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**ON THE CONVERGENCE OF RESOLVENTS OF OPERATORS**

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## ON THE CONVERGENCE OF RESOLVENTS OF OPERATORS

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Let a family of linear operators  $\{A_n\}(n = 1, 2, \dots)$  in a Banach space  $X$  have the resolvents  $\{R(\lambda; A_n)\}$  which is equicontinuous in  $n$ . Suppose that  $\{A_n\}$  is a Cauchy sequence on a dense set. Then the question of convergence arises; when will  $\{R(\lambda; A_n)x\}$  be a Cauchy sequence for all  $x \in X$ ?

This problem is treated in some special cases and an application to the following theorem is presented.

Let  $A$  be the generator of a positive contraction semi-group  $\Sigma$  and let  $B$  be a linear operator with domain  $\mathcal{D}(B) \supset \mathcal{D}(A)$  in a weakly complete Banach lattice  $X$ .

Then  $A + B$  or its closed extension generates a positive contraction semi-group  $\Sigma'$  which dominates  $\Sigma$  if and only if  $A + B$  is dissipative and  $B$  is positive.

In this section we consider the above convergence problem in a Banach space  $X$  (cf. [9], [1], [11]).

Let a family of linear operators  $\{A_n\}(n = 1, 2, \dots)$  satisfy the following conditions:

(1) for some fixed number  $\lambda$ , the resolvent  $R(\lambda; A_n) = (\lambda - A_n)^{-1}$  of  $A_n$  exists which acts on  $X$  to the domain  $\mathcal{D}(A_n)$  of  $A_n$  and satisfies the norm condition  $\|R(\lambda; A_n)\| \leq K_\lambda$ , where  $K_\lambda$  is a positive number independent of  $n$ ,

(2) there is a dense subspace  $\mathcal{M}$  on which  $A = \lim A_n$  exists.

**PROPOSITION 1.** The limit operator  $R_0(\lambda; A) = \lim R(\lambda; A_n)$  exists on  $\overline{\mathcal{N}}$  and satisfies the norm condition  $\|R_0(\lambda; A)\|_{\overline{\mathcal{N}}} \leq K_\lambda$  where  $\mathcal{N} = (\lambda - A)\mathcal{M}$  and  $\overline{\mathcal{N}}$  is its closure.

*Proof.* For any  $x \in \mathcal{M}$  we have

$$\|(\lambda - A_n)x\| \geq K_\lambda^{-1} \|x\|$$

and thus obtain

$$\|(\lambda - A)x\| \geq K_\lambda^{-1} \|x\| - \|A_n x - Ax\|.$$

Letting  $n \rightarrow \infty$ , we have

$$\|(\lambda - A)x\| \geq K_\lambda^{-1} \|x\|.$$

It also follows that we can extend  $(\lambda - A)^{-1}$  to the bounded linear operator  $R_0(\lambda; A)$  on  $\overline{\mathcal{N}}$  which satisfies

$$\| R_0(\lambda; A) \|_{\overline{\mathcal{N}}} = \sup \{ \| R_0(\lambda; A)x \|; \| x \| = 1, x \in \overline{\mathcal{N}} \} \leq K_\lambda .$$

Further, it is easy to see that, for any  $x \in \mathcal{M}$ ,

$$\| R(\lambda; A_n)(\lambda - A)x - x \| \leq K_\lambda \| A_n x - Ax \|^2$$

which implies that  $R_0(\lambda; A) = \lim R(\lambda; A_n)$  on  $\overline{\mathcal{N}}$ .

**REMARK 1.** This proof shows that if  $(\lambda - A)\mathcal{M}$  is dense in  $X$  then the convergence problem is solved.

We next remark some modification of the basic lemma in [1].

**PROPOSITION 2.** The following conditions are equivalent.

- (1)  $\lim_{n, n' \rightarrow \infty} \| R(\lambda; A_n)x - R(\lambda; A_{n'})x \| = 0 \quad (x \in X),$   
 (2)  $\lim_{n, n' \rightarrow \infty} \| R(\lambda; A_n)R(\lambda; A_{n'})x - R(\lambda; A_{n'})R(\lambda; A_n)x \| = 0$   
( $x \in (\lambda - A)\mathcal{M}$ ).

