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**CERTAIN ISOMORPHISMS BETWEEN QUOTIENTS OF A  
GROUP ALGEBRA**

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# CERTAIN ISOMORPHISMS BETWEEN QUOTIENTS OF A GROUP ALGEBRA

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Let  $T$  be the circle group, considered as the additive group of the real numbers modulo  $2\pi$ . Let  $A = A(T)$ , the Banach algebra of functions on  $T$  which have absolutely convergent Fourier series, with the norm of  $f$  in  $A$  equal to  $\sum_n |\hat{f}(n)|$ . If  $E$  is a closed subset of  $T$ , we denote by  $A(E)$  the quotient algebra  $A/I(E)$ , where  $I(E)$  is the closed ideal consisting of those functions in  $A$  which vanish on  $E$ . This paper presents a procedure for constructing perfect sets  $E$  and  $F$ , which are not Helson sets, and a map  $\varphi: F \rightarrow E$  inducing an isomorphism of  $A(E)$  into  $A(F)$ . Thereby we shall obtain cases of an isomorphism of norm one, where  $\varphi$  is the restriction to  $F$  of a discontinuous character of  $T$ , composed with a rotation. In general, our  $\varphi$  will be such a restriction at least on a dense subset of  $F$ , with the norm of the isomorphism not necessarily equal to one.

In the course of this construction we impose a condition of "arithmetic thinness" on the set  $F$ . As we shall prove, this condition is sufficient to imply that  $F$  is a set of uniqueness.

Beurling and Helson [2] established that every automorphism of the algebra  $A$  arises from a rigid motion of the circle—the composition of a rotation,  $x \rightarrow x + x_0$ , and a reflection,  $x \rightarrow x$  or  $x \rightarrow -x$ . One may consider the problem of characterizing the cases in which a homeomorphism  $\varphi$  of one closed set  $F$  onto another,  $E$ , induces an isomorphism of  $A(E)$  into  $A(F)$ . The methods of [2] may be modified to show that if  $E$  and  $F$  are intervals, then  $\varphi(x) = rx + x_0$ , where  $r$  and  $x_0$  are real; but these methods do not solve the problem for more general sets. DeLeeuw and Katznelson [4] showed that whenever the norm of an isomorphism of  $A(E)$  into  $A(F)$  is equal to one, it must arise from a map  $\varphi: F \rightarrow E$  which is the restriction to  $F$  of a character (an additive function of  $T$  into  $T$ ) composed with a rotation; and that if  $F$  is "thick" in one of several senses, then this character must be a continuous one:  $\varphi(x) = nx + x_0$ , where  $x_0$  is real and  $n$  is an integer.

Let us call the map  $\varphi: F \rightarrow E$  *trivial* if, near each point of  $F$ , it is equal to the restriction of a function  $rx + x_0$ , where  $r$  and  $x_0$  are real. What we shall show, in this terminology, is that there exist cases of a nontrivial  $\varphi$  inducing an isomorphism of  $A(E)$  into  $A(F)$ , where  $E$  and  $F$  are not Helson sets. Still, no such case is known in which  $F$  is a set of multiplicity.

2. **Notation and definitions.** The dual group of  $T$  is  $T^\wedge = Z$ , the group of integers; and  $A$  is the Gel'fand representation of  $L^1(Z)$  (cf. [11], Ch. 1 and [7], App. I-IV). For  $f \in A$ , we let

$$\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx,$$

so that  $f(x) = \sum_n \hat{f}(n) e^{inx}$  and the  $A$ -norm is  $\|f\|_A = \sum_n |\hat{f}(n)|$ . The dual of the Banach space  $A = L^1(Z)^\wedge$  is  $PM = L^\infty(Z)^\wedge$ ; each functional  $S \in PM$  is called a *pseudomeasure*. Letting  $(f(x), S)$  or  $(f, S)$  denote the value of  $S$  at  $f$ , we set

$$\hat{S}(n) = \overline{(e^{inx}, S)}; \quad (f, S) = \sum_n \hat{f}(n) \overline{\hat{S}(n)}.$$

The pseudomeasure norm is  $\|S\|_{PM} = \sup_n |\hat{S}(n)|$ .

Let  $C = C(T)$ , the Banach space of the continuous functions on the circle, with the usual norm;  $\|f\|_C \leq \|f\|_A$  if  $f \in A$ . The dual space of  $C$  is  $M = M(T)$ , the space of the finite, regular, complex-valued measures  $\mu$ , with the value of  $\mu$  at  $f$  given by

$$(f, \mu) = \frac{1}{2\pi} \int_0^{2\pi} f(x) d\mu(x)$$

and norm  $\|\mu\|_M$  equal to the total mass. The Fourier-Stieltjes transform of  $\mu \in M$  is the function on  $Z$

$$\hat{\mu}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} d\mu(x).$$

Now  $\mu \in PM$ , with  $\|\mu\|_{PM} \leq \|\mu\|_M$  and

$$(f, \mu) = \sum_n \hat{f}(n) \overline{\hat{\mu}(n)} \quad \text{for } f \in A.$$

The inclusions  $A \subset C$  and  $M \subset PM$  are proper.

Two closed subspaces of  $PM$  are of special interest. One is the space of *pseudofunctions*

$$PF = C_0(Z)^\wedge = \left\{ S \in PM: \lim_{|n| \rightarrow \infty} \hat{S}(n) = 0 \right\};$$

note that the dual of  $PF$  is  $A$ . The other is  $AP = AP(Z)^\wedge$ , consisting of the pseudomeasures  $S$  whose transforms  $\hat{S}$  are almost periodic functions on the integers.  $AP(Z)$  is the closed space generated by the characters  $\{e^{inx}: x \in T\}$  of  $Z$ . For each  $x \in T$ ,  $e^{inx}$  is a character on  $Z$  and is the Fourier-Stieltjes transform of the measure  $\delta_x$  which places mass 1 at  $x$ . Thus  $AP$  contains all the measures with countable support in  $T$ .

