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## **FRACTIONAL POWERS OF OPERATORS. II. INTERPOLATION SPACES**

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## FRACTIONAL POWERS OF OPERATORS, II INTERPOLATION SPACES

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**This is a continuation of an earlier paper "Fractional Powers of Operators" published in this Journal concerning fractional powers  $A^\alpha$ ,  $\alpha \in \mathbb{C}$ , of closed linear operators  $A$  in Banach spaces  $X$  such that the resolvent  $(\lambda + A)^{-1}$  exists for all  $\lambda > 0$  and  $\lambda(\lambda + A)^{-1}$  is uniformly bounded. Various integral representations of fractional powers and relationship between fractional powers and interpolation spaces, due to Lions and others, of  $X$  and domain  $D(A^\alpha)$  are investigated.**

In §1 we define the space  $D_p^\sigma(A)$ ,  $0 < \sigma < \infty$ ,  $1 \leq p \leq \infty$  or  $p = \infty -$ , as the set of all  $x \in X$  such that

$$\lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X),$$

where  $m$  is an integer greater than  $\sigma$  and  $L^p(X)$  is the  $L^p$  space of  $X$ -valued functions with respect to the measure  $d\lambda/\lambda$  over  $(0, \infty)$ .

In §2 we give a new definition of fractional power  $A^\alpha$  for  $\operatorname{Re} \alpha > 0$  and prove the coincidence with the definition given in [2]. Convexity of  $\|A^\alpha x\|$  is shown to be an immediate consequence of the definition. The main result of the section is Theorem 2.6 which says that if  $0 < \operatorname{Re} \alpha < \sigma$ ,  $x \in D_p^\sigma$  is equivalent to  $A^\alpha x \in D_p^{\sigma - \operatorname{Re} \alpha}$ . In particular, we have  $D_1^{\operatorname{Re} \alpha} \subset D(A^\alpha) \subset D_\infty^{\operatorname{Re} \alpha}$ . For the application of fractional powers it is important to know whether the domain  $D(A^\alpha)$  coincides with  $D_p^{\operatorname{Re} \alpha}$  for some  $p$ . We see, as a consequence of Theorem 2.6, that if we have  $D(A^\alpha) = D_p^{\operatorname{Re} \alpha}$  for an  $\alpha$ , it holds for all  $\operatorname{Re} \alpha > 0$ . An example and a counterexample are given. At the end of the section we prove an integral representation of fractional powers.

Section 3 is devoted to the proof of the coincidence of  $D_p^\sigma$  with the interpolation space  $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$  due to Lions-Peetre [4]. We also give a direct proof of the fact that  $D_p^\sigma(A^\alpha) = D_p^{\alpha\sigma}(A)$ .

In §4 we discuss the case in which  $-A$  is the infinitesimal generator of a bounded strongly continuous semi-group  $T_t$ . A new space  $C_{p,m}^\sigma$  is introduced in terms of  $T_t x$  and its coincidence with  $D_p^\sigma$  is shown. Since  $C_{\infty,m}^\sigma$ ,  $\sigma \neq$  integer, coincides with  $C^\sigma$  of [2], this solves a question of [2] whether  $C^\sigma = D^\sigma$  or not affirmatively. The coincidence of  $C_{p,m}^\sigma$  with  $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$  has been shown by Lions-Peetre [4]. Further, another integral representation of fractional powers is obtained.

Finally, § 5 deals with the case in which  $-A$  is the infinitesimal generator of a bounded analytic semi-group  $T_t$ . Analogous results to § 4 are obtained in terms of  $A^\beta T_t x$ .

1. Spaces  $D_p^\sigma$ . Throughout this paper we assume that  $A$  is a closed linear operator with a dense domain  $D(A)$  in a Banach space  $X$  and satisfies

$$(1.1) \quad \|\lambda(\lambda + A)^{-1}\| \leq M, \quad 0 < \lambda < \infty.$$

We defined fractional powers in [2] for operators  $A$  which may not have dense domains. It was shown, however, that if  $\operatorname{Re} \alpha > 0$ ,  $A^\alpha$  is an operator in  $\overline{D(A)}$  and it is determined by a restriction  $A_D$  which has a dense domain in  $\overline{D(A)}$ . Thus our requirement on domain  $D(A)$  is not restrictive as far as we consider exponent  $\alpha$  with positive real part. As a consequence we have

$$(1.2) \quad (\lambda(\lambda + A)^{-1})^m x \rightarrow x, \quad \lambda \rightarrow \infty, m = 1, 2, \dots$$

for all  $x \in X$ . As in [2]  $L$  stands for a bound of  $A(\lambda + A)^{-1} = I - \lambda(\lambda + A)^{-1}$ :

$$(1.3) \quad \|A(\lambda + A)^{-1}\| \leq L, \quad 0 < \lambda < \infty.$$

We will frequently make use of spaces of  $X$ -valued functions  $f(\lambda)$  defined on  $(0, \infty)$ . By  $L^p(X)$  we denote the space of all  $X$ -valued measurable functions  $f(\lambda)$  such that

$$(1.4) \quad \begin{aligned} \|f\|_{L^p} &= \left( \int_0^\infty \|f(\lambda)\|^p d\lambda/\lambda \right)^{1/p} < \infty \text{ if } 1 \leq p < \infty \\ \|f\|_{L^\infty} &= \sup_{0 < \lambda < \infty} \|f(\lambda)\| < \infty \text{ if } p = \infty. \end{aligned}$$

We admit as an index  $p = \infty -$ .  $L^{\infty-}(X)$  represents the subspace of all functions  $f(\lambda) \in L^\infty(X)$  which converge to zero as  $\lambda \rightarrow 0$  and as  $\lambda \rightarrow \infty$ . Since  $d\lambda/\lambda$  is a Haar measure of the multiplicative group  $(0, \infty)$ , an integral kernel  $K(\lambda/\mu)$  with  $\int_0^\infty |K(\lambda)| d\lambda/\lambda < \infty$  defines a bounded integral operator in  $L^p(X)$ ,  $1 \leq p \leq \infty$ .

DEFINITION 1.1. Let  $0 < \sigma < m$ , where  $\sigma$  is a real number and  $m$  an integer, and  $p$  be as above. We denote by  $D_{p,m}^\sigma = D_{p,m}^\sigma(A)$  the space of all  $x \in X$  such that  $\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X)$  with the norm

$$(1.5) \quad \|x\|_{D_{p,m}^\sigma} = \|x\|_X + \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|_{L^p(X)}.$$

$D_{\infty,1}^\sigma$  and  $D_{\infty-,1}^\sigma$  coincide with  $D^\sigma$  and  $D_*^\sigma$  of [2], respectively.

It is easy to see that  $D_{p,m}^\sigma$  is a Banach space. Since  $(A(\lambda + A)^{-1})^m$  is uniformly bounded, only the behavior near infinity of  $(A(\lambda + A)^{-1})^m x$

decides whether  $x$  belongs to  $D_{p,m}^\sigma$  or not.

**PROPOSITION 1.2.** If integers  $m$  and  $n$  are greater than  $\sigma$ , the spaces  $D_{p,m}^\sigma$  and  $D_{p,n}^\sigma$  are identical and have equivalent norms.

*Proof.* It is enough to show that  $D_{p,m}^\sigma = D_{p,m+1}^\sigma$  when  $m > \sigma$ . Because of (1.3) every  $x \in D_{p,m}^\sigma$  belongs to  $D_{p,m+1}^\sigma$ . Since

$$\frac{d}{d\lambda}(\lambda^m(A(\lambda + A)^{-1})^m) = m\lambda^{m-1}(A(\lambda + A)^{-1})^{m+1},$$

we have

$$(1.6) \quad \lambda^\sigma(A(\lambda + A)^{-1})^m x = m\lambda^{\sigma-m} \int_0^\lambda \mu^{m-\sigma} \mu^\sigma(A(\mu + A)^{-1})^{m+1} x d\mu / \mu.$$

This shows

$$\|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|_{L^p(X)} \leq \frac{m}{m - \sigma} \|\lambda^\sigma(A(\lambda + A)^{-1})^{m+1} x\|_{L^p(X)}.$$

**DEFINITION 1.3.** We define  $D_p^\sigma$ ,  $\sigma > 0$ ,  $1 \leq p \leq \infty$ , as the space  $D_{p,m}^\sigma$  with the least integer  $m$  greater than  $\sigma$ . We use  $q_p^\sigma(x)$  to denote the second term of (1.5), so that  $D_p^\sigma$  is a Banach space with the norm  $\|x\| + q_p^\sigma(x)$ .

**PROPOSITION 1.4.** If  $\mu > 0$ ,  $\mu(\mu + A)^{-1}$  maps  $D_p^\sigma$  continuously into  $D_p^{\sigma+1}$ . Furthermore, if  $p \leq \infty -$ , we have for every  $x \in D_p^\sigma$

$$(1.7) \quad \mu(\mu + A)^{-1} x \rightarrow x \quad (D_p^\sigma) \quad \text{as } \mu \rightarrow \infty.$$

*Proof.* Let  $x \in D_p^\sigma$ . Since

$$\begin{aligned} & \|\lambda^{\sigma+1}(A(\lambda + A)^{-1})^{m+1} \mu(\mu + A)^{-1} x\| \\ & \leq \mu \|\lambda(\lambda + A)^{-1}\| \|A(\mu + A)^{-1}\| \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\| \\ & \leq \mu ML \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|, \end{aligned}$$

$\mu(\mu + A)^{-1} x$  belongs to  $D_p^{\sigma+1}$ .