*Proof.* For any  $x \in \mathcal{M}$ ,  $n$  and  $n'$ , we have

$$\begin{aligned} R(\lambda; A_n)x - R(\lambda; A_{n'})x &= R(\lambda; A_n)R(\lambda; A_{n'}) (\lambda - A_{n'})x - R(\lambda; A_{n'})R(\lambda; A_n) (\lambda - A_n)x \\ &= R(\lambda; A_n)R(\lambda; A_{n'}) (\lambda - A)x - R(\lambda; A_{n'})R(\lambda; A_n) (\lambda - A)x \\ &\quad + R(\lambda; A_n)R(\lambda; A_{n'}) (A - A_{n'})x \\ &\quad + R(\lambda; A_{n'})R(\lambda; A_n) (A_n - A)x . \end{aligned}$$

From this relation and  $\overline{\mathcal{M}} = X$ , the assertion is readily verified.

**PROPOSITION 3.** If, for some positive integer  $m$ ,

$$\lim_{n \rightarrow \infty} \| \{(A_n - A)R(\lambda; A_n)\}^m x \| = 0 \quad (x \in \mathcal{M}_1)$$

is satisfied, where  $\mathcal{M}_1$  is dense in  $X$ , then  $(\lambda - A)\mathcal{M}$  is dense in  $X$ .

*Proof.* By virtue of the Hahn-Banach extension theorem, if there exists  $x_0 \in \mathcal{M}_1 - \overline{\mathcal{N}}$ , then so does a bounded linear functional  $F_0$  acting on  $X$  which satisfies the following conditions:

$$F_0(x_0) \neq 0, \quad F_0(x) = 0 \quad (x \in \overline{\mathcal{N}} = \overline{(\lambda - A)\mathcal{M}}).$$

For this  $x_0$  and any  $n$ , we have

$$\begin{aligned} x_0 &= (\lambda - A_n)R(\lambda; A_n)x_0 \\ &= (\lambda - A)R(\lambda; A_n)x_0 - (A_n - A)R(\lambda; A_n)x_0 \\ &= \dots \\ &= (\lambda - A)R(\lambda; A_n)x_0 - (\lambda - A)R(\lambda; A_n)(A_n - A)R(\lambda; A_n)x_0 \\ &\quad + \dots \\ &\quad + (-1)^m \{(A_n - A)R(\lambda; A_n)\}^m x_0 . \end{aligned}$$

This relation implies that

$$F_0(x_0) = (-1)^m F_0(\{(A_n - A)R(\lambda; A_n)\}^m x_0)$$

and for any  $n$

$$0 < |F_0(x_0)| \leq \|F_0\| \|\{(A_n - A)R(\lambda; A_n)\}^m x_0\|$$

which is a contradiction. Consequently we have  $\mathcal{M}_1 \subset \overline{\mathcal{N}}$  and the assertion is proved.

We now concern with a theorem on the perturbation of operators which will be required in the sequel.

**PROPOSITION 4.** Suppose that linear operators  $A$  and  $B$  satisfy the following conditions:

(1) for some number  $\lambda$ , the equation

$$(\lambda - A)y = x \quad (x \in X)$$

has a unique solution  $y = R(\lambda; A)x$ ,

(2) there is a dense subspace  $\mathcal{M}$  such that  $BR(\lambda; A)\mathcal{M} \subset \mathcal{M}$  and

$$\lim_{k \rightarrow \infty} \|\{BR(\lambda; A)\}^k x\| = 0 \quad (x \in \mathcal{M}). \quad (*)$$

Then  $(\lambda - A - B)R(\lambda; A)\mathcal{M}$  is dense in  $X$ .

The proof of this proposition is similar as that of Proposition 3 and is omitted.

**REMARK 2.** Suppose that for some positive integer  $k$

$$(**) \quad \|\{BR(\lambda; A)\}^k\|_{\mathcal{M}} < 1$$

is satisfied, then the condition (\*) in Proposition 4 is satisfied.

**REMARK 3.** Suppose that  $R(\lambda; A)$  satisfies the norm condition  $\|R(\lambda; A)\| \leq K_\lambda$  in Proposition 4 and that there exist positive constants  $a$  and  $b$  such that for any  $x \in \mathcal{M}_1 = R(\lambda; A)\mathcal{M}$

$$\|Bx\| \leq a \|Ax\| + b \|x\|$$

and

$$a |\lambda| K_\lambda + a + b K_\lambda < 1.$$

Then the condition (\*\*) in Remark 2 is satisfied.