*Sets of uniqueness and sets of multiplicity.* (Cf. [7], App. I-IV and Ch. V; and [13], Ch. IX.) For an arbitrary  $S \in PM$ , consider the two series

$$S_1(z) = \sum_{n=0}^{\infty} \hat{S}(n)z^n, S_2 = - \sum_{n=-\infty}^{-1} \hat{S}(n)z^n.$$

The first represents a holomorphic function for  $D_1 = \{|z| < 1\}$ , the second for  $D_2 = \{|z| > 1\}$ . A point  $x \in T$  is called a *regular point* of  $S$  if  $e^{ix}$  has a plane neighborhood  $U$  on which there is a holomorphic function agreeing with  $S_1$  on  $D_1 \cap U$  and with  $S_2$  on  $D_2 \cap U$ . The set of regular points of  $S$ , an open set, is called the *null set* of  $S$ . Its complement is the *support* of  $S$ ; any set containing the support of  $S$  is said to *support*, or *carry*  $S$ .

For a closed set  $E \subset T$ , let

$$\begin{aligned} PM(E) &= \{S \in PM: E \text{ carries } S\}, \\ M(E) &= M \cap PM(E), \\ PF(E) &= PF \cap PM(E). \end{aligned}$$

If  $PF(E) \neq \{0\}$ ,  $E$  is called a set of *multiplicity*; otherwise, a set of *uniqueness*. If  $PF \cap M(E) \neq \{0\}$ ,  $E$  is a set of *multiplicity in the strict sense*; otherwise a set of *uniqueness in the broad sense*. A set of uniqueness in the broad sense may be also a set of multiplicity; for a proof see [10], sections 1 and 3, or [9].

For  $S \in PF$ , a point  $x \in T$  is a regular point if and only if the series  $\sum_{n=-\infty}^{\infty} \hat{S}(n)e^{inx}$  converges to zero throughout a neighborhood of  $x$ . Thus a closed set  $E$  is a set of uniqueness if and only if there exists no nonzero pseudofunction  $S$  such that  $\sum_{n=-\infty}^{\infty} \hat{S}(n)e^{inx}$  converges to zero everywhere in the complement of  $E$ .

*Quotient algebras.* (Cf. [7], Ch. IX, X, XI.) Let  $E$  be a closed subset of  $T$  and let  $I(E)$  be the closed ideal in  $A$  consisting of the functions which vanish on  $E$ . Let  $A(E)$  denote the quotient algebra  $A/I(E)$ , with the usual quotient norm:

$$(2.1) \quad \|f\|_{A(E)} = \inf \{\|f + g\|_A: g \in I(E)\}.$$

We may consider  $A(E)$  as the algebra of restrictions to  $E$  of functions in  $A$ , the *restriction algebra of  $E$* .

The Banach space dual of  $A(E)$  is

$$N(E) = \{S \in PM: (f, S) = 0 \text{ if } f \in I(E)\}.$$

The norm of  $S \in N(E) = A(E)^*$  is precisely the pseudomeasure norm of  $S$ :

$$\|S\|_{N(E)} = \|S\|_{PM} \quad \text{for } S \in N(E).$$

Similarly, let  $C(E)$  be the algebra of restrictions to  $E$  of functions in  $C$ ;

$$\|f\|_{C(E)} = \max \{ \|f(x)\| : x \in E \} \quad \text{for } f \in C(E).$$

The Banach space dual of  $C(E)$  is  $M(E)$ ;

$$\|\mu\|_{M(E)} = \|\mu\|_M \quad \text{for } \mu \in M(E).$$

In general,

$$(2.2) \quad \begin{aligned} A(E) \subset C(E); \quad \|f\|_{C(E)} &\leq \|f\|_{A(E)} \quad \text{if } f \in A(E); \\ M(E) \subset N(E); \quad \|\mu\|_{PM} &\leq \|\mu\|_M \quad \text{if } \mu \in M(E). \end{aligned}$$

The set  $E$  is called a *Helson set* if  $A(E) = C(E)$ , that is, if every continuous function on  $E$  is the restriction to  $E$  of a function in  $A$ . A set  $E$  is a Helson set if and only if there is a constant  $c > 0$  such that

$$\|\mu\|_M \leq c \|\mu\|_{PM} \quad \text{for } \mu \in M(E).$$

The set  $E$  is a set of *synthesis* if  $I(E)$  is the only closed ideal whose hull is  $E$ ; or, equivalently, if  $N(E) = PM(E)$ . This equality does not always hold.

**3. A sketch of the procedure.** Let  $\Phi$  denote an isomorphic mapping of  $A(E)$  into  $A(F)$ . We then have

$$\|\Phi f\|_{A(F)} \leq \|\Phi\| \|f\|_{A(E)} \quad \text{for } f \in A(E).$$

If, as we assume, the functions in the image of  $A(E)$  separate points in  $F$  and do not all vanish at any point of  $F$ , then the mapping  $\Phi$  must arise from a homeomorphism  $\varphi: F \rightarrow E$  by the rule

$$(3.1) \quad \Phi f(x) = f(\varphi(x)) \quad \text{for } x \in F$$

(cf. [8], p. 76). It is evident from (2.1) and (2.2) that for every integer  $n$ , the function  $e^{in\varphi}$  on  $E$  has  $A(E)$ -norm 1. Therefore its image  $e^{in\varphi(x)}$  in  $A(F)$  has  $A(F)$ -norm no greater than  $\|\Phi\|$ . Conversely, for every homeomorphism  $\varphi$  of  $F$  onto  $E$  which is in  $A(F)$ , such that  $\|e^{in\varphi(x)}\|_{A(F)}$  is bounded uniformly in  $n$ , the rule (3.1) defines an isomorphism

$$\Phi: A(F) \rightarrow A(E)$$

with norm  $\|\Phi\| = \sup_n \|e^{in\varphi(x)}\|_{A(F)}$ .

