Pacific Journal of Mathematics

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Vol. 21, No. 1 November 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 1, 1967

FRACTIONAL POWERS OF OPERATORS, II INTERPOLATION SPACES

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This is a continuation of an earlier paper "Fractional Powers of Operators" published in this Journal concerning fractional powers A^{α}, $\alpha \in C$, of closed linear operators A in Banach spaces X such that the resolvent $(\lambda + A)^{-1}$ exists for all $\lambda > 0$ and $\lambda(\lambda + A)^{-1}$ is uniformly bounded. Various integral representations of fractional powers and relationship between fractional powers and interpolation spaces, due to Lions and others, of X and domain $D(A^{\alpha})$ are investigated.

In § 1 we define the space $D_p^{\sigma}(A)$, $0 < \sigma < \infty$, $1 \leqq p \leqq \infty$ or $p = \infty -$, as the set of all $x \in X$ such that

$$\lambda^{\sigma}(A(\lambda + A)^{-1})^m x \in L^p(X) ,$$

where m is an integer greater than σ and $L^p(X)$ is the L^p space of X-valued functions with respect to the measure $d\lambda/\lambda$ over $(0, \infty)$.

In § 2 we give a new definition of fractional power A^{α} for Re $\alpha > 0$ and prove the coincidence with the definition given in [2]. Convexity of $\| A^{\alpha} x \|$ is shown to be an immediate consequence of the definition. The main result of the section is Theorem 2.6 which says that if $0 < \mathrm{Re}\, \alpha < \sigma$, $x \in D_p^{\sigma}$ is equivalent to $A^{\alpha} x \in D_p^{\sigma - \mathrm{Re}\,\alpha}$. In particular, we have $D_1^{\mathrm{Re}\,\alpha} \subset D(A^{\alpha}) \subset D_{\infty -}^{\mathrm{Re}\,\alpha}$. For the application of fractional powers it is important to know whether the domain $D(A^{\alpha})$ coincides with $D_p^{\mathrm{Re}\,\alpha}$ for some p. We see, as a consequence of Theorem 2.6, that if we have $D(A^{\alpha}) = D_p^{\mathrm{Re}\,\alpha}$ for an α, it holds for all Re $\alpha > 0$. An example and a counterexample are given. At the end of the section we prove an integral representation of fractional powers.

Section 3 is devoted to the proof of the coincidence of D_p^{σ} with the interpolation space $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ due to Lions-Peetre [4]. We also give a direct proof of the fact that $D_p^{\sigma}(A^{\alpha}) = D_p^{\alpha\sigma}(A)$.

In § 4 we discuss the case in which $-A$ is the infinitesimal generator of a bounded strongly continuous semi-group T_t. A new space $C_{p,m}^{\sigma}$ is introduced in terms of $T_t x$ and its coincidence with D_p^{σ} is shown. Since $C_{\infty,m}^{\sigma}$, $\sigma \neq$ integer, coincides with C^{σ} of [2], this solves a question of [2] whether $C^{\sigma} = D^{\sigma}$ or not affirmatively. The coincidence of $C_{p,m}^{\sigma}$ with $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ has been shown by Lions-Peetre [4]. Further, another integral representation of fractional powers is obtained.

Finally, § 5 deals with the case in which $-A$ is the infinitesimal generator of a bounded analytic semi-group T_t. Analogous results to § 4 are obtained in terms of $A^\beta T_t x$.

1. Spaces D_p^σ. Throughout this paper we assume that A is a closed linear operator with a dense domain $D(A)$ in a Banach space X and satisfies

$$\|\lambda(\lambda + A)^{-1}\| \leqq M, \qquad 0 < \lambda < \infty . \tag{1.1}$$

We defined fractional powers in [2] for operators A which may not have dense domains. It was shown, however, that if Re $\alpha > 0$, A^α is an operator in $\overline{D(A)}$ and it is determined by a restriction A_D which has a dense domain in $\overline{D(A)}$. Thus our requirement on domain $D(A)$ is not restrictive as far as we consider exponent α with positive real part. As a consequence we have

$$(\lambda(\lambda + A)^{-1})^m x \longrightarrow x, \qquad \lambda \longrightarrow \infty,\ m = 1, 2, \cdots \tag{1.2}$$

for all $x \in X$. As in [2] L stands for a bound of $A(\lambda + A)^{-1} = I - \lambda(\lambda + A)^{-1}$:

$$\|A(\lambda + A)^{-1}\| \leqq L, \qquad 0 < \lambda < \infty . \tag{1.3}$$

We will frequently make use of spaces of X-valued functions $f(\lambda)$ defined on $(0, \infty)$. By $L^p(X)$ we denote the space of all X-valued measurable functions $f(\lambda)$ such that

$$\begin{aligned} \|f\|_{L^p} &= \left(\int_0^\infty \|f(\lambda)\|^p d\lambda/\lambda\right)^{1/p} < \infty \ \text{ if } 1 \leqq p < \infty \\ \|f\|_{L^\infty} &= \sup_{0<\lambda<\infty} \|f(\lambda)\| < \infty \ \text{ if } p = \infty . \end{aligned} \tag{1.4}$$

We admit as an index $p = \infty -$. $L^{\infty-}(X)$ represents the subspace of all functions $f(\lambda) \in L^\infty(X)$ which converge to zero as $\lambda \longrightarrow 0$ and as $\lambda \longrightarrow \infty$. Since $d\lambda/\lambda$ is a Haar measure of the multiplicative group $(0, \infty)$, an integral kernel $K(\lambda/\mu)$ with $\int_0^\infty |K(\lambda)|d\lambda/\lambda < \infty$ defines a bounded integral operator in $L^p(X)$, $1 \leqq p \leqq \infty$.

DEFINITION 1.1. Let $0 < \sigma < m$, where σ is a real number and m an integer, and p be as above. We denote by $D_{p\,m}^\sigma = D_{p,m}^\sigma(A)$ the space of all $x \in X$ such that $\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X)$ with the norm

$$\|x\|_{D_{p,m}^\sigma} = \|x\|_X + \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|_{L^p(X)} . \tag{1.5}$$

$D_{\infty,1}^\sigma$ and $D_{\infty-,1}^\sigma$ coincide with D^σ and D_*^σ of [2], respectively.

It is easy to see that $D_{p,m}^\sigma$ is a Banach space. Since $(A(\lambda + A)^{-1})^m$ is uniformly bounded, only the behavior near infinity of $(A(\lambda + A)^{-1})^m x$

decides whether x belongs to $D^{\sigma}_{p,m}$ or not.

PROPOSITION 1.2. If integers m and n are greater than σ, the spaces $D^{\sigma}_{p,m}$ and $D^{\sigma}_{p,n}$ are identical and have equivalent norms.

Proof. It is enough to show that $D^{\sigma}_{p,m} = D^{\sigma}_{p,m+1}$ when $m > \sigma$. Because of (1.3) every $x \in D^{\sigma}_{p,m}$ belongs to $D^{\sigma}_{p,m+1}$. Since

$$\frac{d}{d\lambda}(\lambda^m(A(\lambda + A)^{-1})^m) = m\lambda^{m-1}(A(\lambda + A)^{-1})^{m+1},$$

we have

$$\lambda^{\sigma}(A(\lambda + A)^{-1})^m x = m\lambda^{\sigma-m}\int_0^{\lambda}\mu^{m-\sigma}\mu^{\sigma}(A(\mu + A)^{-1})^{m+1}x d\mu/\mu . \tag{1.6}$$

This shows

$$\|\lambda^{\sigma}(A(\lambda + A)^{-1})^m x\|_{L^p(X)} \leqq \frac{m}{m-\sigma}\|\lambda^{\sigma}(A(\lambda + A)^{-1})^{m+1}x\|_{L^p(X)} .$$

DEFINITION 1.3. We define D^{σ}_p, $\sigma > 0$, $1 \leqq p \leqq \infty$, as the space $D^{\sigma}_{p,m}$ with the least integer m greater than σ. We use $q^{\sigma}_p(x)$ to denote the second term of (1.5), so that D^{σ}_p is a Banach space with the norm $\|x\| + q^{\sigma}_p(x)$.

PROPOSITION 1.4. If $\mu > 0$, $\mu(\mu + A)^{-1}$ maps D^{σ}_p continuously into $D^{\sigma+1}_p$. Futhermore, if $p \leqq \infty -$, we have for every $x \in D^{\sigma}_p$

$$\mu(\mu + A)^{-1}x \to x \ (D^{\sigma}_p) \quad \text{as} \quad \mu \to \infty . \tag{1.7}$$

Proof. Let $x \in D^{\sigma}_p$. Since

$$\begin{aligned}&\|\lambda^{\sigma+1}(A(\lambda + A)^{-1})^{m+1}\mu(\mu + A)^{-1}x\|\\ &\quad \leqq \mu\|\lambda(\lambda + A)^{-1}\|\,\|A(\mu + A)^{-1}\|\,\|\lambda^{\sigma}(A(\lambda + A)^{-1})^m x\|\\ &\quad \leqq \mu ML\|\lambda^{\sigma}(A(\lambda + A)^{-1})^m x\| ,\end{aligned}$$

$\mu(\mu + A)^{-1}x$ belongs to $D^{\sigma+1}_p$.