*Proof.* For any  $x \in \mathcal{M}$ , we have

$$\begin{aligned} \|BR(\lambda; A)x\| &\leq a \|AR(\lambda; A)x\| + b \|R(\lambda; A)x\| \\ &\leq a \|\lambda R(\lambda; A)x - x\| + bK_\lambda \|x\| \end{aligned}$$

and

$$\|BR(\lambda; A)x\| \leq (a|\lambda|K_\lambda + a + bK_\lambda) \|x\| < \|x\| .$$

Thus the assertion is proved.

**THEOREM 1.** *Suppose that a family of linear operators  $\{A_\varepsilon\}(\varepsilon > 0)$  and a closed linear operator  $A$  from a Banach space  $X$  to  $X$  satisfy the following conditions:*

(1) *for some fixed number  $\lambda$ , the equation*

$$(\lambda - A_\varepsilon)y = x \quad (x \in X)$$

*has a unique solution  $y = R(\lambda; A_\varepsilon)x \in \mathcal{D}(A_\varepsilon)$  and  $\|R(\lambda; A_\varepsilon)\| \leq K_\lambda$ , where  $K_\lambda$  is a positive number independent of  $\varepsilon$ ,*

$$(2) \quad \mathcal{D}(A_\varepsilon) \supset \mathcal{D}(A), \quad \overline{\mathcal{D}(A)} = X,$$

$$(3) \quad A_\varepsilon x = Ax + \varepsilon B_\varepsilon x \quad (x \in \mathcal{D}(A)),$$

$$\|B_\varepsilon x\| \leq K(x) \quad (x \in \mathcal{D}(A)) ,$$

*where  $K(x)$  is a positive number independent of  $\varepsilon$ .*

*Then we have  $\mathcal{R}(\lambda - A) = (\lambda - A)\mathcal{D}(A) = X$ .*

*Proof.* It follows from Proposition 1 that the limit operator  $R_0(\lambda; A)$  exists and bounded on  $\overline{\mathcal{R}(\lambda - A)}$ .

Let  $(\lambda - A)x_n \rightarrow y$  as  $n \rightarrow \infty$ . Then it follows from the boundedness of  $R_0(\lambda; A)$  that  $x_n \rightarrow R_0(\lambda; A)y$  and so that

$$Ax_n \rightarrow \lambda R_0(\lambda; A)y - y$$

as  $n \rightarrow \infty$ . Since  $A$  is closed,  $R_0(\lambda; A)y \in \mathcal{D}(A)$  and  $y \in \mathcal{R}(\lambda - A)$ . Thus we have  $\overline{\mathcal{R}(\lambda - A)} = \mathcal{R}(\lambda - A)$ . It is easy to see that  $\lambda - A_\varepsilon$  is closed and

$$\begin{aligned} (\lambda - A_\varepsilon)R_0(\lambda; A)x &= (\lambda - A)R_0(\lambda; A)x \\ &\quad - \varepsilon B_\varepsilon R_0(\lambda; A)x \quad (x \in \mathcal{R}(\lambda - A)) . \end{aligned}$$

Hence, from the closed graph theorem it follows that  $B_\varepsilon R_0(\lambda; A)$  is a bounded linear operator on  $\mathcal{R}(\lambda - A)$ . Moreover we have, for any  $x \in \mathcal{D}(A)$ ,

$$\|B_\varepsilon R_0(\lambda; A)(\lambda - A)x\| = \|B_\varepsilon x\| \leq K(x) < \infty .$$

Using the resonance theorem it follows that there exists a positive number  $L_\lambda$  which is independent of  $\varepsilon$  such that

$$\| B_\varepsilon R_0(\lambda; A) \|_{\mathcal{D}(\lambda - A)} \leq L_\lambda .$$

Consequently we obtain the basic relation, for any  $x \in \mathcal{D}(A)$ ,

$$\begin{aligned} \|\varepsilon B_\varepsilon x\| &= \|\varepsilon B_\varepsilon R_0(\lambda; A)(\lambda - A)x\| \\ &\leq \varepsilon L_\lambda \|(\lambda - A)x\| \leq \varepsilon L_\lambda \|Ax\| + \varepsilon |\lambda| L_\lambda \|x\| . \end{aligned}$$

Thus the assertion follows from Remark 3.