The adjoint map of  $\Phi$ ,

$$\Phi^*: N(F) \rightarrow N(E),$$

is defined by the condition:

$$(f, \Phi^*S) = (\Phi f, S) \quad \text{for } f \in A(E).$$

Our plan is as follows. We shall describe two sets  $E$  and  $F$  in  $[0, 2\pi)$  and a bicontinuous map  $\varphi$  taking  $F$  onto  $E$ . The set  $F$  will be the intersection  $\bigcap_{k=1}^{\infty} F^k$ , where  $F^k$  is the union of  $J(k)$  closed intervals;  $F_k$  will denote the set of left-hand endpoints of these intervals:  $F_k = \{s_1, \dots, s_{J(k)}\}$ . For  $E$ , the sets  $E^k$  and  $E_k = \{r_1, \dots, r_{J(k)}\}$  will be defined similarly. For each  $k$ , the map  $\varphi$  will take  $F_k$  onto  $E_k$ :  $\varphi(s_j) = r_j$  for  $j = 1, \dots, J(k)$ . We shall require that  $\varphi$  preserve arithmetic relations on  $F_k$ ; that is, whenever  $u_1, \dots, u_{J(k)}$  are integers such that  $\sum_{j=1}^{J(k)} u_j s_j = 0$  modulo  $2\pi$ , then also  $\sum_{j=1}^{J(k)} u_j r_j = 0$  modulo  $2\pi$ .

We shall place on  $F$  an "arithmetic thinness" condition, requiring in particular that it be so "close" to its finite subsets  $F_k$  that every  $S \in PM(F)$  is the limit—in the  $A$ , or weak\*, topology of  $PM$ —of a sequence  $\{\mu_k\}$  of measures supported by the finite sets  $F_k$ . The condition will imply that  $F$  is a set of uniqueness.

We shall also place on  $E$  a relatively mild thinness condition.

Since  $\varphi$  is continuous, the map  $\Phi$  takes  $C(E)$  onto  $C(F)$ , and its adjoint  $\Phi^*$  takes  $M(F)$  onto  $M(E)$ —both isometrically. But as we shall show, the conditions placed on  $\varphi$ ,  $F$ , and  $E$  imply that  $\Phi^*$  extends to a continuous map of  $N(F)$  into  $N(E)$ , that  $\varphi \in A(F)$ , and that the norms  $\|e^{in\varphi(x)}\|_{A(F)}$  are bounded uniformly in  $n$ . Consequently  $\Phi$  maps  $A(E)$  isomorphically into  $A(F)$ .

4. Lemmas about finitely supported measures. In the present section we consider the case of a finite set  $F_0 = \{s_j: j = 1, \dots, J\}$  of  $J$  distinct points, and the measures  $\mu \in M(F_0)$ . Let  $\mu$  assign mass  $a_j$  to the point  $s_j$ . The Fourier-Stieltjes transform of  $\mu$  is

$$(4.1) \quad \hat{\mu}(n) = \sum_{j=1}^J a_j \exp(-ins_j).$$

Its supremum is the pseudomeasure norm  $\|\mu\|_{PM}$  of  $\mu$ .

Every function on a finite set  $F_0$  is the restriction to  $F_0$  of a function in  $A$ , which is to say, a finite set is a Helson set; the  $C(F_0)$  and  $A(F_0)$  norms are equivalent, as of course are the  $M(F_0)$  and  $N(F_0)$  norms. The constant of this equivalence depends on the set. For an arithmetic sequence  $\{a + jb: j = 1, \dots, J\}$  ( $b \neq 0$ ), the constant is of the order of  $J^{1/2}$  (cf. [7], Lemma 2, p. 134, or [13], V. 4.7). As we are about to show, it is never greater than  $J^{1/2}$ .

DEFINITION. Let  $B(s_1, \dots, s_J)$  be the smallest constant  $B$  such that

$$\sum_{j=1}^J |a_n| = \|\mu\|_{\mathcal{M}} \leq B \|\mu\|_{P\mathcal{M}} \quad \text{for every } \mu \in M(F_0).$$

LEMMA 1. In every case,  $B(s_1, \dots, s_J) \leq J^{1/2}$ .

*Proof.*

$$\begin{aligned} |\hat{\mu}(n)|^2 &= \sum_{j=1}^J \sum_{i=1}^J a_i \bar{a}_j \exp[in(s_j - s_i)] \\ &= \sum_{j=1}^J |a_j|^2 + \sum_{i \neq j} a_i \bar{a}_j \exp[in(s_j - s_i)]; \\ \|\mu\|_{P\mathcal{M}}^2 &= \sup_n |\hat{\mu}(n)|^2 \geq \lim_{N \rightarrow \infty} (2N+1)^{-1} \sum_{n=-N}^N |\hat{\mu}(n)|^2 \\ &= \sum_{j=1}^J |a_j|^2 \geq J^{-1} \left( \sum_{j=1}^J |a_j| \right)^2, \end{aligned}$$

the last line by the Cauchy inequality. The lemma is proved.

In general,  $B(s_1, \dots, s_J)$  depends upon the nature of the arithmetic relations among the  $s_j$ 's; a *relation* is an equation

$$\left\| \sum_{j=1}^J u_j s_j \right\| = 0$$

where the  $u_j$ 's are integers and  $\|x\|$  denotes the distance from the real number  $x$  to the nearest integral multiple of  $2\pi$ . If there are no relations among the  $s_j$ 's, that is, if they are independent modulo  $2\pi$  over the rationals, then  $B(s_1, \dots, s_J) = 1$ , by Kronecker's Theorem (cf. [7], App. V).