Let  $p \leq \infty -$ . If  $x \in D(A)$ , then

$$\begin{aligned} & (A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1} x \\ & = (A(\lambda + A)^{-1})^m x - (A(\lambda + A)^{-1})^m (\mu + A)^{-1} Ax \end{aligned}$$

converges to  $(A(\lambda + A)^{-1})^m x$  uniformly in  $\lambda$ . On the other hand,  $(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1}$  is uniformly bounded. Thus it follows that  $(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1} x$  converges to  $(A(\lambda + A)^{-1})^m x$  uniformly in  $\lambda$  for every  $x \in X$ . Since  $\|\lambda^\sigma(A(\lambda + A)^{-1})^m \mu(\mu + A)^{-1} x\| \leq M \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\|$ , this implies (1.7).

**THEOREM 1.5.**  $D_p^\sigma \subset D_q^\tau$  if  $\sigma > \tau$  or if  $\sigma = \tau$  and  $p \leq q$ . The injection is continuous. If  $q \leq \infty -$ ,  $D_p^\sigma$  is dense in  $D_q^\tau$ .

*Proof.* First we prove that  $D_p^\sigma$ ,  $p < \infty$ , is continuously contained in  $D_{\infty-}^\sigma$ .

Let  $x \in D_p^\sigma$ . Applying Hölder's inequality to (1.6), we obtain

$$\|\lambda^\sigma(A(\lambda + A)^{-1})^m x\| \leq \frac{m}{((m - \sigma)p')^{1/p'}} \|\mu^\sigma(A(\mu + A)^{-1})^{m+1} x\|_{L^p(x)},$$

where  $p' = p/(p - 1)$ . Hence  $x \in D_{\infty-}^\sigma$ . Considering the integral over the interval  $(\mu, \lambda)$ , we have similarly

$$\begin{aligned} \|\lambda^\sigma(A(\lambda + A)^{-1})^m x\| &\leq \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}} \|\mu^\sigma(A(\mu + A)^{-1})^m x\| \\ &+ \frac{m}{((m - \sigma)p')^{1/p'}} \left(1 - \frac{\mu^{m-\sigma}}{\lambda^{m-\sigma}}\right) \left(\int_\mu^\lambda \|\tau^\sigma(A(\tau + A)^{-1})^{m+1} x\|^p d\tau/\tau\right)^{1/p}. \end{aligned}$$

The second term tends to zero as  $\mu \rightarrow \infty$  uniformly in  $\lambda > \mu$  and so does the first term as  $\lambda \rightarrow \infty$ . Therefore,  $x \in D_{\infty-}^\sigma$ .

Since  $\lambda^\sigma(A(\lambda + A)^{-1})^m x \in L^p(X) \cap L^{\infty-}(X)$ , it is in any  $L^q(X)$  with  $p \leq q < \infty$ .

If  $\tau < \sigma$ ,  $D_{\infty-}^\sigma$  is contained in  $D_q^\tau$  for any  $q$ . Hence every  $D_q^\sigma$  is contained in  $D_q^\tau$ .

Let  $q \leq \infty -$ . Repeated application of Proposition 1.4 shows that  $D_q^{\tau+m}$  is dense in  $D_q^\tau$  for positive integer  $m$ . Since  $D_p^\sigma$  contains some  $D_q^{\tau+m}$ , it is dense in  $D_q^\tau$ .

**2. Fractional powers.** If  $x \in D_1^\sigma$ , the integral

$$(2.1) \quad A_\alpha^\sigma x = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m - \alpha)} \int_0^\infty \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda$$

converges absolutely for  $0 < \operatorname{Re} \alpha \leq \sigma$  and represents a continuous operator from  $D_1^\sigma$  into  $X$ . Moreover,  $A_\alpha^\sigma x$  is analytic in  $\alpha$  for  $0 < \operatorname{Re} \alpha < \sigma$ .

$A_\alpha^\sigma x$  does not depend on  $m$ . In fact, substitution of (1.6) into (2.1) gives

$$\begin{aligned} A_\alpha^\sigma x &= \frac{\Gamma(m)m}{\Gamma(\alpha)\Gamma(m - \alpha)} \int_0^\infty \mu^{m-1} (A(\mu + A)^{-1})^{m+1} x d\mu \int_\mu^\infty \lambda^{\alpha-m-1} d\lambda \\ &= \frac{\Gamma(m + 1)}{\Gamma(\alpha)\Gamma(m + 1 - \alpha)} \int_0^\infty \mu^{\alpha-1} (A(\mu + A)^{-1})^{m+1} x d\mu. \end{aligned}$$

This shows that  $A_\alpha^\sigma x$  depends only on  $x$  and not on  $D_1^\sigma$  to which  $x$  belongs.

Obviously we have

$$(2.2) \quad A_\sigma^\alpha(\mu(\mu + A)^{-1})^{m+1}x = (\mu(\mu + A)^{-1})^{m+1}A_\sigma^\alpha x, \quad x \in D_1^\alpha.$$

Since the left-hand side and  $(\mu(\mu + A)^{-1})^{m+1}$  are continuous in  $X$ , and  $(\mu(\mu + A)^{-1})^{m+1}$  is one-to-one, it follows that  $A_\sigma^\alpha$  is closable in  $X$ . In view of Theorem 1.5 the smallest closed extension does not depend on  $\sigma$ .

**DEFINITION 2.1.** The fractional power  $A^\alpha$  for  $\text{Re } \alpha > 0$  is the smallest closed extension of  $A_\sigma^\alpha$  for a  $\sigma \geq \text{Re } \alpha$ .

**PROPOSITION 2.2.** If  $\alpha$  is an integer  $m > 0$ ,  $A^\alpha$  coincides with the power  $A^m$ .

To prove the proposition we prepare a lemma.

**LEMMA 2.3.** *If  $m$  is an integer  $m > 0$ ,*

$$(2.3) \quad A^m x = \text{s-lim}_{N \rightarrow \infty} m \int_0^N \lambda^{m-1} (A(\lambda + A)^{-1})^{m+1} x d\lambda.$$

*Proof.* By (1.6) we have

$$m \int_0^N \lambda^{m-1} (A(\lambda + A)^{-1})^{m+1} x = N^m (A(N + A)^{-1})^m x.$$

If  $x \in D(A^m)$ ,  $N^m (A(N + A)^{-1})^m x = (N(N + A)^{-1})^m A^m x$  tends to  $A^m x$  as  $N \rightarrow \infty$  by (1.2). Conversely if  $N^m (A(N + A)^{-1})^m x = A^m (N(N + A)^{-1})^m x$  converges to an element  $y$ ,  $x \in D(A^m)$  and  $y = A^m x$ . For  $A^m$  is closed (see Taylor [5]) and  $(N(N + A)^{-1})^m x$  converges to  $x$ .

*Proof of Proposition 2.2.* If  $x \in D_1^\sigma$ ,  $\sigma > m$ , integral (2.3) converges absolutely. Therefore it follows from Lemma 2.3 that  $x \in D(A^m)$  and  $A^\alpha x = A^m x$ . Thus  $A^m$  is an extension of  $A^\alpha$ . Conversely if  $x \in D(A^m)$ , then  $\mu(\mu + A)^{-1} x \in D(A^{m+1}) \subset D_\infty^{m+1}$  and we have

$$\begin{aligned} A^\alpha(\mu(\mu + A)^{-1})x &= (\mu(\mu + A)^{-1})A^m x \\ &\rightarrow A^m x \quad \text{as } \mu \rightarrow \infty. \end{aligned}$$

Since  $\mu(\mu + A)^{-1} x \rightarrow x$ , it follows that  $x \in D(A^\alpha)$  and  $A^\alpha x = A^m x$ .

The fractional power  $A^\alpha$  defined above coincides with  $A_\dagger^\alpha$  defined in [2]. In fact, if  $m = 1$ , integral (2.1) is the same as integral (4.2) of [2] for  $n = 0$ . Thus

$$(2.4) \quad A^\alpha x = A_\dagger^\alpha x$$

holds for  $0 < \text{Re } \alpha < 1$  if  $x \in D(A)$ . If  $x \in D(A^m)$ ,  $m \geq 1$ , both sides of

(2.4) are analytic for  $0 < \operatorname{Re} \alpha < m$ , so that (2.4) holds there. Since  $D_1^m \subset D(A^m) \subset D_{\infty}^m$  by Lemma 2.3 and (1.2), both  $A^\alpha$  and  $A_+^\alpha$  are the smallest closed extension of their restrictions to  $D(A^m)$ ,  $m > \operatorname{Re} \alpha$ . Thus we have  $A^\alpha = A_+^\alpha$  for all  $\operatorname{Re} \alpha > 0$ .

