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SURFACES HARMONICALLY IMMERSED IN E^3

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In this paper we study surfaces in E^3 which satisfy conditions necessary and/or sufficient to insure their harmonic immersion with respect to a fixed but not necessarily ordinary conformal structure. Our consideration of such surfaces is based upon the notion that surfaces which share some essential property of minimal surfaces are bound to be interesting. Thus our use here of nonstandard conformal structures is simply a device for the identification of such a class of surfaces distinct from others already much studied, such as quasi-minimal surfaces or surfaces of constant mean curvature. In the end, any such endeavor serves to distinguish those facts about minimal surfaces which are special to them from among the many facts which apply to larger classes of surfaces sharing some one vital property of minimal surfaces.

The more quotable results in this paper refer to a conformal structure R_Λ determined by a fixed positive definite linear combination $\Lambda = fI + gII$ of the fundamental forms on the surface, with f and g smooth functions. Specifically, we show that mean curvature H cannot be bounded away from zero on a complete R_Λ-harmonically immersed surface in E^3. This result is less general than it might seem. For we also prove that where $H \neq 0$ on an R_Λ-harmonically immersed surface, $\Lambda \alpha II'$, with II' defined by $\sqrt{H^2 - K}II' = HII - KI$. Included is an example of an R_Λ-harmonically immersed surface on which $H \neq 0$.

1. This section contains an explanation of our notation. Throughout the paper $X: S \rightarrow E^3$ denotes the C^k immersion of an oriented C^k surface S in E^3, with $k \geqq 2$. At certain points below we make the stronger assumption that $k \geqq 4$.

A conformal structure R is said to be defined on S if R is a Riemann surface on the point set S each of whose conformal parameters $w = u + iv$ yields a pair of C^j coordinates u, v on S, with $2 \leqq j \leqq k$. At certain points below we need the stronger assumption that $3 \leqq j \leqq k$. Given a conformal structure R on S, we call u, v R-isothermal coordinates on S provided that $u + iv = w$ is a conformal parameter on R. And quantities on S evaluated only for all possible R-isothermal coordinates are referred to as quantities on R. Thus, we say that S is R-harmonically immersed in E^3 by X if and only if $\Delta X \equiv 0$ on R, that is, if and only if

$$X_{uu} + X_{vv} \equiv 0$$

for R-isothermal coordinates u, v anywhere on S.

Specifically alluded to below are the conformal structures R_1 determined by the first fundamental form I on S; R_2 determined (if Gauss curvature $K > 0$ and mean curvature $H > 0$ on S) by the second fundamental form II on S; and R_2' determined (if $K < 0$ on S) by the form

$$II' = \frac{1}{\sqrt{H^2 - K}}(HII - KI)$$

on S. More generally, R_A denotes the conformal structure on S determined by any fixed positive definite form A on S which is a linear combination of I and II, so that

$$A = fI + gII$$

with f and g C^{k-2} real valued functions on S. Since R_A-isothermal coordinates on S need only be C^j with $j = k - 1$, $k \geqq 4$ will be assumed wherever $j \geqq 3$ is required (as, for example, in using the Codazzi Mainardi equations on R_A). There are, of course, on any S conformal structures R which are not of the form R_A.

Given an arbitrary conformal structure R on S, the expressions

$$\Omega_1 = \{(E - G) - 2iF\}dw^2$$
$$\Omega_2 = \{(L - N) - 2iM\}dw^2$$

associated with the fundamental forms

$$I = Edu^2 + 2Fdudv + Gdv^2$$
$$II = Ldu^2 + 2Mdudv + Ndv^2$$

on R, are quadratic differentials on R ([2], p. 1285). We will use below the formal differential operators

$$\frac{\partial}{\partial w} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{dv}\right), \quad \frac{\partial}{\partial \bar{w}} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right)$$

on R. An arbitrary quadratic differential

$$\Omega = \varphi dw^2$$

on R is said to be R-holomorphic if and only if

$$\varphi_{\bar{w}} \equiv 0$$

for every conformal parameter w on R.

2. Results of a local nature are presented in this section.

LEMMA 1. *$X: S \to E^3$ is R-harmonic if and only if Ω_1 is R-holomorphic while $(L + N) \equiv 0$ on R.*

Proof. The definition of I yields

$$\Omega_1 = 4(X_w \cdot X_w)dw^2$$

so that Ω_1 is R-holomorphic if and only if

$$X_w \cdot X_{w\bar{w}} = 0 ,$$

that is, equivalently, if and only if

$$\Delta X \cdot X_u = \Delta X \cdot X_v = 0$$

for every conformal parameter $w = u + iv$ on R. But the Gauss equations ([6], p. 107) give ΔX as a linear combination of the linearly independent vectors X_u, X_v and the unit normal $\hat{X}$, with $(L + N)$ the coefficient of $\hat{X}$. Thus

$$(1) \qquad \Delta X = (L + N)\hat{X}$$

on R if and only if Ω_1 is R-holomorphic. Moreover, $\Delta X \equiv 0$ on R if and only if Ω_1 is R-holomorphic, while $(L + N) \equiv 0$ on R.