The transform (4.1) is an almost periodic function on the integers: for every  $\varepsilon > 0$ , the integers  $p$  such that

$$(4.2) \quad |\hat{\mu}(m+p) - \hat{\mu}(m)| \leq \varepsilon \|\mu\|_{P\mathcal{M}} \quad \text{for every } m$$

are relatively dense; that is, there is an  $N$  such that every set of  $2N$  consecutive integers contains such a  $p$ . In particular,

$$\max_{|n-m| \leq N} |\hat{\mu}(n)| \geq (1 - \varepsilon) \|\mu\|_{P\mathcal{M}} \quad \text{for every } m.$$

The definition of almost periodicity is customarily stated with just " $\varepsilon$ " on the right-hand side of (4.2). Our version has the feature that the  $N$  depends on  $\varepsilon$  and the set  $F_0$  but not on  $\mu$ . For let  $m$  and  $p$  be integers;

$$\begin{aligned} |\hat{\mu}(m+p) - \hat{\mu}(m)| &= \left| \sum_{j=1}^J a_j [\exp(-i(m+p)s_j) - \exp(-ims_j)] \right| \\ &\leq \left( \sum_{j=1}^J |a_j| \right) \max_{1 \leq j \leq J} |1 - \exp(ips_j)| \\ &\leq B(s_1, \dots, s_J) \|\mu\|_{P\mathcal{M}} \max_{1 \leq j \leq J} |ps_j|. \end{aligned}$$

The solutions  $p$  to the system of inequalities

$$\|ps_j\| < \varepsilon/B, \quad j = 1, \dots, J$$

are relatively dense, and the system does not involve  $\mu$ , so that  $N$  may be selected as claimed. In particular, we have proved:

LEMMA 2. *Given  $\varepsilon > 0$ , there is a number  $N = N(s_1, \dots, s_J; \varepsilon)$  such that for every  $\mu \in M(F_0)$ ,*

$$\max_{|n-m| \leq N} |\hat{\mu}(n)| \geq (1 - \varepsilon) \|\mu\|_{PM} \text{ for every } m.$$

*Note.* There is no bound for  $N$  depending on  $J$  and  $\varepsilon$  alone; the set of points  $\{s_1, \dots, s_J\}$  is critical.

Any two finite sets with the same number of points have isomorphic restriction algebras. Let

$$F_0 = \{s_j: j = 1, \dots, J\}, \quad E_2 = \{r_j: j = 1, \dots, J\}, \\ \varphi(s_j) = r_j.$$

Then  $\varphi$  maps  $F_0$  onto  $E_0$  and induces an isomorphism  $\Phi$  of  $A(E_0)$  onto  $A(F_0)$  as in (3.1). For  $\mu \in M(F_0)$  let  $\mu^*$  denote  $\Phi^*\mu$ , which is the measure on  $E_0$  such that

$$\mu^*(r_j) = \mu(s_j).$$

The norm of  $\Phi^*$  is the supremum of the ratio  $\|\mu^*\|_{PM}/\|\mu\|_{PM}$  for  $\mu \in M(F_0)$ . We know that this ratio is bounded by  $J^{1/2}$ , because

$$\|\mu^*\|_{PM} \leq \|\mu^*\|_M, \|\mu^*\|_M = \|\mu\|_M$$

by the definition of  $\mu^*$ , and  $\|\mu\|_M \leq J^{1/2} \|\mu\|_{PM}$  by Lemma 1.

LEMMA 3. *If  $\varphi$  preserves arithmetic relations on the set  $\{s_1, \dots, s_J\}$ , so that*

$$(4.3) \quad \left\| \sum_{j=1}^J u_j s_j \right\| = 0 \Rightarrow \left\| \sum_{j=1}^J u_j r_j \right\| = 0$$

*for all integral  $(u_1, \dots, u_J)$ , then the range of  $\hat{\mu}$  is dense in that of  $\hat{\mu}^*$ . In particular,*

$$\|\mu^*\|_{PM} \leq \|\mu\|_{PM} \text{ for } \mu \in M(F_0).$$

*Proof.* By Kronecker's Theorem (cf. [3], p. 53 or p. 99) we know that the condition (4.3) insures that for every  $\varepsilon$  and  $m$ , the inequalities

$$\|ns_j - mr_j\| < \varepsilon, \quad j = 1, \dots, J$$



can always be solved simultaneously for  $n$ . Since

$$\begin{aligned} |\hat{\mu}(n) - \hat{\mu}^{\sharp}(m)| &= \left| \left| \sum_{j=1}^J \mu(s_j) [\exp(-ins_j) - \exp(imr_j)] \right| \right| \\ &\leq \|\mu\|_{\mathbf{M}} \cdot \max_{1 \leq j \leq J} \|ns_j - mr_j\|, \end{aligned}$$

the lemma follows.

REMARK. We should prefer a weaker, but still convenient, hypothesis in Lemma 3, giving the weaker conclusion that for some  $c \geq 1$ ,

$$(4.4) \qquad \qquad \qquad \|\mu^{\sharp}\|_{P\mathbf{M}} \leq c \|\mu\|_{P\mathbf{M}} \quad \text{for} \quad \mu \in M(F_0)$$

—where both the hypothesis and the constant  $c$  are independent of  $J$ . For example, perhaps it is true that if (4.3) is required to hold only for those integral  $(u_1, \dots, u_J)$  with  $|u_j| \leq 2$  (or some other bound), then (4.4) follows for some  $c$ . We also should like to have estimates of the function  $N$ , in Lemma 2, better than those provided by the methods of Diophantine approximation theory. But we leave these questions unanswered.

5. Construction of  $E, F$ , and  $\varphi$ . We shall now give in detail our conventions for describing the sets  $E$  and  $F$  and the map  $\varphi: F \rightarrow E$  which were discussed in § 3. We shall describe closed perfect subsets  $E$  and  $F$  of the interval  $[0, 2\pi)$ , and a homeomorphism  $\varphi$  mapping  $F$  onto  $E$ .