Consequently we may employ all results of [2]. In particular, fractional powers satisfy additivity

$$(2.5) \quad A^{\alpha+\beta} = A^\alpha A^\beta, \quad \operatorname{Re} \alpha > 0, \operatorname{Re} \beta > 0$$

in the sense of product of operators and multiplicativity

$$(2.6) \quad (A^\alpha)^\beta = A^{\alpha\beta}, \quad 0 < \alpha < \pi/\omega, \operatorname{Re} \beta > 0,$$

where  $\omega$  is the minimum number such that the resolvent set of  $-A$  contains the sector

$$|\arg \lambda| < \pi - \omega.$$

Such an operator is said to be of type  $(\omega, M(\theta))$  if

$$\sup_{|\arg \lambda| = \theta} \|\lambda(\lambda + A)^{-1}\| \leq M(\theta).$$

Any operator with a dense domain which satisfies (1.1) is of type  $(\omega, M(\theta))$  with  $0 \leq \omega < \pi$ .

Some properties of fractional powers, however, are derived more easily through definition (2.1).

**PROPOSITION 2.4.** If  $0 < \operatorname{Re} \alpha < \sigma$ , there is a constant  $C(\alpha, \sigma, p)$  such that

$$(2.7) \quad \|A^\alpha x\| \leq C(\alpha, \sigma, p) q_p^\sigma(x)^{\operatorname{Re} \alpha / \sigma} \|x\|^{(\sigma - \operatorname{Re} \alpha) / \sigma}, \quad x \in D_p^\sigma.$$

*Proof.* Hölder's inequality gives

$$\begin{aligned} \|A^\alpha x\| &\leq \left| \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \right| \left[ \int_0^N |\lambda^{\alpha-1}| \|(A(\lambda + A)^{-1})^m x\| d\lambda \right. \\ &\quad \left. + \int_N^\infty |\lambda^{\alpha-\sigma}| \|\lambda^\sigma (A(\lambda + A)^{-1})^m x\| d\lambda / \lambda \right] \\ &\leq \left| \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \right| \left[ \frac{L^m N^{\operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \|x\| + \frac{N^{\operatorname{Re} \alpha - \sigma}}{((\sigma - \operatorname{Re} \alpha)p')^{1/p'}} q_p^\sigma(x) \right]. \end{aligned}$$

Taking the minimum of the right-hand side when  $N$  varies  $0 < N < \infty$ , we obtain (2.7).

**PROPOSITION 2.5.** If  $\mu > 0$ , then

$$(2.8) \quad D_p^\sigma(A) = D_p^\sigma(\mu + A)$$

with equivalent norms.

*Proof.* Let  $x \in D_{p,m}^\sigma(A)$  with  $m > \sigma$ . Since

$$\begin{aligned} \|A^k(\lambda + \mu + A)^{-m}x\| &\leq C \|A^m(\lambda + \mu + A)^{-m}x\|^{k/m} \cdot \\ &\|(\lambda + \mu + A)^{-m}x\|^{(m-k)/m}, \quad k = 1, 2, \dots, m-1, \\ \lambda^\sigma((\mu + A)(\lambda + \mu + A)^{-1})^m x \\ &= \lambda^\sigma(\mu^m + m\mu^{m-1}A + \dots + A^m)(\lambda + \mu + A)^{-m}x \end{aligned}$$

belongs to  $L^p(X)$ . The converse is proved in the same way.

**THEOREM 2.6.** *Let  $0 < \operatorname{Re} \alpha < \sigma$ . Then  $x \in D_p^\sigma$  if and only if  $x \in D(A^\alpha)$  and  $A^\alpha x \in D_p^{\sigma - \operatorname{Re} \alpha}$ .*

*Proof.* Let  $x \in D_p^\sigma$  and  $m > \sigma$ . Clearly  $x \in D(A^\alpha)$ . To estimate the integral

$$\begin{aligned} &\lambda^{\sigma - \operatorname{Re} \alpha} (A(\lambda + A)^{-1})^m A^\alpha x \\ &= \frac{\Gamma(m)\lambda^{\sigma - \operatorname{Re} \alpha}}{\Gamma(\alpha)\Gamma(m - \alpha)} \int_0^\infty \mu^{\alpha-1} (A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x d\mu, \end{aligned}$$

we split it into two parts. First,

$$\begin{aligned} &\left\| \lambda^{\sigma - \operatorname{Re} \alpha} \int_0^\lambda \mu^{\alpha-1} (A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x d\mu \right\| \\ &\leq \lambda^{\sigma - \operatorname{Re} \alpha} \int_0^\lambda \mu^{\operatorname{Re} \alpha - 1} d\mu L^m \| (A(\lambda + A)^{-1})^m x \| \\ &= L^m (\operatorname{Re} \alpha)^{-1} \lambda^\sigma \| (A(\lambda + A)^{-1})^m x \| \in L^p. \\ &\left\| \lambda^{\sigma - \operatorname{Re} \alpha} \int_\lambda^\infty \mu^{\alpha-1} (A(\lambda + A)^{-1})^m (A(\mu + A)^{-1})^m x d\mu \right\| \\ &\leq L^m \lambda^{\sigma - \operatorname{Re} \alpha} \int_\lambda^\infty \mu^{\operatorname{Re} \alpha - \sigma} \| \mu^\sigma (A(\mu + A)^{-1})^m x \| d\mu/\mu \end{aligned}$$

also belongs to  $L^p$  because  $\operatorname{Re} \alpha - \sigma < 0$ .

Conversely, let  $A^\alpha x \in D_p^{\sigma - \operatorname{Re} \alpha}$ . If  $n$  is an integer greater than  $\operatorname{Re} \alpha$ , we have

$$\begin{aligned} \|A^{n-\alpha}(\lambda + A)^{-n}\| &\leq C \|A^n(\lambda + A)^{-n}\|^{(n-\operatorname{Re} \alpha)/n} \|(\lambda + A)^{-n}\|^{\operatorname{Re} \alpha/n} \\ &\leq C' \lambda^{-\operatorname{Re} \alpha} \end{aligned}$$

Thus it follows from (2.5) that

$$\begin{aligned} \lambda^\sigma \| (A(\lambda + A)^{-1})^{m+n} x \| &\leq \lambda^\sigma \| A^{n-\alpha}(\lambda + A)^{-n} \| \| (A(\lambda + A)^{-1})^m A^\alpha x \| \\ &\leq C' \lambda^{\sigma - \operatorname{Re} \alpha} \| (A(\lambda + A)^{-1})^m A^\alpha x \| \in L^p. \end{aligned}$$

This completes the proof.



As a corollary we see that if  $\sigma$  is not an integer,  $D_{\infty}^{\sigma}$  and  $D_{\infty-}^{\sigma}$  coincide with  $D^{\sigma}$  and  $D_{*}^{\sigma}$  of [2], respectively.

**THEOREM 2.7.** *If the domain  $D(A^{\alpha})$  contains (is contained in)  $D_p^{\text{Re}\alpha}$  for an  $\text{Re}\alpha > 0$ , then  $D(A^{\alpha})$  contains (is contained in)  $D_p^{\text{Re}\alpha}$  for all  $\text{Re}\alpha > 0$ .*

*Proof.* By virtue of Theorem 6.4 of [2] and Proposition 2.5 we have  $D(A^{\alpha}) = D((\mu + A)^{\alpha})$  and  $D_p^{\text{Re}\alpha}(A) = D_p^{\text{Re}\alpha}(\mu + A)$ ,  $\mu > 0$ ,  $\text{Re}\alpha > 0$ , so that we may assume that  $A$  has a bounded inverse without loss of generality. The theorem is obvious if we show that  $A^{\beta}$ ,  $-\infty < \text{Re}\beta < \text{Re}\alpha$ , is a one-to-one mapping from  $D(A^{\alpha})$  and  $D_p^{\text{Re}\alpha}$  onto  $D(A^{\alpha-\beta})$  and  $D_p^{\text{Re}\alpha-\text{Re}\beta}$ , respectively.

Since  $D(A^{\alpha}) = R(A^{-\alpha})$ ,  $\text{Re}\alpha > 0$  ([2], Theorem 6.4), and since  $A^{\beta-\alpha} = A^{\beta}A^{-\alpha}$  ([2], Theorem 7.3), the statement concerning  $D(A^{\alpha})$  is immediate.

Let  $\text{Re}\beta < 0$ . Then  $x \in D_p^{\text{Re}\alpha-\text{Re}\beta}$  if and only if  $x \in D(A^{-\beta})$  and  $A^{-\beta}x \in D_p^{\text{Re}\alpha}$ . Since  $A^{\beta}$  is a bounded inverse of  $A^{-\beta}$ , we have  $x \in D_p^{\text{Re}\alpha-\text{Re}\beta}$  if and only if  $x$  is in the image of  $D_p^{\text{Re}\alpha}$  by  $A^{\beta}$ . If  $\text{Re}\beta \geq 0$ , choose a number  $\gamma$  so that  $\text{Re}\beta < \gamma < \text{Re}\alpha$ . If  $x \in D_p^{\text{Re}\alpha-\text{Re}\beta}$ ,  $x$  belongs to  $D(A^{-\beta})$ . Thus there is an element  $y$  such that  $x = A^{\beta}y$ . By the former part we have  $A^{-\gamma}x = A^{\beta-\gamma}y \in D_p^{\text{Re}\alpha-\text{Re}\beta+\gamma}$ . Thus  $y$  belongs to  $D_p^{\text{Re}\alpha}$ . On the other hand, if  $y \in D_p^{\text{Re}\alpha}$ , then  $y \in D(A^{\beta})$  and we have  $A^{-\gamma}x = A^{\beta-\gamma}y \in D_p^{\text{Re}\alpha-\text{Re}\beta+\gamma}$ , where  $x = A^{\beta}y$ . Then it follows from the former part that  $x$  belongs to  $D_p^{\text{Re}\alpha-\text{Re}\beta}$ .