**REMARK 4.** Let  $A$  be a closed linear operator with dense domain  $\mathcal{D}(A)$ . Suppose that  $A_\varepsilon = A + \varepsilon B$  generates a strongly continuous semi-group of linear contraction operators for every small  $\varepsilon (0 < \varepsilon < \varepsilon_0)$  and  $\mathcal{D}(A_\varepsilon) \supset \mathcal{D}(A)$ .

Then  $A$  generates a strongly continuous semi-group of linear contraction operators.

*Proof.* Using Theorem 1 and Proposition 1, it follows from the Hille-Yosida theorem. (cf. [3], [11]).

2. The object of this section is to show that some special family of linear operators  $\{A_n\} (n = 1, 2, \dots)$  from a weakly complete Banach lattice  $X$  to  $X$  satisfies the convergence condition and to solve the problem on the perturbation theory for semi-groups of operators which is sited in the introductory part.

Let  $X$  be a Banach lattice with a semi-order  $\geq$  and  $[x, y] (x, y \in X)$  denote a complex-valued (real-valued) function defined on  $X \times X$  called a semi-inner product having the following properties (cf. [4], [6], [7]):

- (1)  $[x + y, z] = [x, z] + [y, z]$ ,
- (2)  $[\lambda x, y] = \lambda [x, y]$ ,
- (3)  $[x, x] = \|x\|^2$ ,
- (4)  $|[x, y]| \leq \|x\| \|y\|$ ,
- (5) if  $y \geq 0$ , then  $[x, y] \geq 0$  for all  $x \geq 0$ ,
- (6)  $[x, x^+] = \|x^+\|^2$ ,

where  $x^+ = \sup(x, 0)$ ,  $x^- = \sup(-x, 0)$ , and  $|x| = \sup(x, -x)$ .

The following theorem is essentially due to Reuter [8].

**PROPOSITION 5.** Suppose that linear operators  $A_0$  and  $A_1$  on a Banach lattice  $X$  satisfy the following conditions:

- (1) for  $n = 0, 1$  and some  $\lambda > 0$ , the equation

$$(\lambda - A_n)y = x \quad (x \in X)$$

has a unique solution  $y = R(\lambda; A_n)x \in \mathcal{D}(A_n)$  and

$$R(\lambda; A_n)x \geq 0 \quad (x \geq 0),$$

(2) there exist dense subspaces  $\mathcal{M}$  and  $\mathcal{M}_1$  such that

$$A_1x \geq A_0x \quad (x \geq 0, x \in \mathcal{M}),$$

$$R(\lambda; A_1)\mathcal{M}_1 \subset \mathcal{M}.$$

Then the following inequality holds:

$$R(\lambda; A_1)x \geq R(\lambda; A_0)x \quad (x \geq 0, x \in \mathcal{M}_1)$$

*Proof.* If  $x \geq 0$  and  $x \in \mathcal{M}_1$ , then  $R(\lambda; A_1)x \geq 0$  and  $R(\lambda; A_1)x \in \mathcal{M}$  and thus we have

$$A_1R(\lambda; A_1)x \geq A_0R(\lambda; A_1)x,$$

$$(\lambda - A_0)R(\lambda; A_1)x \geq (\lambda - A_1)R(\lambda; A_1)x = x.$$

Operating  $R(\lambda; A_0)$ , we obtain

$$R(\lambda; A_1)x \geq R(\lambda; A_0)x.$$

Let  $\sum = \{T_t; t \geq 0\}$  be a one-parameter semi-group of linear operators from a Banach lattice  $X$  to  $X$  satisfying the following conditions:

- (1)  $T_0x = x, T_{t+s}x = T_tT_sx \quad (x \in X, t, s \geq 0),$
- (2)  $\|T_t x\| \leq \|x\| \quad (x \in X, t \geq 0),$
- (3)  $\lim_{t \rightarrow 0^+} T_t x = x \quad (x \in X),$
- (4)  $T_t x \geq 0 \quad (x \geq 0, t \geq 0).$

Such a semi-group is called a strongly continuous semi-group of positive contraction operators.

The following theorem is due to Phillips and is a variant of the Hille-Yosida theorem which will be convenient for purpose. (cf. [7]).