Let  $F = \bigcup_{k=1}^{\infty} F^k$ , where  $F^k$  is the union of  $J(k)$  pairwise disjoint closed intervals, each with length  $d_k > 0$ . We assume once and for all that

$$(5.1) \qquad \qquad \qquad \lim_{k \rightarrow \infty} J(k) = \infty, \qquad \lim_{k \rightarrow \infty} J(k)d_k = 0.$$

Let  $F^k_k$  denote the set of the left-hand endpoints  $s_j$  of the intervals making up  $F^k$ :

$$F^k_k = \{s_1, \dots, s_{J(k)}\}.$$

Thus  $F^k$  is determined by the selection of the set  $F^k_k$  and the number  $d_k$ . In making this selection, we require that

$$s_1 < s_2 < \dots < s_{J(1)};$$

and that for  $k \geq 2$ ,

$$F_{k-1} \subset F_k = \{s_1, \dots, s_{J(k-1)}, \dots, s_{J(k)}\};$$

$$s_{J(k-1)+1} < \dots < s_{J(k)};$$

and

$$F^k \subset F^{k-1}.$$

Thus every point  $s_j$  in  $F_k$  but not in  $F_{k-1}$  ( $J(k-1) < j \leq J(k)$ ) lies in the interval  $[s_i, s_i + d_{k-1} - d_k]$  for some  $s_i \in F_{k-1}$  ( $1 \leq i \leq J(k-1)$ ).

We further require that for every  $k$ , the points of  $F_k$  are at least  $2d_k$  apart, modulo  $2\pi$ . Thus not only are the intervals of  $F^k$  disjoint; but also, each of the intervals contiguous to  $F^k$  in  $[0, 2\pi]$  has length no less than  $d_k$ .

Now let  $E$  be a set constructed in the same manner, except with different choices of the numbers  $d_k$  and the sets of endpoints, and with different notation, as follows:  $E = \bigcap_{k=1}^{\infty} E^k$ , where  $E^k$  is the union of  $J(k)$  intervals with length  $d'_k$  and left-hand endpoints  $r_j$ ;  $E_k = \{r_1, \dots, r_{J(k)}\}$ . We will again have

$$(5.2) \quad \lim_{k \rightarrow \infty} J(k)d'_k = 0.$$

We shall place the points of  $E_k$  in correspondence with those of  $F_k$ , in the following sense: for  $k \geq 2$ , we select the points  $r_j$  for  $J(k-1) < j \leq J(k)$  in such a way that, for each  $i = 1, \dots, J(k-1)$ , the number of these  $r_j$ 's placed in the interval  $[r_i, r_i + d'_{k-1} - d'_k]$  equals the number of  $s_j$ 's (with  $J(k-1) < j \leq J(k)$ ) appearing in the interval  $[s_i, s_i + d_{k-1} - d_k]$ .

For each  $k$ , let  $\varphi_k$  be the continuous increasing function which maps  $[0, 2\pi]$  onto itself such that

$$\varphi_k(0) = 0, \quad \varphi_k(s_j) = r_j \quad (1 \leq j \leq J(k)), \quad \varphi_k(2\pi) = 2\pi;$$

and which is linear on each interval contiguous to the set  $\{0, s_1, \dots, s_{J(k)}, 2\pi\}$ . By (5.1) and (5.2), the sequences  $\{\varphi_k\}$  and  $\{\varphi_k^{-1}\}$  converge uniformly as  $k \rightarrow \infty$  to functions  $\varphi$  and  $\varphi^{-1}$  respectively; which then must be continuous, each the inverse of the other. Therefore  $\varphi$  maps  $F$  homeomorphically onto  $E$ .

**6. Approximating pseudomeasures by finitely supported measures.**

**LEMMA 4.** *Let  $F$  be the set constructed in § 5. By a method to be explained below, it is possible to associate with each  $S \in PM(F)$  a sequence of measures  $\mu_k \in M(F_k)$ , such that*

$$(6.1) \quad |\hat{S}(n) - \hat{\mu}_k(n)| \leq |n|(J(k)d_k)^{1/2} \|S\|_{P_M} \quad \text{for all } k, n.$$

*In particular, by (5.1),*

$$(6.2) \quad \lim_{k \rightarrow \infty} \hat{\mu}_k(n) = \hat{S}(n) \quad \text{for all } n .$$

*Proof.* We shall follow Kahane and Salem ([7], p. 126). For each  $k$ ,  $F^k$  is the union of  $J(k)$  closed intervals. Let us give them names and enumerate from left to right:

$$I_1, I_2, \dots, I_{J(k)} .$$

Without loss of generality we may assume 0 to be the left-hand end-point of  $I_1$ . Then the interval  $[0, 2\pi]$  is the union of the sets

$$I_1, I'_1, I_2, I'_2, \dots, I_{J(k)}, I'_{J(k)}$$

where  $I'_1, \dots, I'_{J(k)}$  are the intervals contiguous to  $F^k$  in  $[0, 2\pi]$ , listed from left to right.