Theorem 6.5 of [2] is obtained as a corollary.

**PROPOSITION 2.8.** For every  $\text{Re}\alpha > 0$

$$(2.9) \quad D_1^{\text{Re}\alpha} \subset D(A^{\alpha}) \subset D_{\infty-}^{\text{Re}\alpha}.$$

*Proof.* It is enough to prove it only in the case  $\alpha = 1$ . The former inclusion is clear from Lemma 2.3. The latter follows from (1.2), for

$$\lambda(A(\lambda + A)^{-1})^2x = \lambda(\lambda + A)^{-1}(1 - \lambda(\lambda + A)^{-1})Ax \rightarrow 0$$

for  $x \in D(A)$  as  $\lambda \rightarrow \infty$ .

**PROPOSITION 2.9.** If there is a complex number  $\text{Re}\alpha > 0$  such that  $D(A^{\alpha}) = D_p^{\text{Re}\alpha}$ , then  $D(A^{\beta}) = D_p^{\text{Re}\beta}$  for all  $\text{Re}\beta > 0$ . In particular,  $D(A^{\alpha})$  coincides with  $D(A^{\beta})$  if  $\text{Re}\alpha = \text{Re}\beta$ . Furthermore, if  $A$  has a bounded inverse,  $A^{it}$  is bounded for all real  $t$ , where  $A^{it}$  is defined in [2].

*Proof.* We need to prove only the last statement. Because of [2], Corollary 7.4 we have

$$A^{it} = A^{1+it}A^{-1},$$

Since  $D(A^{1+it}) = D(A) = R(A^{-1})$ ,  $A^{it}$  is defined everywhere and closed, so that it is bounded.

We proved in [2] that the operator  $A$  of § 14, Example 6 has unbounded purely imaginary powers  $A^{it}$ . The above proposition shows that  $D(A^\alpha)$  cannot be the same as  $D_p^{\text{Re}\alpha}$  for any  $p$ .

However, there are also operators  $A$  for which  $D(A^\alpha)$  coincides with  $D_p^{\text{Re}\alpha}$ .

Let  $X$  be  $L^p(S, B, m)$ , where  $B$  is a Borel field over a set  $S$  and  $m$  a measure on  $B$ , and let  $A(s)$  be a measurable function on  $S$  such that

$$|\arg A(s)| \leq \omega, \text{ a.e. } s$$

for an  $0 \leq \omega < \pi$ . Define

$$Ax(s) = A(s)x(s)$$

for all  $x(s) \in X$  such that  $A(s)x(s) \in X$ . Then it is easy to see that  $A$  is an operator of type  $(\omega, M(\theta))$  if  $p \leq \infty -$ , where  $L^{\infty-}$  denotes the closure of  $D(A)$  in  $L^\infty$ . For this operator  $A$  we have  $D(A) = D_p^1$ , so that  $D(A^\alpha) = D_p^{\text{Re}\alpha}$  for all  $\text{Re } \alpha > 0$ .

In fact, we have

$$(A(\lambda + A)^{-1})^2 x(s) = A(s)^2 x(s) / (\lambda + A(s))^2.$$

Therefore,

$$\begin{aligned} & \int_0^\infty \|\lambda(A(\lambda + A)^{-1})^2 x(s)\|^p d\lambda / \lambda \\ &= \int_0^\infty \lambda^{p-1} d\lambda \int_S \left| \frac{A(s)^2}{(\lambda + A(s))^2} x(s) \right|^p dm(s) \\ &= \int_S |x(s)|^p dm(s) \int_0^\infty \lambda^{p-1} \left| \frac{A(s)}{\lambda + A(s)} \right|^{2p} d\lambda \\ &\sim \|Ax\|^p. \end{aligned}$$

Any normal operator  $A$  of type  $(\omega, M(\theta))$  can be represented as an operator of the above type. Therefore, it satisfies  $D(A^\alpha) = D_2^{\text{Re}\alpha}$  for  $\text{Re } \alpha > 0$ . T. Kato [1] proved that this holds also for any maximal accretive operator  $A$  (see J.-L. Lions [3]).

Now let us complete the definition of fractional powers.

**THEOREM 2.10.** *Let  $0 < \text{Re } \alpha < m$ . If there is a sequence  $N_j \rightarrow \infty$*

such that

$$y = w\text{-}\lim_{j \rightarrow \infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^{N^j} \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda$$

exists, then  $x \in D(A^\alpha)$  and  $y = A^\alpha x$ .

Conversely, if  $x \in D(A^\alpha)$ , then

$$(2.10) \quad A^\alpha x = s\text{-}\lim_{N \rightarrow \infty} \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^N \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda,$$

possibly except for the case in which  $\text{Im } \alpha \neq 0$  and  $\text{Re } \alpha$  is an integer.

*Proof.* The former statement is obtained by modifying the proof of [2], Proposition 4.6. Since  $(\mu(\mu + A)^{-1})^m x \in D_1^{\text{Re } \alpha}$ , we have

$$\begin{aligned} A^\alpha (\mu(\mu + A)^{-1})^m x &= c \int_0^\infty \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m (\mu(\mu + A)^{-1})^m x d\lambda \\ &= (\mu(\mu + A)^{-1})^m w\text{-}\lim_{j \rightarrow \infty} c \int_0^{N^j} \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda \\ &= (\mu(\mu + A)^{-1})^m y. \end{aligned}$$

By virtue of (1, 2), it follows that  $x \in D(A^\alpha)$  and  $y = A^\alpha x$ .

The proof of the latter statement may be reduced to the case in which  $0 < \text{Re } \alpha < 1$  and  $m = 1$ . Suppose that  $x \in D(A^\alpha)$  and an integer  $m > \text{Re } \alpha$ . Substituting (1.6), we have

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda \\ &= m \int_0^N \lambda^{\alpha-m-1} d\lambda \int_0^\lambda \mu^{m-1} (A(\mu + A)^{-1})^{m+1} x d\mu \\ &= \frac{m}{m-\alpha} \int_0^N \left(1 - \frac{\mu^{m-\alpha}}{N^{m-\alpha}}\right) \mu^{\alpha-1} (A(\mu + A)^{-1})^{m+1} x d\mu. \end{aligned}$$

Since  $x \in D(A^\alpha) \subset D_{\infty-\alpha}^{\text{Re } \alpha}$ , it follows that

$$\left\| \int_0^N \frac{\mu^{m-\alpha}}{N^{m-\alpha}} \mu^{\alpha-1} (A(\mu + A)^{-1})^{m+1} x d\mu \right\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Thus the limit (2.10), if it exists, does not depend on  $m > \text{Re } \alpha$ .

Next, let  $\text{Re } \alpha > 1$  and  $m \geq 2$ . Since  $x \in D(A^\alpha)$  belongs to  $D(A)$ , integration by parts yields

$$\begin{aligned} &\int_0^N \lambda^{\alpha-1} (A(\lambda + A)^{-1})^m x d\lambda \\ &= \frac{\alpha-1}{m-1} \int_0^N \lambda^{\alpha-2} (A(\lambda + A)^{-1})^{m-1} A x d\lambda - \frac{N^{\alpha-1}}{m-1} (A(N + A)^{-1})^{m-1} A x. \end{aligned}$$

The second term tends to zero as  $N \rightarrow \infty$  because  $Ax \in D(A^{\alpha-1}) \subset D_{\infty}^{\operatorname{Re}\alpha-1}$ . Therefore, we obtain (2.10) if we can prove it when both  $\alpha$  and  $m$  are reduced by one.

To prove (2.10) in the case  $0 < \operatorname{Re} \alpha < 1$  and  $m = 1$  we assume for a moment that  $A$  has a bounded inverse. Then  $D(A^\alpha)$  is identical with the range of  $A^{-\alpha}$ , which may be represented by the absolutely convergent integral:

$$A^{-\alpha}x = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{-\alpha} (\lambda + A)^{-1} x d\lambda$$

([2], Proposition 5.1). Employing the resolvent equation and (1.6), we get

$$\begin{aligned} & \frac{\Gamma(1)}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^N \lambda^{\alpha-1} A(\lambda + A)^{-1} A^{-\alpha} x d\lambda \\ &= \left( \frac{\sin \pi \alpha}{\pi} \right)^2 \int_0^N \lambda^{\alpha-1} d\lambda \int_0^\infty \mu^{-\alpha} \frac{\lambda(\lambda + A)^{-1} - \mu(\mu + A)^{-1}}{\lambda - \mu} x d\mu \\ &= \left( \frac{\sin \pi \alpha}{\pi} \right)^2 \int_0^N \lambda^{\alpha-1} d\lambda \int_0^\infty \mu^{-\alpha} (\lambda - \mu)^{-1} d\mu \int_\mu^\lambda A(\nu + A)^{-2} x d\nu . \end{aligned}$$

It is enough to show that this converges strongly to the identity, or more weakly that it simply converges, because if it converges, the limit must be  $A^\alpha A^{-\alpha} x = x$ .