LEMMA 2. *If $X: S \to E^3$ is R-harmonic, then*
(i) $K \leqq 0$ on S
(ii) $K = 0$ only where $H = 0$, i.e. only at flat points, and
(iii) $R \equiv R_1$, or else the points at which $R = R_1$ are isolated.

Proof. First, if $X: S \to E^3$ is R-harmonic then

$$(2) \qquad K = \frac{-(L^2 + M^2)}{EG - F^2} \leqq 0 ,$$

on R, since $(L + N) \equiv 0$ on R, thus establishing (i). Next, (2) shows that $K = 0$ on S only where $L = M = -N = 0$ on R, thus establishing (ii). Finally, since the zeros of Ω_1 on R are those points at which $E = G$ and $F = 0$ on R, we conclude that R and R_1 coincide (in assigning identical angle measurements) precisely at the zeros of Ω_1. But the zeros of an R-holomorphic quadratic differential Ω must be isolated unless $\Omega \equiv 0$ on R, thus establishing (iii).

LEMMA 3. *If Ω_1 is R-holomorphic, there exists wherever $R \neq R_1$ a function*

$$\cosh^{-1}(2E - 1) > 0$$

which is R-subharmonic where $K \leqq 0$, R-superharmonic where $K \geqq 0$, and constant only if $K \equiv 0$.

Proof. Since Ω_1 is R-holomorphic, there exist near any point at which $\Omega_1 \neq 0$ special R-isothermal coordinates u, v in terms of which $\Omega_1 = dw^2$ ([1], p. 103), so that

$$I = Edu^2 + (E-1)dv^2 \,. \tag{3}$$

Since such coordinates u, v are uniquely determined up to additive constants and/or simultaneous multiplication by 1 or -1, the function $E > 1$ is well defined on S wherever $\Omega_1 \neq 0$ on R, i.e. wherever $R \neq R_1$. The Theorema Egregium equation applied to (3) yields

$$K = \frac{1}{2\sqrt{E(E-1)}} \Delta \cosh^{-1}(2E-1) \,. \tag{4}$$

Thus $\Delta \cosh^{-1}(2E-1) \geqq 0$ on R where $K \leqq 0$, and $\Delta \cosh^{-1}(2E-1) \leqq 0$ on R where $K \geqq 0$. If $\cosh^{-1}(2E-1)$ is constant, (4) makes $K \equiv 0$.

REMARK. By arguments used to prove the following lemma, the function $E > 1$ just defined is itself R-subharmonic where $R \neq R_1$ and $K \leqq 0$ on S, provided that Ω_1 is R-holomorphic.

LEMMA 4. *If $X: S \rightarrow E^3$ is R-harmonic, there exists wherever $R \neq R_1$ an R-subharmonic function $E > 1$ which is constant only if S is immersed as a portion of a plane.*

Proof. No claims are made unless $R \neq R_1$ somewhere on S. Since $X: S \rightarrow E^3$ is assumed to be R-harmonic, Lemma 1 states that Ω_1 is R-holomorphic, so that the function E referred to in Lemma 3 is available on S wherever $R \neq R_1$. Equation (4) may be rewritten to read

$$\Delta E = \frac{2E-1}{2E(E-1)}\{E_u^2 + E_v^2\} - 2E(E-1)K \,, \tag{5}$$

giving $\Delta E \geqq 0$ for $E > 1$ on R where $R \neq R_1$, since $K \leqq 0$ by Lemma 2. If E is constant, it follows from (4) using Lemma 2 that $K \equiv H \equiv 0$ wherever $R \neq R_1$, a relation which extends easily by continuity to the isolated points on S at which $R = R_1$.