Let $p \leqq \infty -$. If $x \in D(A)$, then

$$\begin{aligned}&(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}x\\ &\quad = (A(\lambda + A)^{-1})^m x - (A(\lambda + A)^{-1})^m(\mu + A)^{-1}Ax\end{aligned}$$

converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ. On the other hand, $(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}$ is uniformly bounded. Thus it follows that $(A(\lambda + A)^{-1})^m\mu(\mu + A)^{-1}x$ converges to $(A(\lambda + A)^{-1})^m x$ uniformly in λ for every $x \in X$. Since $\|\lambda^{\sigma}(A(\lambda+A)^{-1})^m\mu(\mu+A)^{-1}x\| \leqq M\|\lambda^{\sigma}(A(\lambda+A)^{-1})^m x\|$, this implies (1.7).

THEOREM 1.5. *$D_p^\sigma \subset D_q^\tau$ if $\sigma > \tau$ or if $\sigma = \tau$ and $p \leqq q$. The injection is continuous. If $q \leqq \infty -$, D_p^σ is dense in D_q^τ.*

Proof. First we prove that D_p^σ, $p < \infty$, is continuously contained in $D_{\infty-}^\sigma$.

Let $x \in D_p^\sigma$. Applying Hölder's inequality to (1.6), we obtain

$$\| \lambda^\sigma (A(\lambda + A)^{-1})^m x \| \leqq \frac{m}{((m-\sigma)p')^{1/p'}} \| \mu^\sigma (A(\mu + A)^{-1})^{m+1} x \|_{L^p(X)} ,$$

where $p' = p/(p-1)$. Hence $x \in D_\infty^\sigma$. Considering the integral over the interval (μ, λ), we have similarly

$$\| \lambda^\sigma (A(\lambda + A)^{-1})^m x \| \leqq \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}} \| \mu^\sigma (A(\mu + A)^{-1})^m x \|$$

$$+ \frac{m}{((m-\sigma)p')^{1/p'}} \left(1 - \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}}\right) \left(\int_\mu^\lambda \| \tau^\sigma (A(\tau + A)^{-1})^{m+1} x \|^p d\tau/\tau)^{1/p} \right).$$

The second term tends to zero as $\mu \to \infty$ uniformly in $\lambda > \mu$ and so does the first term as $\lambda \to \infty$. Therefore, $x \in D_{\infty-}^\sigma$.

Since $\lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X) \cap L^{\infty-}(X)$, it is in any $L^q(X)$ with $p \leqq q < \infty$.

If $\tau < \sigma$, D_∞^σ is contained in D_q^τ for any q. Hence every D_q^σ is contained in D_q^τ.

Let $q \leqq \infty -$. Repeated application of Proposition 1.4 shows that $D_q^{\tau+m}$ is dense in D_q^τ for positive integer m. Since D_p^σ contains some $D_q^{\tau+m}$, it is dense in D_q^τ.

2. Fractional powers.

If $x \in D_1^\sigma$, the integral

$$A_\sigma^\alpha x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda \tag{2.1}$$

converges absolutely for $0 < \operatorname{Re} \alpha \leqq \sigma$ and represents a continuous operator from D_1^σ into X. Moreover, $A_\alpha^\sigma x$ is analytic in α for $0 < \operatorname{Re} \alpha < \sigma$.

$A_\sigma^\alpha x$ does not depend on m. In fact, substitution of (1.6) into (2.1) gives

$$A_\sigma^\alpha x = \frac{\Gamma(m)m}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \mu^{m-1} (A(\mu + A)^{-1})^{m+1} x d\mu \int_\mu^\infty \lambda^{\alpha-m-1} d\lambda$$

$$= \frac{\Gamma(m+1)}{\Gamma(\alpha)\Gamma(m+1-\alpha)} \int_0^\infty \mu^{\alpha-1} (A(\mu + A)^{-1})^{m+1} x d\mu .$$

This shows that $A_\sigma^\alpha x$ depends only on x and not on D_1^σ to which x belongs.

Obviously we have

$$A_\sigma^\alpha(\mu(\mu + A)^{-1})^{m+1}x = (\mu(\mu + A)^{-1})^{m+1}A_\sigma^\alpha x,\ x \in D_1^\alpha\ . \tag{2.2}$$

Since the left-hand side and $(\mu(\mu + A)^{-1})^{m+1}$ are continuous in X, and $(\mu(\mu + A)^{-1})^{m+1}$ is one-to-one, it follows that A_σ^α is closable in X. In view of Theorem 1.5 the smallest closed extension does not depend on σ.

DEFINITION 2.1. The fractional power A^α for $\operatorname{Re} \alpha > 0$ is the smallest closed extension of A_σ^α for a $\sigma \geqq \operatorname{Re} \alpha$.

PROPOSITION 2.2. If α is an integer $m > 0$, A^α coincides with the power A^m.

To prove the proposition we prepare a lemma.

LEMMA 2.3. *If m is an integer $m > 0$,*

$$A^m x = \operatorname*{s-lim}_{N\to\infty} m\int_0^N \lambda^{m-1}(A(\lambda + A)^{-1})^{m+1}x d\lambda\ . \tag{2.3}$$

Proof. By (1.6) we have

$$m\int_0^N \lambda^{m-1}(A(\lambda + A)^{-1})^{m+1}x = N^m(A(N + A)^{-1})^m x\ .$$

If $x \in D(A^m)$, $N^m(A(N + A)^{-1})^m x = (N(N + A)^{-1})^m A^m x$ tends to $A^m x$ as $N \to \infty$ by (1.2). Conversely if $N^m(A(N + A)^{-1})^m x = A^m(N(N + A)^{-1})^m x$ converges to an element y, $x \in D(A^m)$ and $y = A^m x$. For A^m is closed (see Taylor [5]) and $(N(N + A)^{-1})^m x$ converges to x.

Proof of Proposition 2.2. If $x \in D_1^\sigma$, $\sigma > m$, integral (2.3) converges absolutely. Therefore it follows from Lemma 2.3 that $x \in D(A^m)$ and $A^\alpha x = A^m x$. Thus A^m is an extension of A^α. Conversely if $x \in D(A^m)$, then $\mu(\mu + A)^{-1}x \in D(A^{m+1}) \subset D_{\infty-}^{m+1}$ and we have

$$\begin{aligned} A^\alpha(\mu(\mu + A)^{-1})x &= (\mu(\mu + A)^{-1})A^m x \\ &\to A^m x \quad \text{as} \quad \mu \to \infty\ . \end{aligned}$$

Since $\mu(\mu + A)^{-1}x \to x$, it follows that $x \in D(A^\alpha)$ and $A^\alpha x = A^m x$.

The fractional power A^α defined above coincides with A_+^α defined in [2]. In fact, if $m = 1$, integeral (2.1) is the same as integral (4.2) of [2] for $n = 0$. Thus

$$A^\alpha x = A_+^\alpha x \tag{2.4}$$

holds for $0 < \operatorname{Re} < 1$ if $x \in D(A)$. If $x \in D(A^m)$, $m \geqq 1$, both sides of

(2.4) are analytic for $0 < \operatorname{Re}\alpha < m$, so that (2.4) holds there. Since $D_1^m \subset D(A^m) \subset D_{\infty-}^m$ by Lemma 2.3 and (1.2), both A^α and A_+^α are the smallest closed extension of their restrictions to $D(A^m)$, $m > \operatorname{Re}\alpha$. Thus we have $A^\alpha = A_+^\alpha$ for all $\operatorname{Re}\alpha > 0$.

Consequently we may employ all results of [2]. In particular, fractional powers satisfy additivity

$$A^{\alpha+\beta} = A^\alpha A^\beta, \qquad \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0 \tag{2.5}$$

in the sense of product of operators and multiplicativity

$$(A^\alpha)^\beta = A^{\alpha\beta}, \qquad 0 < \alpha < \pi/\omega, \operatorname{Re}\beta > 0, \tag{2.6}$$

where ω is the minimum number such that the resolvent set of $-A$ contains the sector

$$|\arg\lambda| < \pi - \omega .$$

Such an operator is said to be of type $(\omega, M(\theta))$ if

$$\sup_{|\arg\lambda|=\theta} \|\lambda(\lambda + A)^{-1}\| \leqq M(\theta) .$$

Any operator with a dense domain which satisfies (1.1) is of type $(\omega, M(\theta))$ with $0 \leqq \omega < \pi$.

Some properties of fractional powers, however, are derived more easily through definition (2.1).