First of all, we have

$$\begin{aligned} I_1 &= \int_0^N \lambda^{\alpha-1} d\lambda \int_0^\lambda \mu^{-\alpha} (\lambda - \mu)^{-1} d\mu \int_\mu^\lambda A(\nu + A)^{-2} x d\nu \\ &= \int_0^N A(\nu + A)^{-2} x d\nu \int_\nu^N \lambda^{\alpha-1} d\lambda \int_0^\nu \mu^{-\alpha} (\lambda - \mu)^{-1} d\mu . \end{aligned}$$

Changing variables by  $\lambda = \nu l$ ,  $\mu = \nu m$  and integrating by parts with respect to  $\nu$ , we obtain

$$\begin{aligned} I_1 &= \int_1^\infty l^{\alpha-1} dl \int_0^1 m^{-\alpha} (l - m)^{-1} dm x \\ &\quad - \int_0^N A(\nu + A)^{-1} x d\nu N^\alpha \nu^{-\alpha-1} \int_0^1 m^{-\alpha} (N\nu^{-1} - m)^{-1} dm \\ &= c_1 x - \int_0^1 A(Nn + A)^{-1} x n^{-\alpha-1} dn \int_0^1 m^{-\alpha} (n^{-1} - m)^{-1} dm . \end{aligned}$$

Since  $n^{-\alpha-1} \int_0^1 m^{-\alpha} (n^{-1} - m)^{-1} dm$  is absolutely integrable in  $n$  and since  $A(Nn + A)^{-1} x = x - Nn(Nn + A)^{-1} x$  tends to zero as  $N \rightarrow \infty$ , the second term converges to zero as  $N \rightarrow \infty$ .

Next we write

$$\begin{aligned}
& \int_0^N \lambda^{\alpha-1} d\lambda \int_\lambda^\infty \mu^{-\alpha} (\lambda - \mu)^{-1} d\mu \int_\mu^\lambda A(\nu + A)^{-2} x d\nu \\
&= \int_0^N A(\nu + A)^{-2} x d\nu \int_0^\nu \lambda^{\alpha-1} d\lambda \int_\nu^\infty \mu^{-\alpha} (\mu - \lambda)^{-1} d\mu \\
&\quad + \int_N^\infty A(\nu + A)^{-2} x d\nu \int_0^N \lambda^{\alpha-1} d\lambda \int_\nu^\infty \mu^{-\alpha} (\mu - \lambda)^{-1} d\mu \\
&= I_2 + I_3 .
\end{aligned}$$

Changing variables as above, we have

$$\begin{aligned}
I_2 &= \int_0^N A(\nu + A)^{-2} x d\nu \int_0^1 l^{\alpha-1} dl \int_1^\infty m^{-\alpha} (m - l)^{-1} dm \\
&= c_2 N (N + A)^{-1} x \rightarrow c_2 x \quad \text{as } N \rightarrow \infty .
\end{aligned}$$

Finally,

$$I_3 = \int_1^\infty m^{-\alpha} dm \int_0^1 l^{\alpha-1} (m - l)^{-1} dl \int_N^{mN} A(\nu + A)^{-2} x d\nu$$

tends to zero as  $N \rightarrow \infty$  because  $\int_N^{mN} A(\nu + A)^{-2} x d\nu = mN(mN + A)^{-1} x - N(N + A)^{-1} x$  tends to zero and  $m^{-\alpha} \int_0^1 l^{\alpha-1} (m - l)^{-1} dl$  is absolutely integrable.

Next suppose that  $A$  has not necessarily a bounded inverse. We have, for  $\mu > 0$ ,

$$\begin{aligned}
& (A^\alpha - (\mu + A)^\alpha)(\mu + A)^{-\alpha} x \\
&= \frac{\sin \pi \alpha}{\pi} \left( \int_0^\mu \lambda^{\alpha-1} A + \int_\mu^\infty (\lambda^{\alpha-1} A - (\lambda - \mu)^{\alpha-1} (\mu + A)) (\lambda + A)^{-1} (\mu + A)^{-\alpha} x d\lambda \right)
\end{aligned}$$

because the integral is absolutely convergent and the equality holds for all  $x \in D(A)$  which is dense in  $X$ . This shows together with the above that

$$\begin{aligned}
A^\alpha (\mu + A)^{-\alpha} x &= (\mu + A)^\alpha (\mu + A)^{-\alpha} x \\
&\quad + \text{s-lim}_{N \rightarrow \infty} \frac{\sin \pi \alpha}{\pi} \left( \int_0^\mu \lambda^{\alpha-1} A + \int_\mu^N (\lambda^{\alpha-1} A - (\lambda - \mu)^{\alpha-1} (\mu + A)) \right) \\
&\quad \quad \quad \cdot (\lambda + A)^{-1} (\mu + A)^{-\alpha} x d\lambda \\
&= \text{s-lim}_{N \rightarrow \infty} \frac{\Gamma(1)}{\Gamma(\alpha) \Gamma(1 - \alpha)} \int_0^N \lambda^{\alpha-1} A (\lambda + A)^{-1} (\mu + A)^{-\alpha} x d\lambda .
\end{aligned}$$

**3. Interpolation spaces.** Let  $X$  and  $Y$  be Banach spaces contained in a Hausdorff vector space  $Z$ . Lions and Peetre [4] defined

the mean space  $S(p, \theta, X; p, \theta - 1, Y)$ ,  $1 \leq p \leq \infty$ ,  $0 < \theta < 1$ , of  $X$  and  $Y$  as the space of the means

$$(3.1) \quad x = \int_0^\infty u(\lambda) d\lambda/\lambda ,$$

where  $u(\lambda)$  is a  $Z$ -valued function such that

$$(3.2) \quad \lambda^\theta u(\lambda) \in L^p(X) \text{ and } \lambda^{\theta-1} u(\lambda) \in L^p(Y) .$$

$S(p, \theta, X; p, \theta - 1, Y)$  is a Banach space with the norm

$$(3.3) \quad \|x\|_{S(p, \theta, X, p, \theta-1, Y)} = \inf \left\{ \max (\|\lambda^\theta u(\lambda)\|_{L^p(X)}, \|\lambda^{\theta-1} u(\lambda)\|_{L^p(Y)}); x = \int_0^\infty u(\lambda) d\lambda/\lambda \right\} .$$

**Theorem 3.1.**  $S(p, \theta, X; p, \theta - 1, D(A^m))$ ,  $0 < \theta < 1$ ,  $1 \leq p \leq \infty$ , coincides with  $D_p^{\theta m}(A)$ .

*Proof.* By virtue of Proposition 2.5, we may assume that  $A$  has a bounded inverse without loss of generality. In particular,  $D(A^m)$  is normed by  $\|A^m x\|$ . Further, if we change the variable by  $\lambda' = \lambda^{1/m}$ , condition (3.2) becomes

$$(3.4) \quad \lambda^{m\theta} u(\lambda) \in L^p(X) \text{ and } \lambda^{m(\theta-1)} A^m u(\lambda) \in L_p(X) .$$

Suppose  $x \in D_p^\sigma$  and define

$$u(\lambda) = c \lambda^m A^m (\lambda + A)^{-2m} x ,$$

where  $c = \Gamma(2m)/(\Gamma(m))^2$ . Then

$$\lambda^\sigma u(\lambda) = c(\lambda(\lambda + A^{-1})^m \lambda^\sigma (A(\lambda + A)^{-1})^m x \in L^p(X)$$

and

$$\lambda^{\sigma-m} A^m u(\lambda) = c \lambda^\sigma (A(\lambda + A)^{-1})^{2m} x \in L^p(X) .$$

Thus  $u(\lambda)$  satisfies (3.4) with  $\sigma = m\theta$ . Moreover, it follows from Lemma 2.3 that

$$\begin{aligned} \int_0^\infty u(\lambda) d\lambda/\lambda &= \frac{\Gamma(2m)}{(\Gamma(m))^2} \int_0^\infty \lambda^{m-1} (A(\lambda + A)^{-1})^{2m} A^{-m} x \\ &= x . \end{aligned}$$

Therefore,  $x$  belongs to  $S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$ .

Conversely, let  $x \in S(p, \sigma/m, X; p, \sigma/m - 1, D(A^m))$  so that  $x$  is represented by integral (3.1) with an integrand satisfying (3.4). Then

$$\begin{aligned} \lambda^\sigma (A(\lambda + A)^{-1})^m x &= (A(\lambda + A)^{-1})^m \lambda^\sigma \int_\lambda^\infty \mu^{-\sigma} \mu^\sigma u(\lambda) d\mu/\mu \\ &\quad + (\lambda(\lambda + A)^{-1})^m \lambda^{\sigma-m} \int_0^\lambda \mu^{m-\sigma} \mu^{\sigma-m} A^m u(\lambda) d\mu/\mu. \end{aligned}$$

Since both  $(A(\lambda + A)^{-1})^m$  and  $(\lambda(\lambda + A)^{-1})^m$  are uniformly bounded,  $\lambda^\sigma (A(\lambda + A)^{-1})^m x$  belongs to  $L^p(X)$ , that is,  $x \in D_p^\sigma$ .

**THEOREM 3.2.** *Let  $A$  be an operator of type  $(\omega, M(\theta))$ . Then*

$$D_p^\sigma(A^\alpha) = D_p^{\sigma\alpha}(A), \quad 0 < \alpha < \pi/\omega, \sigma > 0.$$

*Proof.* It is sufficient to prove it in the case  $0 < \alpha < 1$ , because otherwise we have  $A = (A^\alpha)^{1/\alpha}$  with  $0 < 1/\alpha < 1$  (see (2.6)). In view of Theorem 2.6 we may also assume that  $\sigma$  is sufficiently small.