**THEOREM.** (*Phillips*). *A necessary and sufficient condition for a linear operator  $A$  with dense domain to generate a strongly continuous semi-group of positive contraction operators is that  $\mathcal{R}(I - A) = X$  and that  $A$  is dispersive, that is,*

$$[Ax, x^+] \leq 0 \quad (x \in \mathcal{D}(A)).$$

**THEOREM 2.** *Suppose that a family of linear operators  $\{A_n\}$  ( $n = 1, 2, \dots$ ) which generate strongly continuous semi-groups  $\{\sum_n\}$  of positive contraction operators on a weakly complete Banach lattice  $X$  satisfies the following conditions: there exist dense subspaces  $\mathcal{M}, \mathcal{M}_0$  and  $\{\mathcal{M}_n\}$  such that*

- (1)  $\lim_{n, n' \rightarrow \infty} \|A_n x - A_{n'} x\| = 0 \quad (x \in \mathcal{M}),$   
 (2)  $A_{n+1} x \geq A_n x \quad (x \geq 0, x \in \mathcal{M}_n),$   
 (3)  $R(\lambda; A_n) \mathcal{M}_0 \subset \mathcal{M}_n,$   
 (4)  $\mathcal{M}_0^+ = \{x^+; x \in \mathcal{M}_0\} \subset \mathcal{M}_0.$

Then the limit operator  $A = \lim A_n$  on  $\mathcal{M}$  has a closed extension  $\tilde{A}$  which generates a strongly continuous semi-group  $\Sigma$  of positive contraction operators.

Moreover we have

$$T_t x = \lim_{n \rightarrow \infty} T_t^{(n)} x \quad (x \in X, t \geq 0),$$

where  $\Sigma_n = \{T_t^{(n)}; t \geq 0\}$  and  $\Sigma = \{T_t; t \geq 0\}$ .

*Proof.* By the Hille-Yosida theorem (cf. [3], [11]) we find that the conditions (1) and (2) in Proposition 5 and the following norm condition are satisfied for any pair  $\{A_n, A_{n+1}\}$ .

$$\|R(\lambda; A_n)\| \leq \lambda^{-1} \quad (*)$$

Thus we have, for any  $n$ ,

$$R(\lambda; A_{n+1})x \geq R(\lambda; A_n)x \quad (x \geq 0, x \in \mathcal{M}_0).$$

Since  $X$  is weakly complete, the norm condition and this inequality imply that there exists  $y \geq 0$  such that

$$\lim_{n \rightarrow \infty} \|R(\lambda; A_n)x - y\| = 0.$$

From a representation of  $x: x = x^+ - x^-$ , we have, for any  $x \in \mathcal{M}_0$ , using the condition (4),

$$(**) \quad \lim_{n, n' \rightarrow \infty} \|R(\lambda; A_n)x - R(\lambda; A_{n'})x\| = 0$$

and we have this convergence relation for all  $x \in X$  by the condition  $\overline{\mathcal{M}_0} = X$ . We denote  $\tilde{R}(\lambda; A) = \lim R(\lambda; A_n)$ . Then  $\tilde{R}(\lambda; A)$  is positive and satisfies the norm condition (\*). The assertion is now proved by Theorem 2 in [1]. We sketch the proof of this theorem.

Since  $R(\lambda; A_n)$  satisfies the resolvent equation

$$R(\lambda; A_n) - R(\lambda'; A_n) = -(\lambda - \lambda')R(\lambda; A_n)R(\lambda'; A_n)$$

$\tilde{R}(\lambda; A)$  also does. Then we find that  $\tilde{R}(\lambda; A)$  is a one-to-one transformation from  $X$  to  $\mathcal{R}(\tilde{R}(\lambda; A))$  and  $\tilde{A}_\lambda = \lambda - \tilde{R}(\lambda; A)^{-1}$  is independent of  $\lambda$ , that is,

$$\tilde{A}x = \tilde{A}_\lambda x = \tilde{A}_{\lambda'} x \quad (x \in \mathcal{R}),$$



where  $\mathcal{R} = \mathcal{R}(\tilde{R}(\lambda; A)) = \mathcal{R}(\tilde{R}(\lambda'; A))$ .

Then, by the Hille-Yosida theorem, we find that  $\tilde{A}$  generates a strongly continuous semi-group of contraction operators. The positivity and the convergence of semi-groups are verified by the condition (\*\*). It is readily verified that  $\tilde{A}$  is a closed extension of  $A$ .