Proposition 2.4. If $0 < \operatorname{Re}\alpha < \sigma$, there is a constant $C(\alpha, \sigma, p)$ such that

$$\|A^\alpha x\| \leqq C(\alpha, \sigma, p) q_p^\sigma(x)^{\operatorname{Re}\alpha/\sigma} \|x\|^{(\sigma-\operatorname{Re}\alpha)/\sigma}, \; x \in D_p^\sigma . \tag{2.7}$$

Proof. Hölder's inequality gives

$$\begin{aligned}\|A^\alpha x\| \leqq \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \Big[\int_0^N |\lambda^{\alpha-1}| \, \|(A(\lambda + A)^{-1})^m x\| \, d\lambda \\ + \int_N^\infty |\lambda^{\alpha-\sigma}| \, \|\lambda^\sigma (A(\lambda + A)^{-1})^m x\| \, d\lambda/\lambda\Big] \\ \leqq \left|\frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)}\right| \left[\frac{L^m N^{\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \|x\| + \frac{N^{\operatorname{Re}\alpha-\sigma}}{((\sigma - \operatorname{Re}\alpha)p')^{1/p'}} q_p^\sigma(x)\right].\end{aligned}$$

Taking the minimum of the right-hand side when N varies $0 < N < \infty$, we obtain (2.7).

Proposition 2.5. If $\mu > 0$, then

$$D_p^\sigma(A) = D_p^\sigma(\mu + A) \tag{2.8}$$

with equivalent norms.

Proof. Let $x \in D^\sigma_{p,m}(A)$ with $m > \sigma$. Since

$$\begin{aligned} &\| A^k(\lambda + \mu + A)^{-m}x \| \leqq C \| A^m(\lambda + \mu + A)^{-m}x \|^{k/m} \cdot \\ &\qquad \| (\lambda + \mu + A)^{-m}x \|^{(m-k)/m}, \qquad k = 1, 2, \cdots, m-1, \\ &\lambda^\sigma((\mu + A)(\lambda + \mu + A)^{-1})^m x \\ &\qquad = \lambda^\sigma(\mu^m + m\mu^{m-1}A + \cdots + A^m)(\lambda + \mu + A)^{-m}x \end{aligned}$$

belongs to $L^p(X)$. The converse is proved in the same way.

THEOREM 2.6. *Let* $0 < \mathrm{Re}\,\alpha < \sigma$. *Then* $x \in D^\sigma_p$ *if and only if* $x \in D(A^\alpha)$ *and* $A^\alpha x \in D^{\sigma-\mathrm{Re}\alpha}_p$.

Proof. Let $x \in D^\sigma_p$ and $m > \sigma$. Clearly $x \in D(A^\alpha)$. To estimate the integral

$$\begin{aligned} &\lambda^{\sigma-\mathrm{Re}\alpha}(A(\lambda + A)^{-1})^m A^\alpha x \\ &\qquad = \frac{\Gamma(m)\lambda^{\sigma-\mathrm{Re}\alpha}}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \mu^{\alpha-1}(A(\lambda + A)^{-1})^m(A(\mu + A)^{-1})^m x d\mu, \end{aligned}$$

we split it into two parts. First,

$$\begin{aligned} &\left\| \lambda^{\sigma-\mathrm{Re}\alpha} \int_0^\lambda \mu^{\alpha-1}(A(\lambda + A)^{-1})^m(A(\mu + A)^{-1})^m x d\mu \right\| \\ &\qquad \leqq \lambda^{\sigma-\mathrm{Re}\alpha} \int_0^\lambda \mu^{\mathrm{Re}\alpha-1} d\mu L^m \| (A(\lambda + A)^{-1})^m x \| \\ &\qquad = L^m(\mathrm{Re}\,\alpha)^{-1}\lambda^\sigma \| (A(\lambda + A)^{-1})^m x \| \in L^p. \end{aligned}$$

$$\begin{aligned} &\left\| \lambda^{\sigma-\mathrm{Re}\alpha} \int_\lambda^\infty \mu^{\alpha-1}(A(\lambda + A)^{-1})^m(A(\mu + A)^{-1})^m x d\mu \right\| \\ &\qquad \leqq L^m \lambda^{\sigma-\mathrm{Re}\alpha} \int_\lambda^\infty \mu^{\mathrm{Re}\alpha-\sigma} \| \mu^\sigma(A(\mu + A)^{-1})^m x \| d\mu/\mu \end{aligned}$$

also belongs to L^p because $\mathrm{Re}\,\alpha - \sigma < 0$.

Conversely, let $A^\alpha x \in D^{\sigma-\mathrm{Re}\alpha}_p$. If n is an integer greater than $\mathrm{Re}\,\alpha$, we have

$$\begin{aligned} \| A^{n-\alpha}(\lambda + A)^{-n} \| &\leqq C \| A^n(\lambda + A)^{-n} \|^{(n-\mathrm{Re}\alpha)/n} \| (\lambda + A)^{-n} \|^{\mathrm{Re}\alpha/n} \\ &\leqq C'\lambda^{-\mathrm{Re}\alpha} \end{aligned}$$

Thus it follows from (2.5) that

$$\begin{aligned} \lambda^\sigma \| (A(\lambda + A)^{-1})^{m+n}x \| &\leqq \lambda^\sigma \| A^{n-\alpha}(\lambda + A)^{-n} \| \| (A(\lambda + A)^{-1})^m A^\alpha x \| \\ &\leqq C'\lambda^{\sigma-\mathrm{Re}\alpha} \| (A(\lambda + A)^{-1})^m A^\alpha x \| \in L^p. \end{aligned}$$

This completes the proof.

As a corollary we see that if σ is not an integer, D^{σ}_{∞} and $D^{\sigma}_{\infty-}$ coincide with D^{σ} and D^{σ}_{*} of [2], respectively.

THEOREM 2.7. *If the domain* $D(A^{\alpha})$ *contains (is contained in)* $D^{\mathrm{Re}\,\alpha}_{p}$ *for an* $\mathrm{Re}\,\alpha > 0$, *then* $D(A^{\alpha})$ *contains (is contained in)* $D^{\mathrm{Re}\,\alpha}_{p}$ *for all* $\mathrm{Re}\,\alpha > 0$.

Proof. By virtue of Theorem 6.4 of [2] and Proposition 2.5 we have $D(A^{\alpha}) = D((\mu + A)^{\alpha})$ and $D^{\mathrm{Re}\,\alpha}_{p}(A) = D^{\mathrm{Re}\,\alpha}_{p}(\mu + A)$, $\mu > 0$, $\mathrm{Re}\,\alpha > 0$, so that we may assume that A has a bounded inverse without loss of generality. The theorem is obvious if we show that A^{β}, $-\infty < \mathrm{Re}\,\beta < \mathrm{Re}\,\alpha$, is a one-to-one mapping from $D(A^{\alpha})$ and $D^{\mathrm{Re}\,\alpha}_{p}$ onto $D(A^{\alpha-\beta})$ and $D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}_{p}$, respectively.

Since $D(A^{\alpha}) = R(A^{-\alpha})$, $\mathrm{Re}\,\alpha > 0$ ([2], Theorem 6.4), and since $A^{\beta-\alpha} = A^{\beta}A^{-\alpha}$ ([2], Theorem 7.3), the statemant concerning $D(A^{\alpha})$ is immediate.

Let $\mathrm{Re}\,\beta < 0$. Then $x \in D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}_{p}$ if and only if $x \in D(A^{-\beta})$ and $A^{-\beta}x \in D^{\mathrm{Re}\,\alpha}_{p}$. Since A^{β} is a bounded inverse of $A^{-\beta}$, we have $x \in D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}_{p}$ if and only if x is in the image of $D^{\mathrm{Re}\,\alpha}_{p}$ by A^{β}. If $\mathrm{Re}\,\beta \geqq 0$, choose a number γ so that $\mathrm{Re}\,\beta < \gamma < \mathrm{Re}\,\alpha$. If $x \in D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}_{p}$, x belongs to $D(A^{-\beta})$. Thus there is an element y such that $x = A^{\beta}y$. By the former part we have $A^{-\gamma}x = A^{\beta-\gamma}y \in D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta+\gamma}_{p}$. Thus y belongs to $D^{\mathrm{Re}\,\alpha}_{p}$. On the other hand, if $y \in D^{\mathrm{Re}\,\alpha}_{p}$, then $y \in D(A^{\beta})$ and we have $A^{-\gamma}x = A^{\beta-\gamma}y \in D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta+\gamma}_{p}$, where $x = A^{\beta}y$. Then it follows from the former part that x belongs to $D^{\mathrm{Re}\,\alpha-\mathrm{Re}\,\beta}_{p}$.

Theorem 6.5 of [2] is obtained as a corollary.