By [2] Proposition 10.2 we have

$$\lambda^\sigma A^\alpha (\lambda + A)^{-1} x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^{\sigma+1} \tau^{\alpha-\sigma}}{\lambda^2 + 2\lambda\tau^\alpha \cos \pi\alpha + \tau^{2\alpha}} \tau^\alpha A(\tau + A)^{-1} x d\tau/\tau.$$

Since the kernel

$$\frac{(\lambda^{-1}\tau^\alpha)^{1-\sigma}}{1 + 2(\lambda^{-1}\tau^\alpha) \cos \pi\alpha + (\lambda^{-1}\tau^\alpha)^2}, \quad 0 < \sigma < 1,$$

defines a bounded integral operator in  $L^p(X)$ ,  $D_p^{\sigma\alpha}(A)$  is contained in  $D_p^\sigma(A^\alpha)$ .

If  $\alpha = 1/m$  with an odd integer  $m$ , we have conversely

$$D_p^\sigma(A^{1/m}) \subset D_p^{\sigma/m}(A).$$

In fact, let  $x \in D_p^\sigma(A^{1/m})$ . Since

$$\lambda^\sigma A(\lambda^m + A)^{-1} = \lambda^\sigma \prod_{i=1}^m (A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}) x,$$

where  $\varepsilon_i$  are roots of  $(-\varepsilon)^m = -1$  with  $\varepsilon_1 = 1$ , and since

$$A^{1/m}(\varepsilon_i \lambda + A^{1/m})^{-1}, \quad i = 2, \dots, m,$$

are uniformly bounded,  $\lambda^\sigma A(\lambda^m + A)^{-1} x \in L^p(X)$ . Changing the variable by  $\lambda' = \lambda^m$ , we get  $\lambda^{\sigma/m} A(\lambda + A)^{-1} x \in L^p(X)$ .

In a general case choose an odd number  $m$  such that  $0 < 1/m < \alpha$ . Since  $A^{1/m} = (A^\alpha)^{1/(\alpha m)}$ , we have

$$D_p^{\alpha\sigma}(A) \subset D_p^\sigma(A^\alpha) \subset D_p^{\alpha\sigma m}(A^{1/m}) \subset D_p^{\alpha\sigma}(A).$$

Another less computational proof will be obtained from the Lions-Peetre theory and Proposition 2.8.

**4. Infinitesimal generators of bounded semi-groups.** Throughout this section we assume that  $T_t, t \geq 0$ , is a bounded strongly continuous semi-group of operators in  $X$  and  $-A$  is its infinitesimal generator:

$$(4.1) \quad T_t = \exp(-tA), \quad \|T_t\| \leq M.$$

$A$  is an operator of type  $(\pi/2, M(\theta))$ .

**DEFINITION 4.1.** Let  $0 < \sigma < m$ , where  $\sigma$  is a real number and  $m$  an integer, and let  $1 \leq p \leq \infty$ . We denote by  $C_{p,m}^\sigma = C_{p,m}^\sigma(A)$  the set of all elements  $x \in X$  such that

$$(4.2) \quad t^{-\sigma}(I - T_t)^m x \in L^p(X).$$

As is easily seen,  $C_{p,m}^\sigma$  is a Banach space with the norm

$$\|x\|_{\sigma_{p,m}^\sigma} = \|x\| + \|t^{-\sigma}(I - T_t)^m x\|_{L^p(X)}.$$

Since  $(I - T_t)^m$  is uniformly bounded, condition (4.2) is equivalent to that  $t^{-\sigma}(I - T_t)^m x$  belongs to  $L^p(X)$  near the origin. In particular, we have

$$(4.3) \quad C_{p,m}^\sigma(A) = C_{p,m}^\sigma(\mu + A), \mu > 0.$$

$C_{\infty,1}^\sigma$  and  $C_{\infty,-1}^\sigma$  coincide with  $C^\sigma$  and  $C_*^\sigma$  of [2], respectively, and  $C_{\infty,1}^\sigma$  consists of all elements  $x$  such that  $T_t x$  is (weakly) uniformly Hölder continuous with exponent  $\sigma$ .

**PROPOSITION 4.2.** If  $x \in C_{p,m}^\sigma$ , then  $x$  belongs to  $D(A^\alpha)$  for all  $0 < \operatorname{Re} \alpha < \sigma$ , and

$$(4.4) \quad A^\alpha x = \frac{1}{K_{\alpha,m}} \int_0^\infty t^{-\alpha-1}(I - T_t)^m x dt, \quad 0 < \operatorname{Re} \alpha < \sigma,$$

where

$$K_{\alpha,m} = \int_0^\infty t^{-\alpha-1}(1 - e^{-t})^m dt.$$

*Proof.* If  $0 < \operatorname{Re} \alpha < \sigma$ , the right-hand side of (4.4) converges absolutely and represents an analytic function of  $\alpha$ .

If  $x \in D(A)$ , then we have by [2] Proposition 11.4

$$\int_0^\infty t^{-\alpha-1}(I - T_t)^m x dt$$



$$\begin{aligned}
&= \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} \int_0^{\infty} t^{-\alpha-1} (I - T_{kt}) x dt \\
&= \Gamma(-\alpha) \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} k^{\alpha} A^{\alpha} x, \quad 0 < \operatorname{Re} \alpha < 1.
\end{aligned}$$

The coefficient of  $A^{\alpha}x$  does not depend on  $A$ . Taking  $A = 1$ , we see that it is equal to  $K_{\alpha, m}$ .

Next let  $0 < \operatorname{Re} \alpha < \min(\sigma, 1)$  and  $x \in C_{p, m}^{\sigma}$ . Then integral (4.4) with  $x$  replaced by  $\mu(\mu + A)^{-1}x$ ,  $\mu > 0$ , exists and converges to the integral (4.4) as  $\mu \rightarrow \infty$ . Thus  $A^{\alpha}\mu(\mu + A)^{-1}x$  converges to the integral (4.4). Since  $A^{\alpha}$  is closed and  $\mu(\mu + A)^{-1}x \rightarrow x$  as  $\mu \rightarrow \infty$ , it follows that  $x \in D(A^{\alpha})$  and (4.4) holds.

In the general case the assertion is obtained by [2], Proposition 8.4 or by repeating an argument as above.

Lions and Peetre [4] gave another proof when  $\alpha$  is an integer.

**THEOREM 4.3.**  $C_{p, m}^{\sigma}$  coincides with  $D_p^{\sigma}$  with equivalent norms.

*Proof.* First we note that

$$(4.5) \quad (I - T_t)x = AI_t x, \quad x \in X,$$

where

$$(4.6) \quad I_t x = \int_0^t T_s x ds.$$

Obviously we have

$$(4.7) \quad \|I_t\| \leq M_t, \quad t > 0.$$

Let  $x \in C_{p, m}^{\sigma}$ . Then  $(\lambda + A)^{-m}x$ ,  $\lambda > 0$ , belongs to  $C_{p, 2m}^{\sigma+m}$  since

$$\begin{aligned}
&t^{-\sigma-m} \|(I - T_t)^{2m}(\lambda + A)^{-m}x\| \\
&\leq t^{-m} \|I_t^m\| \|(A(\lambda + A)^{-1})^m\| t^{-\sigma} \|(I - T_t)^m x\|.
\end{aligned}$$

Hence we have by Proposition 4.2

$$\begin{aligned}
(A(\lambda + A)^{-1})^m x &= c \int_0^{\infty} t^{-m-1} (I - T_t)^{2m} (\lambda + A)^{-m} x \\
&= c \int_0^{1/\lambda} (A(\lambda + A)^{-1})^m t^{-m-1} I_t^m (I - T_t)^m x dt \\
&\quad + c \int_{1/\lambda}^{\infty} (\lambda + A)^{-m} t^{-m-1} (I - T_t)^{2m} x dt,
\end{aligned}$$

where  $c = K_{m, 2m}^{-1}$ . Therefore,

$$\begin{aligned}
\lambda^{\sigma} \|(A(\lambda + A)^{-1})^m x\| &\leq c L^m M^m \lambda^{\sigma} \int_0^{1/\lambda} t^{\sigma} t^{-\sigma} \|(I - T_t)^m x\| dt/t \\
&\quad + c M^m (2M)^m \lambda^{\sigma-m} \int_{1/\lambda}^{\infty} t^{\sigma-m} t^{-\sigma} \|(I - T_t)^m x\| dt/t.
\end{aligned}$$

This shows that  $x \in D_{p,m}^\sigma$ .

Conversely, let  $x \in D_{p,m}^\sigma$ . Since

$$(A(\lambda + A)^{-1})^{2m} I_t^m x = (\lambda + A)^{-m} (I - T_t)^m (A(\lambda + A)^{-1})^m x,$$

it follows that  $I_t^m x \in D_{p,2m}^{\sigma+2m}$ . Thus by Proposition 2.2 we get

$$\begin{aligned} (I - T_t)^m x &= A^m I_t^m x = c \int_0^\infty \lambda^{m-1} (A(\lambda + A)^{-1})^{2m} I_t^m x \\ &= c \int_0^{1/t} I_t^m \lambda^{m-1} (A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + c \int_{1/t}^\infty (I - T_t)^m \lambda^{m-1} (\lambda + A)^{-m} (A(\lambda + A)^{-1})^m x d\lambda, \end{aligned}$$

where  $c = \Gamma(2m)/(\Gamma(m))^2$ . By the same computation as above we conclude that  $x \in C_{p,m}^\sigma$ .