**REMARK 5.** Suppose that a family of linear operators  $\{A_n\}$  ( $n = 1, 2, \dots$ ) which generate strongly continuous semi-groups of positive contraction operators on a weakly complete Banach lattice  $X$  satisfies the following conditions:

$$(1) \quad \lim_{n, n' \rightarrow \infty} \|A_n x - A_{n'} x\| = 0 \quad (x \in \mathcal{M}),$$

where  $\mathcal{M}$  is a dense subspace in  $X$ ,

$$(2) \quad A_{n+1} x \geq A_n x \quad (x \geq 0, x \in \mathcal{D}(A_n)),$$

$$(3) \quad \mathcal{D}(A_{n+1}) \supset \mathcal{D}(A_n).$$

Then the assertion in Theorem 2 is true.

**REMARK 6.** In Theorem 2, the condition (1) can be replaced by the following condition:

$$(1') \quad \|A_n^2 x\| \leq K(x) \quad (x \in \mathcal{M}_2),$$

where  $K(x)$  is a positive number independent of  $n$  and  $\mathcal{M}_2$  is dense in  $X$ .

*Proof.* We remark that the convergence of the family of resolvents in Theorem 2 does not depend on (1). Then we have, for any  $x \in \mathcal{M}_2$ ,

$$\begin{aligned} \|A_n x - A_{n'} x\| &\leq \lambda \|R(\lambda; A_n) A_n x - R(\lambda; A_{n'}) A_{n'} x\| \\ &\quad + \|A_n x - \lambda R(\lambda; A_n) A_n x\| \\ &\quad + \|A_{n'} x - \lambda R(\lambda; A_{n'}) A_{n'} x\| \\ &\leq \lambda^2 \|R(\lambda; A_n) x - R(\lambda; A_{n'}) x\| \\ &\quad + \|R(\lambda; A_n) A_n^2 x\| + \|R(\lambda; A_{n'}) A_{n'}^2 x\| \\ &\leq \lambda^2 \|R(\lambda; A_n) x - R(\lambda; A_{n'}) x\| + 2\lambda^{-1} K(x). \end{aligned}$$

Letting  $\lambda \rightarrow \infty$ , we have, for any  $\varepsilon > 0$ ,

$$\|A_n x - A_{n'} x\| \leq \lambda^2 \|R(\lambda; A_n) x - R(\lambda; A_{n'}) x\| + \varepsilon$$

and the assertion is proved by (\*\*).

From Remark 4 in [1] it follows that

**REMARK 7.** Suppose that there exists a dense subspace  $\mathcal{M}_2$  such that  $\tilde{R}(\lambda; A)\mathcal{M}_2 \subset \mathcal{M}$  in Theorem 2, then  $\tilde{A}$  is the closure of  $A$ .

We next concern with the generation of contraction semi-groups which dominate a given semi-group and give an alternative form of a theorem of Reuter, Miyadera and Olubummo (cf. [8], [5], [6], [7]).

Given a one-parameter semi-group  $\Sigma = \{T_t; t \geq 0\}$  of positive contraction operators, if  $\Sigma' = \{T'_t; t \geq 0\}$  is another one, we say that  $\Sigma'$  dominates  $\Sigma$ , if

$$T'_t x \geq T_t x \quad (x \geq 0, t \geq 0).$$

In applications, it is important to know whether a given semi-group  $\Sigma$  is dominated by any other semi-group  $\Sigma'$ .

The following lemmas in a Banach space will be required.

LEMMA. (*Lumer and Phillips*). *If  $A$  with dense domain is a dissipative operator, that is,*

$$\operatorname{Re} [Ax, x] \leq 0 \quad (x \in \mathcal{D}(A)),$$

*then  $A$  has a closed extension.*

PROPOSITION 6. Suppose that a linear operator  $A$  which generates a strongly continuous semi-group of contraction operators on a Banach space  $X$  and a linear operator  $B$  with domain  $\mathcal{D}(B) \supset \mathcal{D}(A)$  satisfy the following condition:  $A + B$  has a closed extension. Then

$$\|BR(\lambda; A)\| \leq K$$

where  $K$  is a positive number independent of  $\lambda > 1$  and

$$\lim_{\lambda \rightarrow \infty} \|BR(\lambda; A)x\| = 0 \quad (x \in X).$$

The proof of Proposition 6 is readily verified by using the resolvent equation and is omitted.