PROPOSITION 2.8. For every $\mathrm{Re}\,\alpha > 0$

$$D^{\mathrm{Re}\,\alpha}_{1} \subset D(A^{\alpha}) \subset D^{\mathrm{Re}\,\alpha}_{\infty-} . \tag{2.9}$$

Proof. It is enough to prove it only in the case $\alpha = 1$. The former inclusion is clear from Lemma 2.3. The latter follows from (1.2), for

$$\lambda(A(\lambda + A)^{-1})^{2}x = \lambda(\lambda + A)^{-1}(1 - \lambda(\lambda + A)^{-1})Ax \to 0$$

for $x \in D(A)$ as $\lambda \to \infty$.

PROPOSITION 2.9. If there is a complex number $\mathrm{Re}\,\alpha > 0$ such that $D(A^{\alpha}) = D^{\mathrm{Re}\,\alpha}_{p}$, then $D(A^{\beta}) = D^{\mathrm{Re}\,\beta}_{p}$ for all $\mathrm{Re}\,\beta > 0$. In particular, $D(A^{\alpha})$ coinsides with $D(A^{\beta})$ if $\mathrm{Re}\,\alpha = \mathrm{Re}\,\beta$. Furthermore, if A has a bounded inverse, A^{it} is bounded for all real t, where A^{it} is defined in [2].

Proof. We need to prove only the last statement. Because of [2], Corollary 7.4 we have

$$A^{it} = A^{1+it}A^{-1} ,$$

Since $D(A^{1+it}) = D(A) = R(A^{-1})$, A^{it} is defined everywhere and closed, so that it is bounded.

We proved in [2] that the operator A of § 14, Example 6 has unbounded purely imaginary powers A^{it}. The above proposition shows that $D(A^{\alpha})$ cannot be the same as $D_p^{\mathrm{Re}\,\alpha}$ for any p.

However, there are also operators A for which $D(A^{\alpha})$ coincides with $D_p^{\mathrm{Re}\,\alpha}$.

Let X be $L^p(S, B, m)$, where B is a Borel field over a set S and m a measure on B, and let $A(s)$ be a measurable function on S such that

$$|\arg A(s)| \leqq \omega, \text{ a.e.} s$$

for an $0 \leqq \omega < \pi$. Define

$$Ax(s) = A(s)x(s)$$

for all $x(s) \in X$ such that $A(s)x(s) \in X$. Then it is easy to see that A is an operator of type $(\omega, M(\theta))$ if $p \leqq \infty -$, where $L^{\infty -}$ denotes the closure of $D(A)$ in L^{∞}. For this operator A we have $D(A) = D_p^1$, so that $D(A^{\alpha}) = D_p^{\mathrm{Re}\,\alpha}$ for all $\mathrm{Re}\,\alpha > 0$.

In fact, we have

$$(A(\lambda + A)^{-1})^2 x(s) = A(s)^2 x(s)/(\lambda + A(s))^2 .$$

Therefore,

$$\begin{aligned}
&\int_0^{\infty} \| \lambda(A(\lambda + A)^{-1})^2 x(s) \|^p d\lambda/\lambda \\
&= \int_0^{\infty} \lambda^{p-1} d\lambda \int_S \left| \frac{A(s)^2}{(\lambda + A(s))^2} x(s) \right|^p dm(s) \\
&= \int_S |x(s)|^p dm(s) \int_0^{\infty} \lambda^{p-1} \left| \frac{A(s)}{\lambda + A(s)} \right|^{2p} d\lambda \\
&\sim \| Ax \|^p .
\end{aligned}$$

Any normal operator A of type $(\omega, M(\theta))$ can be represented as an operator of the above type. Therefore, it satisfies $D(A^{\alpha}) = D_2^{\mathrm{Re}\,\alpha}$ for $\mathrm{Re}\,\alpha > 0$. T. Kato [1] proved that this holds also for any maximal accretive operator A (see J.-L. Lions [3]).

Now let us complete the definition of fractional powers.

THEOREM 2.10. *Let $0 < \mathrm{Re}\,\alpha < m$. If there is a sequence $N_j \to \infty$*

such that

$$y = w\text{-}\lim_{j\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{N_j} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda$$

exists, then $x \in D(A^\alpha)$ *and* $y = A^\alpha x$.

Conversely, if $x \in D(A^\alpha)$, *then*

$$A^\alpha x = s\text{-}\lim_{N\to\infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{N} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda\ , \tag{2.10}$$

possibly except for the case in which Im $\alpha \neq 0$ *and* Re α *is an integer.*

Proof. The former statement is obtained by modifying the proof of [2], Proposition 4.6. Since $(\mu(\mu+A)^{-1})^m x \in D_1^{\mathrm{Re}\,\alpha}$, we have

$$\begin{aligned} A^\alpha(\mu(\mu+A)^{-1})^m x &= c\int_0^\infty \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m(\mu(\mu+A)^{-1})^m x d\lambda \\ &= (\mu(\mu+A)^{-1})^m\, w\text{-}\lim_{j\to\infty} c\int_0^{N_j} \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &= (\mu(\mu+A)^{-1})^m y\ . \end{aligned}$$

By virtue of (1, 2), it follows that $x \in D(A^\alpha)$ and $y = A^\alpha x$.

The proof of the latter statement may be reduced to the case in which $0 < \mathrm{Re}\,\alpha < 1$ and $m = 1$. Suppose that $x \in D(A^\alpha)$ and an integer $m > \mathrm{Re}\,\alpha$. Substituting (1.6), we have

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &\quad = m\int_0^N \lambda^{\alpha-m-1} d\lambda \int_0^\lambda \mu^{m-1}(A(\mu+A)^{-1})^{m+1} x d\mu \\ &\quad = \frac{m}{m-\alpha}\int_0^N \left(1 - \frac{\mu^{m-\alpha}}{N^{m-\alpha}}\right)\mu^{\alpha-1}(A(\mu+A)^{-1})^{m+1} x d\mu\ . \end{aligned}$$

Since $x \in D(A^\alpha) \subset D_{\infty-}^{\mathrm{Re}\,\alpha}$, it follows that

$$\left\| \int_0^N \frac{\mu^{m-\alpha}}{N^{m-\alpha}}\ \mu^{\alpha-1}(A(\mu+A)^{-1})^{m+1} x d\mu \right\| \longrightarrow 0 \quad \text{as} \quad N \longrightarrow \infty\ .$$

Thus the limit (2.10), if it exists, does not depend on $m > \mathrm{Re}\,\alpha$.

Next, let $\mathrm{Re}\,\alpha > 1$ and $m \geqq 2$. Since $x \in D(A^\alpha)$ belongs to $D(A)$, integration by parts yields

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1}(A(\lambda+A)^{-1})^m x d\lambda \\ &\quad = \frac{\alpha-1}{m-1}\int_0^N \lambda^{\alpha-2}(A(\lambda+A)^{-1})^{m-1} A x d\lambda - \frac{N^{\alpha-1}}{m-1}(A(N+A)^{-1})^{m-1} A x\ . \end{aligned}$$

The second term tends to zero as $N \to \infty$ because $Ax \in D(A^{\alpha-1}) \subset D_{\infty-}^{\mathrm{Re}\alpha-1}$. Therefore, we obtain (2.10) if we can prove it when both α and m are reduced by one.

To prove (2.10) in the case $0 < \mathrm{Re}\,\alpha < 1$ and $m = 1$ we assume for a moment that A has a bounded inverse. Then $D(A^\alpha)$ is identical with the range of $A^{-\alpha}$, which may be represented by the absolulely convergent integral:

$$A^{-\alpha}x = \frac{\sin \pi x}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda + A)^{-1}x d\lambda$$

([2], Proposition 5.1). Employing the resolvent equation and (1.6), we get

$$\begin{aligned}
&\frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^N \lambda^{\alpha-1}A(\lambda + A)^{-1}A^{-\alpha}x d\lambda \\
&\quad = \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\infty \mu^{-\alpha}\frac{\lambda(\lambda + A)^{-1} - \mu(\mu + A)^{-1}}{\lambda - \mu}x d\mu \\
&\quad = \left(\frac{\sin \pi\alpha}{\pi}\right)^2 \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\infty \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu \int_\mu^\lambda A(\nu + A)^{-2}x d\nu \, .
\end{aligned}$$

It is enough to show that this converges strongly to the identity, or more weakly that it simply converges, because if it converges, the limit must be $A^\alpha A^{-\alpha}x = x$.