In particular,  $C_{p,m}^\sigma$  does not depend on  $m$ . We denote  $C_{p,m}^\sigma$  with the least  $m > \sigma$  by  $C_p^\sigma$ . Because of Theorem 2.6,  $C_p^\sigma$  coincides with  $C^\sigma$  of [2] if  $\sigma$  is not an integer.

**THEOREM 4.4.** *Let  $0 < \operatorname{Re} \alpha < m$ . If there is a sequence  $\varepsilon_j \rightarrow 0$  such that*

$$(4.8) \quad y = w\text{-}\lim_{j \rightarrow \infty} \frac{1}{K_{\alpha,m}} \int_{\varepsilon_j}^\infty t^{-\alpha-1} (I - T_t)^m x dt$$

*exists, then  $x \in D(A^\alpha)$  and  $y = A^\alpha x$ .*

*Conversely, if  $x \in D(A^\alpha)$ , then*

$$(4.9) \quad A^\alpha x = s\text{-}\lim_{\varepsilon \rightarrow 0} \frac{1}{K_{\alpha,m}} \int_\varepsilon^\infty t^{-\alpha-1} (I - T_t)^m x dt.$$

*Proof.* The former part is proved in the same way as Theorem 2.10.

To prove the latter part, let us assume for a moment that  $T_t$  satisfies

$$\|T_t\| \leq M e^{-\mu t}, \quad t > 0,$$

for a  $\mu > 0$ . Then  $A^\alpha$  is the inverse of  $A^{-\alpha}$  which can be represented by the absolutely convergent integral

$$(4.10) \quad A^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty s^{\alpha-1} T_s x ds$$

([2], Theorem 7.3 and Proposition 11.1).

Now it is enough to prove that

$$\frac{1}{K_{\alpha,m} \Gamma(\alpha)} \int_\varepsilon^\infty t^{-\alpha-1} (I - T_t)^m dt \int_0^\infty s^{\alpha-1} T_s x ds$$

converges strongly as  $\varepsilon \rightarrow 0$ , because the limit must coincide with  $A^\alpha A^{-\alpha} x = x$ .

We have

$$\begin{aligned} I_\varepsilon &= \int_\varepsilon^\infty t^{-\alpha-1} (I - T_t)^m dt \int_0^\infty s^{\alpha-1} T_s x ds \\ &= \sum_{k=1}^m (-1)^{k+1} \binom{m}{k} k^\alpha \int_{k\varepsilon}^\infty t^{-\alpha-1} (I - T_t) dt \int_0^\infty s^{\alpha-1} T_s x ds . \end{aligned}$$

Now

$$\begin{aligned} &\int_{k\varepsilon}^\infty t^{-\alpha-1} T_t dt \int_0^\infty s^{\alpha-1} T_s x ds \\ &= \int_{k\varepsilon}^\infty t^{-\alpha-1} dt \int_t^\infty (s-t)^{\alpha-1} T_s x ds \\ &= \int_{k\varepsilon}^\infty T_s x ds \int_{k\varepsilon}^s t^{-\alpha-1} (s-t)^{\alpha-1} dt \\ &= \frac{1}{\alpha(k\varepsilon)^\alpha} \int_{k\varepsilon}^\infty (s-k\varepsilon)^\alpha T_s x ds / s . \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_{k=1}^m (-1)^{k+1} \binom{m}{k} k^\alpha \int_{k\varepsilon}^\infty t^{-\alpha-1} dt \int_0^\infty s^{\alpha-1} T_s x ds \\ &= \frac{1}{\alpha\varepsilon^\alpha} \int_0^\infty s^{\alpha-1} T_s x ds , \end{aligned}$$

so that we obtain

$$I_\varepsilon = \frac{1}{\alpha\varepsilon^\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_{k\varepsilon}^\infty (s-k\varepsilon)^\alpha T_s x ds / s .$$

Since  $T_s x \rightarrow x$  as  $s \rightarrow 0$ , it follows that

$$\begin{aligned} &\frac{1}{\alpha\varepsilon^\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_{k\varepsilon}^{m\varepsilon} (s-k\varepsilon)^\alpha T_s x ds / s \\ &= \frac{1}{\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_k^m (s-k)^\alpha T_{\varepsilon s} x ds / s \\ &\rightarrow \frac{1}{\alpha} \sum_{k=0}^m (-1)^k \binom{m}{k} \int_k^m (s-k)^\alpha ds / s x \text{ as } \varepsilon \rightarrow 0 . \end{aligned}$$

On the other hand, the Taylor expansion up to order  $m$  gives

$$\begin{aligned} f_\varepsilon(s) &= \sum_{k=0}^m (-1)^k \binom{m}{k} (s-k\varepsilon)^\alpha \\ &= \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{m!} (s-k\varepsilon)^{\alpha-m} (-k\varepsilon)^m , \end{aligned}$$

where  $0 < k' < k$ . Hence we have

$$\begin{aligned} & \frac{1}{\alpha \varepsilon^\alpha} \int_{m\varepsilon}^{\infty} f_\varepsilon(s) T_s x ds / s \\ &= \frac{(\alpha - 1) \cdots (\alpha - m + 1)}{m!} \sum_{k=0}^m (-1)^{k+m} \binom{m}{k} k^m \int_m^{\infty} (s - k')^{\alpha-m} T_{\varepsilon s} x ds / s. \end{aligned}$$

Since  $(s - k')^{\alpha-m} s^{-1}$  is absolutely integrable, this converges to a constant times  $x$  as  $\varepsilon \rightarrow 0$ .

To prove (4.9) in the general case, it is sufficient to show that

$$\begin{aligned} (4.11) \quad & (A^\alpha - (\mu + A)^\alpha)(\mu + A)^{-\alpha} x \\ &= \frac{1}{K_{\alpha, m}} \int_0^{\infty} t^{-\alpha-1} \{(I - T_t)^m - (I - e^{-\mu t} T_t)^m\} (\mu + A)^{-\alpha} x dt, \\ & \mu > 0, x \in X, \end{aligned}$$

and that the integral converges absolutely.

By Theorem 2.6, (4.5) and a similar decomposition of  $I - e^{-\mu t} T_t$  we have

$$(I - T_t)^m (I - e^{-\mu t} T_t)^n x = O(t^\sigma), x \in C_\infty^\sigma, m + n > \sigma.$$

Since  $(\mu + A)^{-\alpha} x \in D(A^\alpha) \subset C_\infty^{\text{Re } \alpha}$ , it follows that

$$\begin{aligned} & \{(I - T_t)^m - (I - e^{-\mu t} T_t)^m\} x \\ &= (e^{-\mu t} - 1) T_t \{(I - T_t)^{m-1} + \cdots + (I - e^{-\mu t} T_t)^{m-1}\} x \\ &= O(t^{\min(\text{Re } \alpha, m-1)+1}). \end{aligned}$$

This shows that integral (4.11) is absolutely convergent. (4.11) is valid for all  $x \in D(A)$  which is dense in  $X$ . Therefore, (4.11) holds for all  $x \in X$ .

### 5. Infinitesimal generators of bounded analytic semi-groups.

Let  $T_t$  be a semi-group of operators analytic in a sector  $|\arg t| < \pi/2 - \omega$ ,  $0 \leq \omega < \pi/2$ , and uniformly bounded in each smaller sector  $|\arg t| \leq \pi/2 - \omega - \varepsilon$ ,  $\varepsilon > 0$ . We call such a semi-group a bounded analytic semi-group.

It is known that the negative of an operator  $A$  generates a bounded analytic semi-group if and only if  $A$  is of type  $(\omega, M(\theta))$  for some  $0 \leq \omega < \pi/2$ . A bounded strongly continuous semi-group  $T_t$  has a bounded analytic extension if there is a complex number  $\text{Re } \alpha > 0$  such that

$$(5.1) \quad \|A^\alpha T_t\| \leq C t^{-\text{Re } \alpha}, t > 0,$$

with a constant  $C$  independent of  $t$ . Conversely, if  $T_t$  is bounded analytic,

(5.1) holds for all  $\operatorname{Re} \alpha > 0$  ([2], Theorems 12.1 and 12.2).

We assume throughout this section that  $-A$  is the infinitesimal generator of a bounded analytic semi-group  $T_t$ .

**DEFINITION 5.1.** Let  $0 < \sigma < \operatorname{Re} \beta$  and  $1 \leq p \leq \infty$ . We denote by  $B_{p,\beta}^\sigma = B_{p,\beta}^\sigma(A)$  the set of all  $x \in X$  such that

$$(5.2) \quad t^{\operatorname{Re} \beta - \sigma} A^\beta T_t x \in L^p(X).$$

$B_{p,\beta}^\sigma$  is a Banach space with the norm

$$\|x\|_{B_{p,\beta}^\sigma} = \|x\| + \|t^{\operatorname{Re} \beta - \sigma} A^\beta T_t x\|_{L^p(X)}.$$

**PROPOSITION 5.2.** Let  $0 < \operatorname{Re} \alpha < \sigma$ . Then every  $x \in B_{p,\beta}^\sigma$  belongs to  $D(A^\alpha)$  and

$$(5.3) \quad A^\alpha x = \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{\beta - \alpha - 1} A^\beta T_t x dt,$$

where the integral converges absolutely.