THEOREM 3. *In a weakly complete Banach lattice  $X$  let  $A$  be the generator of a positive contraction semi-group  $\Sigma$  and let  $B$  be a linear operator with domain  $\mathcal{D}(B) \supset \mathcal{D}(A)$ . Then  $A_1 = A + B$  or its closed extension generates a positive contraction semi-group  $\Sigma'$  which dominates  $\Sigma$  if and only if*

$$(1) \quad \operatorname{Re} [A_1 x, x] \leq 0 \quad (x \in \mathcal{D}(A)),$$

$$(2) \quad Bx \geq 0 \quad (x \geq 0, x \in \mathcal{D}(A)).$$

*Proof.* To prove the sufficiency of the conditions (1) and (2), we approximate  $A_1$  by a sequence of linear operators  $\{A_{n,\lambda}\}$  in the following way. Define a sequence of linear operators  $\{A_{n,\lambda}\}$  by

$$A_{n,\lambda} = A + (n - \lambda)BR(n; A) \quad (n \geq \lambda)$$

and  $\{B_{n,\lambda}\}$  by

$$\begin{aligned} B_{n,\lambda} &= A_{n+1,\lambda} - A_{n,\lambda} \\ &= BR(n+1; A)(\lambda - A)R(n; A) \quad (n \geq \lambda). \end{aligned}$$

Then it follows from Lemma (Lumer and Phillips) and Proposition 6 that there is a positive integer  $L$  independent of  $n$  and  $\lambda$  such that  $\|B_{n,\lambda}\| \leq L$ .

If we assume that the resolvent  $R(\lambda; A_{n,\lambda})$  exists which acts on  $X$  and is positive for some  $\lambda$  and  $n$  ( $n \geq \lambda$ ), then we have, for any  $x \geq 0$ ,

$$\begin{aligned} \lambda \|R(\lambda; A_{n,\lambda})x\|^2 &= [\lambda R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\leq [\lambda R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\quad - \operatorname{Re} [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x]. \end{aligned}$$

Using Theorem (Phillips), we remark that  $A$  is a dispersive operator. Thus we have

$$\begin{aligned} \operatorname{Re} [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &= \operatorname{Re} [AR(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\quad + \operatorname{Re} [BR(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &= [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x]. \end{aligned}$$

Hence we obtain

$$\begin{aligned} \lambda \|R(\lambda; A_{n,\lambda})x\|^2 \\ &\leq [\lambda R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] - [A_1 R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &= [x, R(\lambda; A_{n,\lambda})x] - [BR(n; A)(\lambda - A)R(\lambda; A_{n,\lambda})x, R(\lambda; A_{n,\lambda})x] \\ &\leq [x, R(\lambda; A_{n,\lambda})x], \end{aligned}$$

where the last inequality holds by virtue of the formula

$$\begin{aligned} (\lambda - A)R(\lambda; A_{n,\lambda})x \\ &= x + (n - \lambda)BR(n; A)R(\lambda; A_{n,\lambda})x. \end{aligned}$$

Thus we obtain, for any  $x \geq 0$  and then for any  $x \in X$ ,

$$\lambda \|R(\lambda; A_{n,\lambda})x\| \leq \|x\|.$$

By induction on  $n$  we next show that the resolvent  $R(\lambda; A_{n,\lambda})$  exists which acts on  $X$  and is positive for any  $\lambda > L$  and any  $n \geq \lambda$ . It is clear that  $R(\lambda; A_{\lambda,\lambda}) = R(\lambda; A)$  is a positive operator for any  $\lambda > L$ . Suppose that  $R(\lambda; A_{n,\lambda})$  is positive for any  $\lambda > L$  and some  $n$ , then we have  $\|B_{n,\lambda}R(\lambda; A_{n,\lambda})\| < 1$ . It follows from this norm condition

that  $R(\lambda; A_{n+1, \lambda})$  exists which acts on  $X$  and is given by the following formula (cf. [3], [11]):

$$R(\lambda; A_{n+1, \lambda}) = \sum_{k=0}^{\infty} R(\lambda; A_{n, \lambda}) [B_{n, \lambda} R(\lambda; A_{n, \lambda})]^k .$$

Since  $B_{n, \lambda} R(\lambda; A_{n, \lambda})$  is positive, it follows that

$$R(\lambda; A_{n+1, \lambda})x \geq R(\lambda; A_{n, \lambda})x \geq 0 \quad (x \geq 0) .$$

Hence, using the weakly completeness of  $X$ , we have for any  $x \geq 0$  and then  $x \in X$ ,

$$\lim_{n, n' \rightarrow \infty} \| R(\lambda; A_{n, \lambda})x - R(\lambda; A_{n', \lambda})x \| = 0 .$$

To show that  $\{R(\lambda'; A_{n, \lambda})x\} (0 < \lambda' < \lambda)$  is also a Cauchy sequence for any  $x \in X$ , we make use of the relation

$$R(\lambda - \mu; A_{n, \lambda}) = \sum_{k=1}^{\infty} \mu^{k-1} R(\lambda; A_{n, \lambda})^k ,$$

where, provided that  $|\mu| < \lambda$ , the right hand side converges uniformly in  $n$  (cf. [3], [11]). It also follows from this formula that  $\lambda' R(\lambda'; A_{n, \lambda})$  is positive and is a contraction operator for any  $\lambda' (0 < \lambda' < \lambda)$ .