Let  $S \in PM(F)$  with  $\hat{S}(0) = 0$ . The formal integral of  $S$  is the  $L^2$  function

$$\sigma(x) \sim \sum_{n \neq 0} \frac{\hat{S}(n)}{in} e^{inx}$$

with norm

$$(6.3) \quad \|\sigma\|_2 \leq \left( \sum_{n \neq 0} n^{-2} \right)^{1/2} \|S\|_{PM} .$$

The function  $\sigma(x)$  will be constant on each interval  $I'_j$ . Let  $\sigma_k(x)$  be the step function which on  $I_j \cup I'_j$  has the same constant value that  $\sigma(x)$  has on  $I'_j$ . In § 5 we stipulated that each  $I'_j$  must have length no less than the length of  $I_j$ , which is  $d_k$ . Therefore

$$\int_{F^k} |\sigma_k(x)| = \int_{F^k} |\sigma(x + d_k)| ,$$

and hence both the quantities  $\int_{F^k} |\sigma_k(x)|$  and  $\int_{F^k} |\sigma(x)|$  are majorized by  $(J(k)d_k)^{1/2} \|\sigma\|_2$ . The measure  $\mu_k = d\sigma_k$  is supported by the finite set  $F^k$ , and

$$\hat{S}(n) - \hat{\mu}_k(n) = \frac{in}{2\pi} \int_0^{2\pi} [\sigma(x) - \sigma_k(x)] e^{-inx} dx .$$

Since the integrand is zero on the complement of  $F^k$ , we have

$$\begin{aligned} |\hat{S}(n) - \hat{\mu}_k(n)| &\leq \frac{|n|}{2\pi} \left( \int_{F^k} |\sigma(x)| + \int_{F^k} |\sigma_k(x)| \right) \\ &\leq \frac{|n|}{\pi} (J(k)d_k)^{1/2} \|\sigma\|_2 , \end{aligned}$$

which with (6.3) implies (6.1).

If  $\hat{S}(0) \neq 0$ , let  $x$  be a point in  $F_1$  (and hence in every  $F_k$ ), and consider  $T = S - \hat{S}(0)\delta_x$  instead of  $S$ . Then  $\hat{T}(n) = \hat{S}(n) - \hat{S}(0)e^{-in x}$ ,  $\hat{T}(0) = 0$ . Associate  $\mu'_k$  with  $T$  by the above process; (6.1) will then hold for  $S$  if we take  $\mu_k = \mu'_k + \hat{S}(0)\delta_x$ . The proof of Lemma 4 is complete.

**7. A thinness condition for the set  $F$ .** We shall now make use of Lemma 4 to study the implications of a certain thinness requirement, which we call

*Condition I.*

$$\lim_{k \rightarrow \infty} (J(k)d_k)^{1/2} N(s_1, \dots, s_{J(k)}; \alpha) = 0,$$

where  $0 < \alpha < 1$ , and where  $N$  is the function of Lemma 2. Condition I may be enforced in the construction of the set  $F$  without restricting the quantity of arithmetic relations among the points  $\{s_j\}$ , since at each step,  $d_k$  may be chosen after  $N_k$  is evaluated. Let us illustrate that Condition I does not imply that  $F$  is a Helson set. Let  $\{p_k\}$  be a positive sequence,  $\sum_{k=1}^{\infty} p_k < 1$ , and consider the set consisting of the sums  $\{\sum_{k=1}^{\infty} \varepsilon_k p_k; \varepsilon_k = 0 \text{ or } 1\}$ . Such a set is called a *symmetric set*. By replacing  $\{p_k\}$  with a subsequence tending to zero fast enough, we obtain a set satisfying Condition I. But no symmetric set can be a Helson set (cf. [7], Ch. XI, Th. VIII).

**THEOREM 1.** *Let  $F$  be a set constructed as in §5, obeying Condition I. If  $S \in PM(F)$  and  $\{\mu_k\}$  is the sequence associated with  $S$  as in Lemma 4, then*

$$(7.1) \quad \limsup_{k \rightarrow \infty} \|\mu_k\|_{PM} \leq (1 - \alpha)^{-1} \|S\|_{PM}.$$

Also,

$$(7.2) \quad \limsup_{|n| \rightarrow \infty} |\hat{S}(n)| \geq (1 - \alpha) \|S\|_{PM} \text{ for every } S \in PM(F).$$

*Proof.* For convenience let us write

$$N_k = N(s_1, \dots, s_{J(k)}; \alpha); \\ \varepsilon_k = N_k (J(k)d_k)^{1/2}.$$

Then by (6.1),

$$(7.3) \quad |\hat{S}(n) - \hat{\mu}_k(n)| \leq \varepsilon_k |n| N_k^{-1} \|S\|_{PM} \text{ for all } k, n;$$

and by Condition I,  $\lim_{k \rightarrow \infty} \varepsilon_k = 0$ . By the definition of  $N_k$ , there is an  $n_0$  such that  $|n_0| \leq N_k$  and

$$\begin{aligned} \|\mu_k\|_{PM}(1-\alpha) &\leq |\hat{\mu}_k(n_0)| \leq |\hat{S}(n_0)| + \varepsilon_k \|S\|_{PM} \\ &\leq (1 + \varepsilon_k) \|S\|_{PM}; \end{aligned}$$

(7.1) follows.

Let  $\eta > 0$  and pick  $m_0$  such that  $|\hat{S}(m_0)| \geq \|S\|_{PM}(1-\eta)$ . Let  $k$  be large enough so that  $|m_0| \leq N_k$ . There is an  $n_k$  (cf. Lemma 2) between, say,  $7N_k$  and  $9N_k$  such that  $|\hat{\mu}_k(n_k)| \geq (1-\alpha)\|\mu_k\|_{PM}$ . So:

$$|\hat{S}(n_k)| \geq |\hat{\mu}_k(n_k)| - 9\varepsilon_k \|S\|_{PM};$$

but

$$\begin{aligned} |\hat{\mu}_k(n_k)| &\geq (1-\alpha)\|\mu_k\|_{PM} \geq (1-\alpha)|\hat{\mu}_k(m_0)| \\ &\geq (1-\alpha)(|\hat{S}(m_0)| - \varepsilon_k \|S\|_{PM}) \\ &\geq (1-\alpha)\|S\|_{PM}(1-\eta - \varepsilon_k). \end{aligned}$$

So

$$|\hat{S}(n_k)| > \|S\|_{PM}[(1-\alpha)(1-\eta - \varepsilon_k) - 9\varepsilon_k].$$

Since  $n_k \geq 7N_k$  we know  $\lim_{k \rightarrow \infty} n_k = \infty$ . Therefore

$$\limsup_{|n| \rightarrow \infty} |\hat{S}(n)| \geq \|S\|_{PM}(1-\alpha)(1-\eta),$$

where  $\eta$  is arbitrary; (7.2) follows, and the theorem is proved.