First of all, we have

$$\begin{aligned}
I_1 &= \int_0^N \lambda^{\alpha-1}d\lambda \int_0^\lambda \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu \int_\mu^\lambda A(\nu + A)^{-2}x d\nu \\
&= \int_0^N A(\nu + A)^{-2}x d\nu \int_\nu^N \lambda^{\alpha-1}d\lambda \int_0^\nu \mu^{-\alpha}(\lambda - \mu)^{-1}d\mu \, .
\end{aligned}$$

Changing variables by $\lambda = \nu l$, $\mu = \nu m$ and integrating by parts with respect to ν, we obtain

$$\begin{aligned}
I_1 &= \int_1^\infty l^{\alpha-1}dl \int_0^1 m^{-\alpha}(l - m)^{-1}dm x \\
&\quad - \int_0^N A(\nu + A)^{-1}x d\nu N^\alpha \nu^{-\alpha-1}\int_0^1 m^{-\alpha}(N\nu^{-1} - m)^{-1}dm \\
&= c_1 x - \int_0^1 A(Nn + A)^{-1}x n^{-\alpha-1}dn \int_0^1 m^{-\alpha}(n^{-1} - m)^{-1}dm \, .
\end{aligned}$$

Since $n^{-\alpha-1}\int_0^1 m^{-\alpha}(n^{-1} - m)^{-1}dm$ is absolutely integrable in n and since $A(Nn + A)^{-1}x = x - Nn(Nn + A)^{-1}x$ tends to zero as $N \to \infty$, the second term converges to zero as $N \to \infty$.

Next we write

$$\begin{aligned}\int_0^N \lambda^{\alpha-1}d\lambda\int_\lambda^\infty \mu^{-\alpha}(\lambda-\mu)^{-1}d\mu\int_\mu^\lambda A(\nu+A)^{-2}xd\nu \\ = \int_0^N A(\nu+A)^{-2}xd\nu\int_0^\nu \lambda^{\alpha-1}d\lambda\int_\nu^\infty \mu^{-\alpha}(\mu-\lambda)^{-1}d\mu \\ + \int_N^\infty A(\nu+A)^{-2}xd\nu\int_0^N \lambda^{\alpha-1}d\lambda\int_\nu^\infty \mu^{-\alpha}(\mu-\lambda)^{-1}d\mu \\ = I_2 + I_3 .\end{aligned}$$

Changing variables as above, we have

$$\begin{aligned}I_2 &= \int_0^N A(\nu+A)^{-2}xd\nu\int_0^1 l^{\alpha-1}dl\int_1^\infty m^{-\alpha}(m-l)^{-1}dm \\ &= c_2N(N+A)^{-1}x \longrightarrow c_2x \quad \text{as} \quad N\longrightarrow\infty .\end{aligned}$$

Finally,

$$I_3 = \int_1^\infty m^{-\alpha}dm\int_0^1 l^{\alpha-1}(m-l)^{-1}dl\int_N^{mN} A(\nu+A)^{-2}xd\nu$$

tends to zero as $N\longrightarrow\infty$ because $\int_N^{mN} A(\nu+A)^{-2}xd\nu = mN(mN+A)^{-1}x - N(N+A)^{-1}x$ tends to zero and $m^{-\alpha}\int_0^1 l^{\alpha-1}(m-l)^{-1}dl$ is absolutely integrable.

Next suppose that A has not necessarily a bounded inverse. We have, for $\mu > 0$,

$$\begin{aligned}&(A^\alpha - (\mu+A)^\alpha)(\mu+A)^{-\alpha}x \\ &= \frac{\sin\pi\alpha}{\pi}\Big(\int_0^\mu \lambda^{\alpha-1}A + \int_\mu^\infty(\lambda^{\alpha-1}A - (\lambda-\mu)^{\alpha-1}(\mu+A)\Big)(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda\end{aligned}$$

because the integral is absolutely convergent and the equality holds for all $x\in D(A)$ which is dense in X. This shows together with the above that

$$\begin{aligned}A^\alpha(\mu+A)^{-\alpha}x &= (\mu+A)^\alpha(\mu+A)^{-\alpha}x \\ &\quad + \operatorname*{s-lim}_{N\to\infty}\frac{\sin\pi\alpha}{\pi}\Big(\int_0^\mu \lambda^{\alpha-1}A + \int_\mu^N(\lambda^{\alpha-1}A - (\lambda-\mu)^{\alpha-1}(\mu+A)\Big) \\ &\qquad \cdot(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda \\ &= \operatorname*{s-lim}_{N\to\infty}\frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^N \lambda^{\alpha-1}A(\lambda+A)^{-1}(\mu+A)^{-\alpha}xd\lambda .\end{aligned}$$

3. Interpolation spaces. Let X and Y be Banach spaces contained in a Hausdorff vector space Z. Lions and Peetre [4] defined the mean space $S(p, \theta, X; p, \theta - 1, Y)$, $1 \leqq p \leqq \infty$, $0 < \theta < 1$, of X and Y as the space of the means

$$x = \int_0^\infty u(\lambda) d\lambda/\lambda\ , \tag{3.1}$$

where $u(\lambda)$ is a Z-valued function such that

$$\lambda^\theta u(\lambda) \in L^p(X) \text{ and } \lambda^{\theta-1} u(\lambda) \in L^p(Y)\ . \tag{3.2}$$

$S(p, \theta, X; p, \theta - 1, Y)$ is a Banach space with the norm

$$\begin{aligned} &\| x \|_{S(p,\theta,X,p,\theta-1,Y)} \\ &= \inf \left\{ \max (\| \lambda^\theta u(\lambda) \|_{L^p(X)}, \| \lambda^{\theta-1} u(\lambda) \|_{L^p(Y)}); x = \int_0^\infty u(\lambda) d\lambda/\lambda \right\}. \end{aligned} \tag{3.3}$$

Theorem 3.1. *$S(p, \theta, X; p, \theta - 1, D(A^m))$, $0 < \theta < 1$, $1 \leqq p \leqq \infty$, coincides with $D_p^{\theta m}(A)$.*

Proof. By virtue of Proposition 2.5, we may assume that A has a bounded inverse without loss of generality. In particular, $D(A^m)$ is normed by $\| A^m x \|$. Further, if we change the variable by $\lambda' = \lambda^{1/m}$, condition (3.2) becomes

$$\lambda^{m\theta} u(\lambda) \in L^p(X) \text{ and } \lambda^{m(\theta-1)} A^m u(\lambda) \in L_p(X)\ . \tag{3.4}$$

Suppose $x \in D_p^\sigma$ and define

$$u(\lambda) = c\lambda^m A^m (\lambda + A)^{-2m} x\ ,$$

where $c = \Gamma(2m)/(\Gamma(m))^2$. Then

$$\lambda^\sigma u(\lambda) = c(\lambda(\lambda + A^{-1})^m \lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X)$$

and

$$\lambda^{\sigma - m} A^m u(\lambda) = c\lambda^\sigma (A(\lambda + A)^{-1})^{2m} x \in L^p(X)\ .$$

Thus $u(\lambda)$ satisfies (3.4) with $\sigma = m\theta$. Moreover, it follows from Lemma 2.3 that

$$\begin{aligned} \int_0^\infty u(\lambda) d\lambda/\lambda &= \frac{\Gamma(2m)}{(\Gamma(m))^2} \int_0^\infty \lambda^{m-1} (A(\lambda + A)^{-1})^{2m} A^{-m} x \\ &= x\ . \end{aligned}$$

Therefore, x belongs to $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$.

Conversely, let $x \in S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ so that x is represented by integral (3.1) with an integrand satisfying (3.4). Then

$$\lambda^\sigma (A(\lambda + A)^{-1})^m x = (A(\lambda + A)^{-1})^m \lambda^\sigma \int_\lambda^\infty \mu^{-\sigma} \mu^\sigma u(\lambda) d\mu/\mu$$
$$+ (\lambda(\lambda + A)^{-1})^m \lambda^{\sigma - m} \int_0^\lambda \mu^{m-\sigma} \mu^{\sigma-m} A^m u(\lambda) d\mu/\mu .$$

Since both $(A(\lambda + A)^{-1})^m$ and $(\lambda(\lambda + A)^{-1})^m$ are uniformly bounded, $\lambda^\sigma (A(\lambda + A)^{-1})^m x$ belongs to $L^p(X)$, that is, $x \in D_p^\sigma$.

THEOREM 3.2. *Let A be an operator of type $(\omega, M(\theta))$. Then*

$$D_p^\sigma(A^\alpha) = D_p^{\sigma\alpha}(A) , \qquad 0 < \alpha < \pi/\omega, \sigma > 0 .$$

Proof. It is sufficient to prove it in the case $0 < \alpha < 1$, because otherwise we have $A = (A^\alpha)^{1/\alpha}$ with $0 < 1/\alpha < 1$ (see (2.6)). In view of Theorem 2.6 we may also assume that σ is sufficiently small.

By [2] Proposition 10.2 we have

$$\lambda^\sigma A^\alpha (\lambda + A^\alpha)^{-1} x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^{\sigma+1} \tau^{\alpha - \alpha\sigma}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} \tau^{\alpha\sigma} A(\tau + A)^{-1} x d\tau/\tau .$$

Since the kernel

$$\frac{(\lambda^{-1}\tau^\alpha)^{1-\sigma}}{1 + 2(\lambda^{-1}\tau^\alpha) \cos \pi\alpha + (\lambda^{-1}\tau^\alpha)^2} , \qquad 0 < \sigma < 1 ,$$

defines a bounded integral operator in $L^p(X)$, $D_p^{\sigma\alpha}(A)$ is contained in $D_p^\sigma(A^\alpha)$.