LEMMA 5. *If Ω_1 is R_A-holomorphic with $A = fI + gII$, then where $R_A \neq R_1$ there exist, locally, special R_A-isothermal lines-of-curvature coordinates u, v in terms of which*

$$g(k_2 - k_1)I = (f + gk_2)du^2 + (f + gk_1)dv^2 \,, \tag{6}$$

while the principal curvatures k_1 *and* k_2 *cannot be equal.*

Proof. Where $R_A \neq R_1$, special R_A-isothermal coordinates u, v may be locally introduced in terms of which I has the form (3) while

$$A = \mu(du^2 + dv^2) \tag{7}$$

with $\mu > 0$. Moreover, since $g \neq 0$, it is easy to check that $M = 0$, making u, v lines-of-curvature coordinates as well, so that

$$II = k_1 E du^2 + k_2(E-1)dv^2 \, .$$

Now (7) yields

$$0 < \mu = E(f + gk_1) = (E-1)(f + gk_2) \, ,$$

making $k_1 = k_2$ impossible since $E > 1$ and $g \neq 0$. Finally, (6) follows by simple arithmetic. Note that here,

$$\frac{f + gk_2}{g(k_2 - k_1)}$$

is the expression E referred to in Lemmas 3 and 4.

LEMMA 6. *If* $X: S \to E^3$ *is* R_A*-harmonic, then*

$$R_A = R_2'$$

where $R_A \neq R_1$, *so that* $R_A = R_1$ *precisely where* $H = 0$, *making zeros of* H *isolated unless* $H \equiv 0$.

Proof. By Lemma 1, Ω_1 is R_A-holomorphic, and where $R_A \neq R_1$, we may introduce the special R_A-isothermal lines-of-curvature coordinates provided by Lemma 5, in terms of which (trivially)

$$HM = KF = 0$$

while, by Lemma 1,

$$L = -N \, .$$

That $L \neq 0$ follows since, by Lemma 5, $K = H = 0$ is possible only where $R_A = R_1$. Thus Lemma 4 of [2] indicates that our coordinates are R_2'-isothermal, making $R_A = R_2'$ wherever $R_A \neq R_1$. Of course, $H \neq 0$ where $R_A \neq R_1$, since $L \neq 0$ and

$$H = \frac{L}{2E(1-E)} \, .$$

On the other hand, it is well known that $H = 0$ if $\Delta X = 0$ on R_1 at a point of S.

EXAMPLE 1. There is (at least locally) an R'_2-harmonic imbedding $X: S \to E^3$ with $H \neq 0$. As an example, take for S any portion of the u, v plane, and require that X produce the fundamental forms

$$I = \left(\frac{e^u + 2}{2}\right)du^2 + \left(\frac{e^u}{2}\right)dv^2$$
$$II = \frac{1}{2}\sqrt{\frac{e^u}{2 + e^u}}\,(du^2 - dv^2)\ .$$

Since the Codazzi-Mainardi and Theorema Egregium equations are satisfied, the fundamental existence theorem of surface theory gives the local existence of X. On the other hand, the requirements of Lemma 1 are fulfilled with $H \neq 0$.

EXAMPLE 2. It is an easy matter to find R-harmonic imbeddings $X: S \to E^3$ for an R not of the form R_A. If S is the x, y-plane, consider the imbedding $X: S \to E^3$ defined by

$$X(x, y) = (x, y, xy)\ .$$

Use for R the conformal structure determined by the metric $dx^2 + dy^2$ on S. The asymptotic coordinates x, y on the imbedded surface are not R_1-isothermal, nor are they R'_2-isothermal by the Remark, p. 1284 of [2]. Lemma 6 thus establishes that R is not of the form R_A. We note in passing that Ω_2 happens also to be R-holomorphic for this $X: S \to E^3$.

REMARK. Using Lemma 6 and the facts on p. 1284 of [2], it is easy to check that no immersion $X: S \to E^3$ of the form

$$X(x, y) = (x, y, f(x, y))$$

with S a portion of the x, y-plane can be R_A-harmonic with x, y R_A-isothermal coordinates unless $f \equiv c$, i.e. unless S is immersed as a piece of plane.

LEMMA 7. *If $X: S \to E^3$ is R'_2-harmonic, then near any point at which $R'_2 \neq R_1$ there exist special R'_2-isothermal coordinates u, v in terms of which*

$$(8)\qquad \begin{aligned}(k_1 + k_2)I &= k_2du^2 - k_1dv^2\\ (k_1 + k_2)II &= k_1k_2(du^2 - dv^2)\ .\end{aligned}$$

Proof. Use Lemma 5, setting

$$f=\frac{-K}{\sqrt{H^2-K}}\,, \qquad g=\frac{H}{\sqrt{H^2-K}}\,.$$

Note that by (8), k_2 has the sign of H on S, so that $|k_2|>|k_1|$. Here, of course,

$$\frac{k_2}{k_1+k_2}$$

is the function E referred to in Lemma 4.