*Proof.* Since  $A^\beta T_t x$  is of order  $t^{\sigma - \operatorname{Re} \beta}$  as  $t \rightarrow 0$  and of order  $t^{-\operatorname{Re} \beta + \sigma}$  as  $t \rightarrow \infty$  in the sense of  $L^p(X)$ , the integral converges absolutely for  $0 < \operatorname{Re} \alpha < \sigma$ .

To prove (5.3), first let  $x \in D(A^\beta)$ . Then it follows from [2], Proposition 11.1 and Theorem 7.3 that

$$\begin{aligned} & \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{\beta - \alpha - 1} A^\beta T_t x dt \\ &= s\text{-}\lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{\beta - \alpha - 1} e^{-\varepsilon t} T_t A^\beta x dt \\ &= s\text{-}\lim_{\varepsilon \rightarrow 0} (\varepsilon + A)^{\alpha - \beta} A^\beta x \\ &= s\text{-}\lim_{\varepsilon \rightarrow 0} A^{\beta - \alpha} (\varepsilon + A)^{\alpha - \beta} A^\alpha x. \end{aligned}$$

Because of [2], Propositions 6.2 and 6.3,  $A^{\beta - \alpha} (\varepsilon + A)^{\alpha - \beta}$  converges strongly to the identity on  $\overline{R(A)}$  as  $\varepsilon \rightarrow 0$ . Since  $A^\alpha X$  is contained in  $\overline{R(A)}$  ([2], Proposition 4.3), (5.3) holds for all  $x \in D(A^\beta)$ . In the general case (5.3) is proved by approximating  $x \in B_{p,\beta}^\sigma$  by  $(\mu(\mu + A)^{-1})^m x$ ,  $m > \operatorname{Re} \beta$ , which belongs to  $D(A^\beta)$ .

**THEOREM 5.3.**  $B_{p,\beta}^\sigma$  coincides with  $D_p^\sigma$ . In particular,  $B_{p,\beta}^\sigma$  does not depend on  $\beta$ .

*Proof.* Let  $x \in B_{p,\beta}^\sigma$ . If  $m$  is an integer greater than  $\operatorname{Re} \beta$ ,  $x$  belongs to  $B_{p,m}^\sigma$ , for

$$t^{m-\sigma}A^mT_t x = t^{m-\beta}A^{m-\beta}T_{t/2} \cdot t^{\beta-\sigma}A^\beta T_{t/2}x$$

and  $t^{m-\beta}A^{m-\beta}T_{t/2}$  is uniformly bounded. Since

$$t^{m-\sigma}A^{2m}T_t(\lambda + A)^{-m}x = (A(\lambda + A)^{-1})^m t^{m-\sigma}A^mT_t x ,$$

$(\lambda + A)^{-m}x$  belongs to  $B_{p,2m}^{\sigma+m}$ . Hence it follows from Proposition 5.2 that

$$\begin{aligned} A^m(\lambda + A)^{-m}x &= c \int_0^\infty t^m A^{2m}T_t(\lambda + A)^{-m}x dt/t \\ &= c(A(\lambda + A)^{-1})^m \int_0^{1/\lambda} t^m A^m T_t x dt/t \\ &\quad + c(\lambda + A)^{-m} \int_{1/\lambda}^\infty t^m A^{2m} T_t x dt/t , \end{aligned}$$

where  $c = \Gamma(m)^{-1}$ . The rest of the proof is the same as that of Theorem 4.3.

Conversely, assume that  $x \in D_{p,m}^\sigma = D_{p,2m}^\sigma$ . Since  $T_t x, t > 0$ , belongs to any  $D_{p,m}^\sigma$ , we have by (2.1)

$$\begin{aligned} A^\beta T_t x &= c \int_0^\infty \lambda^{\beta-1} (A(\lambda + A)^{-1})^{2m} T_t x d\lambda \\ &= c T_t \int_0^{1/t} \lambda^{\beta-1} (A(\lambda + A)^{-1})^{2m} x d\lambda \\ &\quad + c A^m T_t \int_{1/t}^\infty \lambda^{\beta-1} (\lambda + A)^{-m} (A(\lambda + A)^{-1})^m x d\lambda , \end{aligned}$$

where  $c = \Gamma(2m)/(\Gamma(\beta)\Gamma(2m - \beta))$ . Arguing as before, we get  $x \in B_{p,\beta}^\sigma$ .

**THEOREM 5.4** *Let  $0 < \text{Re } \alpha < \text{Re } \beta$ . If*

$$(5.4) \quad y = w\text{-}\lim_{\varepsilon_j \rightarrow 0} \frac{1}{\Gamma(\beta - \alpha)} \int_{\varepsilon_j}^\infty t^{\beta-\alpha-1} A^\beta T_t x dt$$

*exists, then  $x \in D(A^\alpha)$  and  $y = A^\alpha x$ . If  $x \in D(A^\alpha)$ , then*

$$(5.5) \quad A^\alpha x = s\text{-}\lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\beta - \alpha)} \int_\varepsilon^\infty t^{\beta-\alpha-1} A^\beta T_t x dt .$$

*Proof.* The former part is proved in the same way as Theorem 2.10. Let us prove the latter assuming that  $\mu - A$  generates a bounded analytic semi-group for a  $\mu > 0$ .  $D(A^\alpha)$  is the same as the range  $R(A^{-\alpha})$  in this case, and we have  $A^\beta T_t A^{-\alpha} x = A^{\beta-\alpha} T_t x$  by the additivity of fractional powers. So it is sufficient to prove the following:

$$(5.6) \quad x = s\text{-}\lim_{\varepsilon \rightarrow 0} \frac{1}{\Gamma(\beta)} \int_\varepsilon^\infty t^{\beta-1} A^\beta T_t x dt, \quad x \in X ,$$

when  $\operatorname{Re} \beta > 0$ .

First we note that if  $\operatorname{Re} \alpha > 0$ , then

$$(5.7) \quad t^\alpha A^\alpha T_t x \rightarrow 0 \text{ as } t \rightarrow 0 \text{ or as } t \rightarrow \infty$$

for each  $x \in X$ , because (5.7) holds for  $x \in D(A)$  and  $t^\alpha A^\alpha T_t$  is uniformly bounded.

Let  $\beta$  be equal to an integer  $m$ . Since  $d/dt A^\beta T_t x = -A^{\beta+1} T_t x$ , we have, by integrating by parts,

$$\begin{aligned} & \int_\varepsilon^\infty t^{m-1} A^m T_t x dt \\ &= \varepsilon^{m-1} A^{m-1} T_\varepsilon x + (m-1) \int_\varepsilon^\infty t^{m-2} A^{m-1} T_t x dt. \end{aligned}$$

(5.7) shows that the first term tends to zero as  $\varepsilon \rightarrow 0$  if  $m > 1$ . When  $m = 1$ , we have

$$\int_\varepsilon^\infty A T_t x dt = T_\varepsilon x \rightarrow x \text{ as } \varepsilon \rightarrow 0.$$

Thus (5.6) holds if  $\beta$  is an integer.

If  $\beta$  is not an integer, take an integer  $m > \operatorname{Re} \beta$ . We have

$$\begin{aligned} A^\beta T_t x &= A^{\beta-m} A^m T_t x \\ &= \frac{1}{\Gamma(m-\beta)} \int_t^\infty (s-t)^{m-\beta-1} A^m T_s x ds, \quad t > 0, \end{aligned}$$

by [2], Proposition 11.1. Therefore,

$$\begin{aligned} & \frac{1}{\Gamma(\beta)} \int_\varepsilon^\infty t^{\beta-1} A^\beta T_t x dt \\ &= \frac{1}{\Gamma(\beta)\Gamma(m-\beta)} \int_\varepsilon^\infty A^m T_s x ds \int_\varepsilon^s t^{\beta-1} (s-t)^{m-\beta-1} dt \\ &= \frac{1}{\Gamma(m)} \int_\varepsilon^\infty s^{m-1} A^m T_s x ds \\ &\quad - \frac{\varepsilon^m}{\Gamma(\beta)\Gamma(m-\beta)} \int_1^\infty A^m T_{\varepsilon\sigma} x d\sigma \int_0^1 \tau^{\beta-1} (\sigma-\tau)^{m-\beta-1} d\tau. \end{aligned}$$

The first term tends to  $x$  as  $\varepsilon \rightarrow 0$ . The second term converges to zero, because

$$\int_1^\infty \sigma^{-m} d\sigma \int_0^1 \tau^{\beta-1} (\sigma-\tau)^{m-\beta-1} d\tau$$

is absolutely convergent and  $(\varepsilon\sigma)^m A^m T_{\varepsilon\sigma} x$  tends to zero as  $\varepsilon \rightarrow 0$ .

The proof in the general case is obtained from the absolutely convergent integral representation:

$$\begin{aligned} & (A^\alpha - (\mu + A)^\alpha)(\mu + A)^{-\alpha}x \\ &= \frac{1}{\Gamma(\beta - \alpha)} \int_0^\infty t^{\beta-\alpha-1} (A^\beta - e^{-\mu t}(\mu + A)^\beta) T_t (\mu + A)^{-\alpha} x dt . \end{aligned}$$

The absolute convergence follows from [2], Propositions 6.2 and 6.3.

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