If $\alpha = 1/m$ with an odd integer m, we have conversely

$$D_p^\sigma(A^{1/m}) \subset D_p^{\sigma/m}(A) .$$

In fact, let $x \in D_p^\sigma(A^{1/m})$. Since

$$\lambda^\sigma A(\lambda^m + A)^{-1} = \lambda^\sigma \prod_{i=1}^m (A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}) x ,$$

where ε_i are roots of $(-\varepsilon)^m = -1$ with $\varepsilon_1 = 1$, and since

$$A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}, \qquad i = 2, \cdots, m,$$

are uniformly bounded, $\lambda^\sigma A(\lambda^m + A)^{-1} x \in L^p(X)$. Changing the variable by $\lambda' = \lambda^m$, we get $\lambda^{\sigma/m} A(\lambda + A)^{-1} x \in L^p(X)$.

In a general case choose an odd number m such that $0 < 1/m < \alpha$. Since $A^{1/m} = (A^\alpha)^{1/(\alpha m)}$, we have

$$D_p^{\alpha\sigma}(A) \subset D_p^\sigma(A^\alpha) \subset D_p^{\alpha\sigma m}(A^{1/m}) \subset D_p^{\alpha\sigma}(A) .$$

Another less computational proof will be obtained from the Lions-Peetre theory and Proposition 2.8.

4. Infinitesimal generators of bounded semi-groups. Throughout this section we assume that T_t, $t \geqq 0$, is a bounded strongly continuous semi-group of operators in X and $-A$ is its infinitesimal generator:

$$T_t = \exp(-tA), \qquad \| T_t \| \leqq M . \tag{4.1}$$

A is an operator of type $(\pi/2, M(\theta))$.

DEFINITION 4.1. Let $0 < \sigma < m$, where σ is a real number and m an integer, and let $1 \leqq p \leqq \infty$. We denote by $C^\sigma_{p,m} = C^\sigma_{p,m}(A)$ the set of all elements $x \in X$ such that

$$t^{-\sigma}(I - T_t)^m x \in L^p(X) . \tag{4.2}$$

As is easily seen, $C^\sigma_{p,m}$ is a Banach space with the norm

$$\| x \|_{C^\sigma_{p,m}} = \| x \| + \| t^{-\sigma}(I - T_t)^m x \|_{L^p(X)} .$$

Since $(I - T_t)^m$ is uniformly bounded, condition (4.2) is equivalent to that $t^{-\sigma}(I - T_t)^m x$ belongs to $L^p(X)$ near the origin. In particular, we have

$$C^\sigma_{p,m}(A) = C^\sigma_{p,m}(\mu + A), \ \mu > 0 . \tag{4.3}$$

$C^\sigma_{\infty,1}$ and $C^\sigma_{\infty-,1}$ coincide with C^σ and C^σ_* of [2], respectively, and $C^\sigma_{\infty,1}$ consists of all elements x such that $T_t x$ is (weakly) uniformly Hölder continuous with exponent σ.

PROPOSITION 4.2. If $x \in C^\sigma_{p,m}$, then x belongs to $D(A^\alpha)$ for all $0 < \operatorname{Re} \alpha < \sigma$, and

$$A^\alpha x = \frac{1}{K_{\alpha,m}} \int_0^\infty t^{-\alpha-1}(I - T_t)^m x dt, \qquad 0 < \operatorname{Re} \alpha < \sigma , \tag{4.4}$$

where

$$K_{\alpha,m} = \int_0^\infty t^{-\alpha-1}(1 - e^{-t})^m dt .$$

Proof. If $0 < \operatorname{Re} \alpha < \sigma$, the right-hand side of (4.4) converges absolutely and represents an analytic function of α.

If $x \in D(A)$, then we have by [2] Proposition 11.4

$$\int_0^\infty t^{-\alpha-1}(I - T_t)^m x dt$$

$$= \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k}\int_0^\infty t^{-\alpha-1}(I - T_{kt})x dt$$

$$= \Gamma(-\alpha) \sum_{k=1}^{m} (-1)^{k+1}\binom{m}{k} k^\alpha A^\alpha x, \qquad 0 < \operatorname{Re}\alpha < 1 .$$

The coefficient of $A^\alpha x$ does not depend on A. Taking $A = 1$, we see that it is equal to $K_{\alpha,m}$.

Next let $0 < \operatorname{Re}\alpha < \min(\sigma, 1)$ and $x \in C^\sigma_{p,m}$. Then integral (4.4) with x replaced by $\mu(\mu + A)^{-1}x$, $\mu > 0$, exists and converges to the integral (4.4) as $\mu \to \infty$. Thus $A^\alpha \mu(\mu + A)^{-1}x$ converges to the integral (4.4). Since A^α is closed and $\mu(\mu + A)^{-1}x \to x$ as $\mu \to \infty$, it follows that $x \in D(A^\alpha)$ and (4.4) holds.

In the general case the assertion is obtained by [2], Proposition 8.4 or by repeating an argument as above.

Lions and Peetre [4] gave another proof when α is an integer.

THEOREM 4.3. *$C^\sigma_{p,m}$ coincides with D^σ_p with equivalent norms.*

Proof. First we note that

$$(I - T_t)x = AI_t x , \qquad x \in X , \tag{4.5}$$

where

$$I_t x = \int_0^t T_s x ds . \tag{4.6}$$

Obviously we have

$$\| I_t \| \leqq M_t , \qquad t > 0 . \tag{4.7}$$

Let $x \in C^\sigma_{p\ m}$. Then $(\lambda + A)^{-m}x$, $\lambda > 0$, belongs to $C^{\sigma+m}_{p,2m}$ since

$$t^{-\sigma-m} \| (I - T_t)^{2m}(\lambda + A)^{-m}x \|$$
$$\leqq t^{-m} \| I_t^m \| \, \| (A(\lambda + A)^{-1})^m \| \, t^{-\sigma} \| (I - T_t)^m x \| .$$

Hence we have by Proposition 4.2

$$(A(\lambda + A)^{-1})^m x = c\int_0^\infty t^{-m-1}(I - T_t)^{2m}(\lambda + A)^{-m}x$$
$$= c\int_0^{1/\lambda} (A(\lambda + A)^{-1})^m t^{-m-1} I_t^m (I - T_t)^m x dt$$
$$+ c\int_{1/\lambda}^\infty (\lambda + A)^{-m} t^{-m-1}(I - T_t)^{2m} x dt ,$$

where $c = K^{-1}_{m,2m}$. Therefore,

$$\lambda^\sigma \| (A(\lambda + A)^{-1})^m x \| \leqq cL^m M^m \lambda^\sigma \int_0^{1/\lambda} t^\sigma t^{-\sigma} \| (I - T_t)^m x \| dt/t$$
$$+ cM^m (2M)^m \lambda^{\sigma-m} \int_{1/\lambda}^\infty t^{\sigma-m} t^{-\sigma} \| (I - T_t)^m x \| dt/t .$$

This shows that $x \in D^{\sigma}_{p,m}$.

Conversely, let $x \in D^{\sigma}_{p,m}$. Since

$$(A(\lambda + A)^{-1})^{2m} I_t^m x = (\lambda + A)^{-m}(I - T_t)^m (A(\lambda + A)^{-1})^m x ,$$

it follows that $I_t^m x \in D^{\sigma+m}_{p,2m}$. Thus by Proposition 2.2 we get

$$\begin{aligned}(I - T_t)^m x = A^m I_t^m x &= c\int_0^\infty \lambda^{m-1}(A(\lambda + A)^{-1})^{2m} I_t^m x \\ &= c\int_0^{1/t} I_t^m \lambda^{m-1}(A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + c\int_{1/t}^\infty (I - T_t)^m \lambda^{m-1}(\lambda + A)^{-m}(A(\lambda + A)^{-1})^m x d\lambda ,\end{aligned}$$

where $c = \Gamma(2m)/(\Gamma(m))^2$. By the same computation as above we conclude that $x \in C^{\sigma}_{p,m}$.

In particular, $C^{\sigma}_{p,m}$ does not depend on m. We denote $C^{\sigma}_{p,m}$ with the least $m > \sigma$ by C^{σ}_{p}. Because of Theorem 2.6, C^{σ}_{∞} coincides with C^{σ} of [2] if σ is not an integer.