By Theorem 1, Condition I has several important consequences for the set  $F$ , which we now list as corollaries.

**COROLLARY 1.** *For each  $S \in PM(F)$ , the associated sequence  $\{\mu_k\}$  converges to  $S$  in the  $A$  topology of  $PM$ .*

*Proof.* This result is evident from (6.2) and (7.1).

**COROLLARY 2.** *The set  $F$  is a set of synthesis.*

*Proof.* We need to show that  $PM(F) = N(F)$ . Let  $S \in PM(F)$ . Each  $\mu_k$  is in  $N(F)$ , that is,  $(f, \mu_k) = 0$  for every  $f \in I(F)$ . But  $(f, S) = \lim_{k \rightarrow \infty} (f, \mu_k)$  for every  $f \in A$ , by Corollary 1. Therefore  $(f, S) = 0$  for every  $f \in I(F)$ , so  $S \in N(F)$ .

**COROLLARY 3.** *The set  $F$  is a set of uniqueness.*

*Proof.* The result (7.2) easily implies that

$$\limsup_{|n| \rightarrow \infty} |\hat{S}(n)| > 0 \quad \text{for every } S \in PM(F).$$

**COROLLARY 4.** *If the sequence  $\{B(s_1, \dots, s_{J(k)}): k = 1, 2, \dots\}$ ,*

where  $B$  is the function of Lemma 1, is bounded, then  $F$  is a Helson set.

*Proof.* In this case (7.1) implies that the sequence  $\{\|\mu_k\|_M\}$  is bounded. This fact, together with (6.2) or Corollary 1 proves that  $\{\mu_k\}$  converges in the  $C$  topology of  $M$ . It must converge to some  $\mu \in M$ , and  $\mu = S$ ; thus  $PM(F) = M(F)$  and  $F$  is a Helson set. This result is due to Kahane and Salem ([7], p. 126).

COROLLARY 5. *If Condition I holds for every  $\alpha > 0$ , then*

$$\lim_{k \rightarrow \infty} \|\mu_k\|_{PM} = \|S\|_{PM};$$

$$\limsup_{|n| \rightarrow \infty} |\hat{S}(n)| = \|S\|_{PM} \text{ for every } S \in PM(F).$$

*Proof.* Statements (7.1) and (7.2) hold for every  $\alpha > 0$ .

REMARK. Let  $B$  be a Banach space,  $B^*$  the dual space,  $\Gamma$  a subspace of  $B^*$ ; and for  $f \in B$  define

$$\|f\|_1 = \sup \left\{ \frac{|(f, g)|}{\|g\|_{B^*}} : g \in \Gamma, g \neq 0 \right\}.$$

If this norm is equivalent to the  $B$  norm in  $B$ , then of course  $\Gamma$  is  $B$ -dense in  $B^*$ , but as Dixmier [5] pointed out, the converse is false. An illustration of this fact is provided by the set  $F$  constructed by Rudin ([12], or [7], p. 103), which is not a Helson set but which has  $\|\mu\|_M = \|\mu\|_{PM}$  for all those  $\mu \in M(F)$  which have finite support. The space  $\Gamma$  consisting of these measures is  $A(F)$ -dense in  $N(F)$  (as it is for arbitrary  $F$ ), but the  $A(F)$  norm is not equivalent to the norm  $\|f\|_1$ , which in this case equals the  $C(F)$  norm.

In the case of the set  $F$  of Theorem 1, however, the finitely supported measures are  $A(F)$ -sequentially dense in  $N(F)$  and

$$\|f\|_1 \geq \|f\|_{A(F)}(1 - \alpha) \text{ for } f \in A(F).$$

Even these conditions do not reflect the full strength of the approximation of  $S \in N(F)$  by the sequence  $\{\mu_k\}$ ; for we have the further fact that  $\hat{S}$  is well approximated by  $\hat{\mu}_k$  throughout an almost-period of  $\hat{\mu}_k$ .

8. An isomorphism of  $A(E)$  into  $A(F)$ . To establish the isomorphism, we shall place the following three requirements on the set  $F$ , the set  $E$ , and the mapping  $\varphi$ , respectively:

*Condition I.*

$$\lim_{k \rightarrow \infty} (J(k)d_k)^{1/2} N(s_1, \dots, s_{J(k)}; \alpha) = 0 ,$$

where  $0 < \alpha < 1$ , and where  $N$  is the function of Lemma 2;

*Condition II.*

$$\sum_{k=1}^{\infty} d'_k B(r_1, \dots, r_{J(k+1)}) < \infty ,$$

where  $B$  is the function of Lemma 1; and

*Condition III.*

$$\left\| \sum_{j=1}^{J(k)} u_j s_j \right\| = 0 \Rightarrow \left\| \sum_{j=1}^{J(k)} u_j r_j \right\| = 0$$

for all integers  $u_1, \dots, u_{J(k)}$  and for every  $k$ .

Condition II is a relatively mild requirement. By Lemma 1, it holds if

$$\sum_{k=1}^{\infty} d'_k J(k+1)^{1/2} < \infty .$$

It is satisfied, for example, by a symmetric set of constant ratio  $\xi < 1/2$ :

$$\left\{ (1 - \xi)^{\xi^{-1}} \sum_{k=1}^{\infty} \varepsilon_k \xi^k : \varepsilon_k = 0 \text{ or } 1 \text{ for each } k \right\} .$$

To describe this set we may take  $J(k) = 2^k, d'_k = \xi^k$ .

