countable subring of the ring direct product $\prod M_n$. Suppose that R contains $\sum M_n + \{1\}$. There is a pure subgroup G of \bar{B} which contains B such that*

$$\frac{E(G)}{J(E(G))} \cong R.$$

Proof. Let $G = G(R)$ and apply 5.10, 5.13 and 2.12.

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Dallas O. Banks, <i>Lower bounds for the eigenvalues of a vibrating string whose density satisfies a Lipschitz condition</i>	393
Ralph Joseph Bean, <i>Decompositions of E^3 which yield E^3</i>	411
Robert Bruce Brown, <i>On generalized Cayley-Dickson algebras</i>	415
Richard Dowell Byrd, <i>Complete distributivity in lattice-ordered groups</i>	423
Roger Countryman, <i>On the characterization of compact Hausdorff X for which $C(X)$ is algebraically closed</i>	433
Cecil Craig, Jr. and A. J. Macintyre, <i>Inequalities for functions regular and bounded in a circle</i>	449
Takesi Isiwata, <i>Mappings and spaces</i>	455
David Lewis Outcalt, <i>Power-associative algebras in which every subalgebra is an ideal</i>	481
Sidney Charles Port, <i>Equilibrium systems of stable processes</i>	487
Jack Segal, <i>Quasi dimension type. I. Types in the real line</i>	501
Robert William Stringall, <i>Endomorphism rings of primary abelian groups</i>	535
William John Sweeney, <i>"The δ-Poincaré estimate"</i>	559
L. Tzafriri, <i>Operators commuting with Boolean algebras of projections of finite multiplicity</i>	571