THEOREM 4.4. *Let $0 < \operatorname{Re}\alpha < m$. If there is a sequence $\varepsilon_j \to 0$ such that*

$$y = w\text{-}\lim_{j\to\infty} \frac{1}{K_{\alpha,m}} \int_{\varepsilon_j}^\infty t^{-\alpha-1}(I - T_t)^m x dt \tag{4.8}$$

exists, then $x \in D(A^\alpha)$ and $y = A^\alpha x$.

Conversely, if $x \in D(A^\alpha)$, then

$$A^\alpha x = s\text{-}\lim_{\varepsilon\to 0} \frac{1}{K_{\alpha,m}} \int_{\varepsilon}^\infty t^{-\alpha-1}(I - T_t)^m x dt . \tag{4.9}$$

Proof. The former part is proved in the same way as Theorem 2.10.

To prove the latter part, let us assume for a moment that T_t satisfies

$$\| T_t \| \leqq M e^{-\mu t}, \qquad t > 0 ,$$

for a $\mu > 0$. Then A^α is the inverse of $A^{-\alpha}$ which can be represented by the absolutely convergent integral

$$A^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} T_s x ds \tag{4.10}$$

([2], Theorem 7.3 and Proposition 11.1).

Now it is enough to prove that

$$\frac{1}{K_{\alpha,m}\Gamma(\alpha)} \int_\varepsilon^\infty t^{-\alpha-1}(I - T_t)^m dt \int_0^\infty s^{\alpha-1} T_s x ds$$

converges strongly as $\varepsilon \to 0$, because the limit must coincide with $A^\alpha A^{-\alpha}x = x$.

We have

$$I_\varepsilon = \int_\varepsilon^\infty t^{-\alpha-1}(I - T_t)^m dt \int_0^\infty s^{\alpha-1} T_s x ds$$
$$= \sum_{k=1}^m (-1)^{k+1}\binom{m}{k} k^\alpha \int_{k\varepsilon}^\infty t^{-\alpha-1}(I - T_t)dt \int_0^\infty s^{\alpha-1} T_s x ds .$$

Now

$$\int_{k\varepsilon}^\infty t^{-\alpha-1} T_t dt \int_0^\infty s^{\alpha-1} T_s x ds$$
$$= \int_{k\varepsilon}^\infty t^{-\alpha-1} dt \int_t^\infty (s - t)^{\alpha-1} T_s x ds$$
$$= \int_{k\varepsilon}^\infty T_s x ds \int_{k\varepsilon}^s t^{-\alpha-1}(s - t)^{\alpha-1} dt$$
$$= \frac{1}{\alpha(k\varepsilon)^\alpha} \int_{k\varepsilon}^\infty (s - k\varepsilon)^\alpha T_s x ds/s .$$

Furthermore,

$$\sum_{k=1}^m (-1)^{k+1}\binom{m}{k} k^\alpha \int_{k\varepsilon}^\infty t^{-\alpha-1} dt \int_0^\infty s^{\alpha-1} T_s x ds$$
$$= \frac{1}{\alpha\varepsilon^\alpha} \int_0^\infty s^{\alpha-1} T_s x ds ,$$

so that we obtain

$$I_\varepsilon = \frac{1}{\alpha\varepsilon^\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_{k\varepsilon}^\infty (s - k\varepsilon)^\alpha T_s x ds/s .$$

Since $T_s x \to x$ as $s \to 0$, it follows that

$$\frac{1}{\alpha\varepsilon^\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_{k\varepsilon}^{m\varepsilon} (s - k\varepsilon)^\alpha T_s x ds/s$$

$$= \frac{1}{\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_k^m (s - k)^\alpha T_{\varepsilon s} x ds/s$$
$$\to \frac{1}{\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_k^m (s - k)^\alpha ds/s x \text{ as } \varepsilon \to 0 .$$

On the other hand, the Taylor expansion up to order m gives

$$f_\varepsilon(s) = \sum_{k=0}^m (-1)^k \binom{m}{k} (s - k\varepsilon)^\alpha$$
$$= \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\alpha(\alpha - 1) \cdots (\alpha - m + 1)}{m!} (s - k'\varepsilon)^{\alpha-m} (-k\varepsilon)^m ,$$

where $0 < k' < k$. Hence we have

$$\frac{1}{\alpha\varepsilon^{\alpha}}\int_{m\varepsilon}^{\infty} f_{\varepsilon}(s) T_s x ds/s$$
$$= \frac{(\alpha - 1)\cdots(\alpha - m + 1)}{m!}\sum_{k=0}^{m}(-1)^{k+m}\binom{m}{k}k^m\int_{m}^{\infty}(s - k')^{\alpha-m}T_{\varepsilon s}x ds/s .$$

Since $(s - k')^{\alpha-m}s^{-1}$ is absolutely integrable, this converges to a constant times x as $\varepsilon \to 0$.

To prove (4.9) in the general case, it is sufficient to show that

$$(4.11)\qquad (A^{\alpha} - (\mu + A)^{\alpha})(\mu + A)^{-\alpha}x$$
$$= \frac{1}{K_{\alpha,m}}\int_0^{\infty} t^{-\alpha-1}\{(I - T_t)^m - (I - e^{-\mu t}T_t)^m\}(\mu + A)^{-\alpha}x dt ,$$
$$\mu > 0,\ x \in X ,$$

and that the integral converges absolutely.

By Theorem 2.6, (4.5) and a similar decomposition of $I - e^{-\mu t}T_t$ we have

$$(I - T_t)^m(I - e^{-\mu t}T_t)^n x = O(t^{\sigma}),\ x \in C_{\infty}^{\sigma},\ m + n > \sigma .$$

Since $(\mu + A)^{-\alpha}x \in D(A^{\alpha}) \subset C_{\infty}^{\mathrm{Re}\,\alpha}$, it follows that

$$\{(I - T_t)^m - (I - e^{-\mu t}T_t)^m\}x$$
$$= (e^{-\mu t} - 1)T_t\{(I - T_t)^{m-1} + \cdots + (I - e^{-\mu t}T_t)^{m-1}\}x$$
$$= O(t^{\min(\mathrm{Re}\,\alpha, m-1)+1}) .$$

This shows that integral (4.11) is absolutely convergent. (4.11) is valid for all $x \in D(A)$ which is dense in X. Therefore, (4.11) holds for all $x \in X$.

5. Infinitesimal generators of bounded analytic semi-groups. Let T_t be a semi-group of operators analytic in a sector $|\arg t| < \pi/2 - \omega$, $0 \leqq \omega < \pi/2$, and uniformly bounded in each smaller sector $|\arg t| \leqq \pi/2 - \omega - \varepsilon$, $\varepsilon > 0$. We call such a semi-group a bounded analytic semi-group.

It is known that the negative of an operator A generates a bounded analytic semi-group if and only if A is of type $(\omega, M(\theta))$ for some $0 \leqq \omega < \pi/2$. A bounded strongly continuous semi-group T_t has a bounded analytic extension if there is a complex number $\mathrm{Re}\,\alpha > 0$ such that

$$(5.1)\qquad \|A^{\alpha}T_t\| \leqq Ct^{-\mathrm{Re}\,\alpha},\ t > 0,$$

with a constant C independent of t. Conversely, if T_t is bounded analytic, (5.1) holds for all $\operatorname{Re}\alpha > 0$ ([2], Theorems 12.1 and 12.2).

We assume throughout this section that $-A$ is the infinitesimal generator of a bounded analytic semi-group T_t.

DEFINITION 5.1. Let $0 < \sigma < \operatorname{Re}\beta$ and $1 \leqq p \leqq \infty$. We denote by $B^\sigma_{p,\beta} = B^\sigma_{p,\beta}(A)$ the set of all $x \in X$ such that

$$t^{\operatorname{Re}\beta-\sigma}A^\beta T_t x \in L^p(X). \tag{5.2}$$

$B^\sigma_{p,\beta}$ is a Banach space with the norm

$$\| x \|_{B^\sigma_{p,\beta}} = \| x \| + \| t^{\operatorname{Re}\beta-\sigma}A^\beta T_t x \|_{L^p(X)} .$$

PROPOSITION 5.2. Let $0 < \operatorname{Re}\alpha < \sigma$. Then every $x \in B^\sigma_{p,\beta}$ belongs to $D(A^\alpha)$ and

$$A^\alpha x = \frac{1}{\Gamma(\beta-\alpha)}\int_0^\infty t^{\beta-\alpha-1}A^\beta T_t x dt , \tag{5.3}$$

where the integral converges absolutely.

Proof. Since $A^\beta T_t x$ is of order $t^{\sigma-\operatorname{Re}\beta}$ as $t \to 0$ and of order $t^{-\operatorname{Re}\beta+\varepsilon}$ as $t \to \infty$ in the sense of $L^p(X)$, the integral converges absolutely for $0 < \operatorname{Re}\alpha < \sigma$.