3. Results in the large are presented in this section.

THEOREM 1. *If $K \geqq 0$ on a complete S in E^3 while Ω_1 is R-holomorphic and never vanishes on R, then $K \equiv 0$ on S.*

Proof. If $K \neq 0$ somewhere on S, then by Lemma 5 of [4], S is simply connected. Next, since an R-holomorphic quadratic differential must vanish identically if R is (conformally) a sphere, we can apply Lemma 5 of [4] once more to conclude that R is (conformally) the finite plane. Finally, since $K \geqq 0$ on S, Lemma 3 above yields the R-superharmonic function $\cosh^{-1}(2E-1)>0$ which must be constant on R. Thus E is constant, and using (4), $K \equiv 0$ on S.

Theorem 1 includes the elementary fact that there is no complete, umbilic free S in E^3 on which $K \equiv c > 0$. For, as shown in [3], $K \equiv c > 0$ on S in E^3 if and only if Ω_1 is R_2-holomorphic with zeros of Ω_1 on R_2 corresponding to umbilics on S. In the following theorem, $\Lambda = fI + gII$ as usual, while the principal curvatures on the surface S in question are numbered so that $|f+gk_2|>|f+gk_1|$ everywhere.

THEOREM 2. *If $K \leqq 0$ on a complete S in E^3 while Ω_1 is R_Λ-holomorphic and never vanishes on R_Λ then*

$$\frac{f+gk_2}{g(k_2-k_1)} \tag{9}$$

cannot be bounded from above unless $K \equiv 0$ on S.

Proof. Since Ω_1 is R_Λ-holomorphic and never vanishes on R_Λ, there are no points on S at which $R_\Lambda = R_1$, and therefore no umbilic points on S. Thus, special R_Λ-isothermal coordinates u, v in terms of which (6) is valid are locally available anywhere on S. Moreover, for all such choices of u, v,

$$I < \frac{f+gk_2}{g(k_2-k_1)}(du^2+dv^2)\,. \tag{10}$$

On the other hand, the form

$$\Gamma = (du^2 + dv^2)$$

is well defined independently of the various choices of the special R_A-isothermal coordinates u, v. Since I is complete, (10) implies that

$$\frac{f + gk_2}{g(k_2 - k_1)}\Gamma$$

is complete as well. Multiplication of a complete metric by a positive real valued function bounded away from zero will again yield a complete metric, so that if (9) is bounded from above, Γ itself is complete on S. Lifting the flat, complete, R_A-conformal metric Γ to the universal covering surface Σ of S, we conclude that Σ is R_A-conformally the finite plane ([5], p. 394). But then, by the Remark following Lemma 3 above, the function (9) lifted to Σ is an R_A-subharmonic function bounded from above, and therefore must be constant. It follows finally from (4) that $K \equiv 0$ on S.

COROLLARY. *H cannot be bounded away from zero on a complete, R'_2-harmonically immersed surface S in E^3.*

Proof. If S is R'_2-harmonically immersed in E^3 with $H \neq 0$, then the R'_2-holomorphic Ω_1 never vanishes. We apply Theorem 2 with

$$f = \frac{-K}{\sqrt{H^2 - K}}, \qquad g = \frac{H}{\sqrt{H^2 - K}}$$

so that

$$\frac{k_2}{k_1 + k_2} > 1 \tag{11}$$

is the expression (9). But (11) is bounded from above if and only if H is bounded away from zero. Thus, if H is bounded away from zero, $K \equiv 0$ on S, a contradiction here since R'_2 is only defined where $K < 0$ on S. In view of Lemma 6, we may reword the corollary to state that *H cannot be bounded away from zero on a complete R_A-harmonically immersed surface in E^3.*

It seems natural to ask whether there exists in E^3 a complete R'_2-harmonically immersed S on which H never vanishes. Regardless of the answer to this question, our Corollary may have its best explanation in light of the Milnor conjecture ([4], p. 8) which contains the guess that H cannot be bounded away from zero on a complete surface in E^3 on which $K \leqq 0$ unless $K \equiv 0$.

REFERENCES

1. L. Bers, "Quasiconformal mappings and Teichmüllers theorem," *Conference on Analytic Functions*, Princeton University Press, Princeton, 1960.
2. T. Klotz, *Another conformal structure on immersed surfaces of negative curvature*, Pacific J. Math. **13** (1964), 1281–1288.
3. ———, *Some uses of the second conformal structure on strictly convex surfaces*, Proc. Amer. Math. Soc. **14** (1963), 793–799.
4. T. Klotz and R. Osserman, *On complete surfaces in E^3 with constant mean curvature*, to appear in Comment. Math. Helv.
5. R. Osserman, *On complete minimal surfaces*, Arch. Rat. Mech. Anal. **13** (1963), 392–404.
6. D. Struik, *Classical Differential Geometry*, Addison-Wesley, Cambridge, 1950.

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