Let  $k$  be a positive integer such that  $k > L$ . We define, for any  $\lambda \leq k$ ,

$$\tilde{R}(\lambda; A_k)x = \lim_{n \rightarrow \infty} R(\lambda; A_{n, k})x \quad (x \in X) .$$

Then it is easy to see that  $\{\tilde{R}(\lambda; A_k); \lambda \leq k\}$  satisfies the resolvent equation and the norm condition  $\lambda \|\tilde{R}(\lambda; A_k)\| \leq 1$ .

Moreover  $\{\tilde{R}(\lambda; A_k)\}_k$  is a consistent family of resolvents in the following sense:

$$\tilde{R}(\lambda; A_{k'})x = \tilde{R}(\lambda; A_k)x \quad (x \in X, \lambda < k < k') .$$

In fact, we have the inequality

$$\begin{aligned} & \| \tilde{R}(\lambda; A_{k'})x - \tilde{R}(\lambda; A_k)x \| \\ & \leq \| \tilde{R}(\lambda; A_{k'})x - R(\lambda; A_{n, k'})x \| \\ & \quad + [1 + \lambda^{-1}(k' - k)L] \| R(\lambda; A_{n, k})x - \tilde{R}(\lambda; A_k)x \| \\ & \quad + \lambda^{-1}(k' - k) \| BR(n; A)\tilde{R}(\lambda; A_k)x \| \end{aligned}$$

and letting  $n \rightarrow \infty$ , we obtain the desired result.

Since  $\{\tilde{R}(\lambda; A_k)\}_k$  is consistent, we have a family of resolvents

$$\{\tilde{R}(\lambda; A_1); \tilde{R}(\lambda; A_1) = \tilde{R}(\lambda; A_k) \quad (\lambda \leq k)$$

which satisfies the norm condition  $\lambda \|\tilde{R}(\lambda; A_1)\| \leq 1$ .

Then, using the same method as that in the proof of Theorem 2, we find that  $\tilde{A}_1 = \lambda - \tilde{R}(\lambda; A_1)^{-1}$  generates a strongly continuous semi-group  $\Sigma'$  of positive contraction operators which dominates  $\Sigma$  and that  $\tilde{A}_1$  is a closed extension of  $A_1$ .

We now prove the inverse part. Let  $\Sigma = \{T_t; t \geq 0\}$  and  $\Sigma' = \{T'_t; t \geq 0\}$ . Then the condition (1) follows from

$$\begin{aligned} \operatorname{Re} [A_1 x, x] &= \lim_{t \rightarrow 0^+} \operatorname{Re} [t^{-1}(T'_t x - x), x] \\ &= \lim_{t \rightarrow 0^+} t^{-1} \operatorname{Re} \{[T'_t x, x] - [x, x]\} \\ &\leq 0 \quad (x \in \mathcal{D}(A)), \end{aligned}$$

and (2) follows from

$$A_1 x = \lim_{t \rightarrow 0^+} t^{-1}(T'_t x - x) \geq \lim_{t \rightarrow 0^+} t^{-1}(T_t x - x) = Ax \quad (x \geq 0, x \in \mathcal{D}(A)).$$

Thus the assertion is proved.

REMARK 8. In Theorem 3 any one of the following conditions can take the place of the condition (1).

$$(1') \quad [A_1 x, x] \leq 0 \quad (x \geq 0, x \in \mathcal{D}(A))$$

and  $A_1$  has a closed extension,

$$(1'') \quad [A_1 x, x] \leq 0 \quad (x \geq 0, x \in \mathcal{D}(A))$$

and  $BR(\lambda; A)$  is a bounded linear operator for any  $\lambda > 0$ .

The contents of this section will be discussed in [2] by virtue of the notation of Gâteaux differentials.

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