To prove (5.3), first let $x \in D(A^\beta)$. Then it follows from [2], Proposition 11.1 and Theorem 7.3 that

$$\begin{aligned}
&\frac{1}{\Gamma(\beta-\alpha)}\int_0^\infty t^{\beta-\alpha-1}A^\beta T_t x dt \\
&= s\text{-}\lim_{\varepsilon\to 0}\frac{1}{\Gamma(\beta-\alpha)}\int_0^\infty t^{\beta-\alpha-1}e^{-\varepsilon t}T_t A^\beta x dt \\
&= s\text{-}\lim_{\varepsilon\to 0}(\varepsilon + A)^{\alpha-\beta}A^\beta x \\
&= s\text{-}\lim_{\varepsilon\to 0} A^{\beta-\alpha}(\varepsilon + A)^{\alpha-\beta}A^\alpha x .
\end{aligned}$$

Because of [2], Propositions 6.2 and 6.3, $A^{\beta-\alpha}(\varepsilon+A)^{\alpha-\beta}$ converges strongly to the identity on $\overline{R(A)}$ as $\varepsilon \to 0$. Since $A^\alpha X$ is contained in $\overline{R(A)}$ ([2], Proposition 4.3), (5.3) holds for all $x \in D(A^\beta)$. In the general case (5.3) is proved by approximating $x \in B^\sigma_{p,\beta}$ by $(\mu(\mu+A)^{-1})^m x$, $m > \operatorname{Re}\beta$, which belongs to $D(A^\beta)$.

THEOREM 5.3. *$B^\sigma_{p,\beta}$ coincides with D^σ_p. In particular, $B^\sigma_{p,\beta}$ does not depend on β.*

Proof. Let $x \in B^\sigma_{p,\beta}$. If m is an integer greater than $\operatorname{Re}\beta$, x belongs to $B^\sigma_{p,m}$, for

$$t^{m-\sigma}A^m T_t x = t^{m-\beta}A^{m-\beta}T_{t/2} \cdot t^{\beta-\sigma}A^\beta T_{t/2}x$$

and $t^{m-\beta}A^{m-\beta}T_{t/2}$ is uniformly bounded. Since

$$t^{m-\sigma}A^{2m}T_t(\lambda + A)^{-m}x = (A(\lambda + A)^{-1})^m t^{m-\sigma}A^m T_t x ,$$

$(\lambda + A)^{-m}x$ belongs to $B^{\sigma+m}_{p,2m}$. Hence it follows from Proposition 5.2 that

$$\begin{aligned} A^m(\lambda + A)^{-m}x &= c\int_0^\infty t^m A^{2m} T_t(\lambda + A)^{-m}x dt/t \\ &= c(A(\lambda + A)^{-1})^m \int_0^{1/\lambda} t^m A^m T_t x dt/t \\ &\quad + c(\lambda + A)^{-m}\int_{1/\lambda}^\infty t^m A^{2m} T_t x dt/t , \end{aligned}$$

where $c = \Gamma(m)^{-1}$. The rest of the proof is the same as that of Theorem 4.3.

Conversely, assume that $x \in D^\sigma_{p,m} = D^\sigma_{p,2m}$. Since $T_t x$, $t > 0$, belongs to any $D^\sigma_{p,m}$, we have by (2.1)

$$\begin{aligned} A^\beta T_t x &= c\int_0^\infty \lambda^{\beta-1}(A(\lambda + A)^{-1})^{2m} T_t x d\lambda \\ &= c\, T_t \int_0^{1/t} \lambda^{\beta-1}(A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + cA^m T_t \int_{1/t}^\infty \lambda^{\beta-1}(\lambda + A)^{-m}(A(\lambda + A)^{-1})^m x d\lambda , \end{aligned}$$

where $c = \Gamma(2m)/(\Gamma(\beta)\Gamma(2m - \beta))$. Arguing as before, we get $x \in B^\sigma_{p,\beta}$.

THEOREM 5.4 *Let* $0 < \operatorname{Re}\alpha < \operatorname{Re}\beta$. *If*

$$y = w\text{-}\lim_{\varepsilon_j \to 0} \frac{1}{\Gamma(\beta - \alpha)}\int_{\varepsilon_j}^\infty t^{\beta-\alpha-1}A^\beta T_t x dt \tag{5.4}$$

exists, then $x \in D(A^\alpha)$ *and* $y = A^\alpha x$. *If* $x \in D(A^\alpha)$, *then*

$$A^\alpha x = s\text{-}\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\beta - \alpha)}\int_\varepsilon^\infty t^{\beta-\alpha-1}A^\beta T_t x dt . \tag{5.5}$$

Proof. The former part is proved in the same way as Theorem 2.10. Let us prove the latter assuming that $\mu - A$ generates a bounded analytic semi-group for a $\mu > 0$. $D(A^\alpha)$ is the same as the range $R(A^{-\alpha})$ in this case, and we have $A^\beta T_t A^{-\alpha}x = A^{\beta-\alpha}T_t x$ by the additivity of fractional powers. So it is sufficient to prove the following:

$$x = s\text{-}\lim_{\varepsilon \to 0} \frac{1}{\Gamma(\beta)}\int_\varepsilon^\infty t^{\beta-1}A^\beta T_t x dt, \qquad x \in X , \tag{5.6}$$

when $\operatorname{Re}\beta > 0$.

First we note that if $\operatorname{Re}\alpha > 0$, then

$$t^{\alpha}A^{\alpha}T_t x \longrightarrow 0 \text{ as } t \longrightarrow 0 \text{ or as } t \longrightarrow \infty \tag{5.7}$$

for each $x \in X$, because (5.7) holds for $x \in D(A)$ and $t^{\alpha}A^{\alpha}T_t$ is uniformly bounded.

Let β be equal to an integer m. Since $d/dtA^{\beta}T_t x = -A^{\beta+1}T_t x$, we have, by integrating by parts,

$$\begin{aligned}&\int_{\varepsilon}^{\infty} t^{m-1}A^m T_t x dt\\ &\quad = \varepsilon^{m-1}A^{m-1}T_{\varepsilon}x + (m-1)\int_{\varepsilon}^{\infty} t^{m-2}A^{m-1}T_t x dt\,.\end{aligned}$$

(5.7) shows that the first term tends to zero as $\varepsilon \longrightarrow 0$ if $m > 1$. When $m = 1$, we have

$$\int_{\varepsilon}^{\infty} AT_t x dt = T_{\varepsilon}x \longrightarrow x \text{ as } \varepsilon \longrightarrow 0\,.$$

Thus (5.6) holds if β is an integer.

If β is not an integer, take an integer $m > \operatorname{Re}\beta$. We have

$$\begin{aligned}A^{\beta}T_t x &= A^{\beta-m}A^m T_t x\\ &= \frac{1}{\Gamma(m-\beta)}\int_t^{\infty}(s-t)^{m-\beta-1}A^m T_s x ds, \qquad t > 0\,,\end{aligned}$$

by [2], Proposition 11.1. Therefore,

$$\begin{aligned}&\frac{1}{\Gamma(\beta)}\int_{\varepsilon}^{\infty} t^{\beta-1}A^{\beta}T_t x dt\\ &= \frac{1}{\Gamma(\beta)\Gamma(m-\beta)}\int_{\varepsilon}^{\infty} A^m T_s x ds\int_{\varepsilon}^{s} t^{\beta-1}(s-t)^{m-\beta-1}dt\\ &= \frac{1}{\Gamma(m)}\int_{\varepsilon}^{\infty} s^{m-1}A^m T_s x ds\\ &\quad - \frac{\varepsilon^m}{\Gamma(\beta)\Gamma(m-\beta)}\int_1^{\infty} A^m T_{\varepsilon\sigma}x d\sigma\int_0^1 \tau^{\beta-1}(\sigma-\tau)^{m-\beta-1}d\tau\,.\end{aligned}$$

The first term tends to x as $\varepsilon \longrightarrow 0$. The second term converges to zero, because

$$\int_1^{\infty}\sigma^{-m}d\sigma\int_0^1 \tau^{\beta-1}(\sigma-\tau)^{m-\beta-1}d\tau$$

is absolutely convergent and $(\varepsilon\sigma)^m A^m T_{\varepsilon\sigma}x$ tends to zero as $\varepsilon \longrightarrow 0$.

The proof in the general case is obtained from the absolutely convergent integral representation:

$$(A^\alpha - (\mu + A)^\alpha)(\mu + A)^{-\alpha} x$$
$$= \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{\beta-\alpha-1} (A^\beta - e^{-\mu t} (\mu + A)^\beta) T_t (\mu + A)^{-\alpha} x dt \ .$$

The absolute convergence follows from [2], Propositions 6.2 and 6.3.

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Received November 19, 1965.

STANFORD UNIVERSITY
AND
UNIVERSITY OF TOKYO

PACIFIC JOURNAL OF MATHEMATICS

Printed in Japan by International Academic Printing Co., Ltd., Tokyo Japan

Pacific Journal of Mathematics
Vol. 21, No. 1 November, 1967