**THEOREM 2.** *Let the sets  $F$  and  $E$  and the mapping  $\varphi$ , constructed as in § 5, obey Conditions I, II, and III, respectively. Then by the rule (3.1), the mapping  $\varphi$  induces the isomorphism  $\Phi$  of  $A(E)$  into  $A(F)$ , with the norm no greater than  $(1 - \alpha)^{-1}$ . If Condition I holds for every  $\alpha > 0$ , the isomorphism is norm-decreasing.*

*Proof.* Using (3.1) for  $f \in C(E)$ , we see that the homeomorphism  $\varphi$  of  $F$  onto  $E$  induces the isometric isomorphisms

$$\begin{aligned} \Phi: C(E) &\rightarrow C(F) ; \\ \Phi^*: M(F) &\rightarrow M(E) . \end{aligned} \tag{8.1}$$

By Lemma 3, Condition III implies that the restrictions of  $\Phi^*$  to measures on the sets of endpoints,

$$\Phi^*: M(F_k) \rightarrow M(E_k) , \quad k = 1, 2, \dots ,$$

are continuous with respect to the pseudomeasure norms; in fact

$$(8.2) \quad \|\mu^*\|_{PM} \leq \|\mu\|_{PM} \quad \text{for } \mu \in M(F_k), \quad k = 1, 2, \dots,$$

where  $\mu^*$  denotes  $\Phi^*\mu$ . We shall now show that if  $S \in N(F)$ , and  $\{\mu_k\}$  is the sequence associated with  $S$  by Lemma 4, then Condition II implies that the sequence  $\{\hat{\mu}_k^*(m): k = 1, 2, \dots\}$  is a Cauchy sequence for every  $m$ ; and we shall then define  $S^* = \Phi^*S$  by the conditions  $\hat{S}^*(n) = \lim_{k \rightarrow \infty} \hat{\mu}_k^*(n)$ . Let

$$a_j = \mu_k^*(r_j) = \mu_k(s_j), \quad \text{for } 1 \leq j \leq J(k).$$

Similarly, let

$$b_i = \mu_{k+1}^*(r_i) = \mu_{k+1}(s_i), \quad \text{for } 1 \leq i \leq J(k+1).$$

Then

$$\begin{aligned} \hat{\mu}_k^*(m) &= \sum_{j=1}^{J(k)} a_j \exp(-imr_j); \\ \hat{\mu}_{k+1}^*(m) &= \sum_{i=1}^{J(k+1)} b_i \exp(-imr_i) \\ &= \sum_{j=1}^{J(k)} \sum \{b_i \exp(-imr_i): r_i - r_j < d'_k\} \\ &= \sum_{j=1}^{J(k)} \sum \{b_i [\exp(-imr_j) + \exp(-imr_i) \\ &\quad - \exp(-imr_j)]: r_i - r_j < d'_k\} \\ &= \hat{\mu}_k^*(m) + \sum_{j=1}^{J(k)} \sum \{b_i [\exp(-imr_i) \\ &\quad - \exp(-imr_j)]: r_i - r_j < d'_k\}, \end{aligned}$$

since

$$a_j = \sum \{b_i: r_i - r_j < d'_k\}$$

by the definition of  $\mu_k$  and  $\mu_{k+1}$ . Therefore

$$\begin{aligned} |\hat{\mu}_{k+1}^*(m) - \hat{\mu}_k^*(m)| &\leq \min \{2, |m| d'_k\} \sum_{i=1}^{J(k+1)} |b_i| \\ &= \mathcal{O}(d'_k B(r_1, \dots, r_{J(k+1)})) \quad \text{as } k \rightarrow \infty \end{aligned}$$

because

$$\sum_{i=1}^{J(k+1)} |b_i| = \|\mu_{k+1}^*\|_M \leq B(r_1, \dots, r_{J(k+1)}) \|\mu_{k+1}^*\|_{PM}$$

and  $\|\mu_k^*\|_{PM} = \mathcal{O}(\|S\|_{PM})$  by (8.2) and (7.1). Therefore, Condition II on the set  $E$  implies that  $\{\hat{\mu}_k^*(m): k=1, 2, \dots\}$  is a Cauchy sequence for each  $m$ . Let  $\hat{S}^*(m)$  be its limit. Then



$$|\hat{S}^*(m)| = \left| \lim_{k \rightarrow \infty} \hat{\mu}_k^*(m) \right| \leq (1 - \alpha)^{-1} \|S\|_{PM};$$

thus  $\|S^*\|_{PM} \leq (1 - \alpha)^{-1} \|S\|_{PM}.$

Since  $S^* = \lim_{k \rightarrow \infty} \mu_k^*$  in the  $A$  topology of  $PM$ , we know  $S^* \in N(E)$ . The map  $S \rightarrow S^*$  is an extension of (8.1) to a continuous map of  $N(F)$  into  $N(E)$ , with norm no greater than  $(1 - \alpha)^{-1}$ .

To show that  $\varphi$  induces an isomorphism of  $A(E)$  into  $A(F)$ , it suffices to show that  $e^{i\varphi} \in A(F)$ . For then

$$(S, e^{i\varphi}) = \lim_{k \rightarrow \infty} (\mu_k, e^{i\varphi}) = \hat{S}^*(m)$$

and hence

$$\begin{aligned} |(S, e^{i\varphi})| &\leq (1 - \alpha)^{-1} \|S\|_{PM} \quad \text{for all } S \in N(F), \\ \text{so } \|e^{i\varphi}\|_{A(F)} &\leq (1 - \alpha)^{-1} \quad \text{for all } m. \end{aligned}$$

We already know that  $\varphi$  induces a continuous linear function  $G$  on  $N(F)$ :

$$(8.3) \quad G(S) = \hat{S}^*(1) = \lim_{k \rightarrow \infty} (\mu_k, e^{i\varphi}).$$

Since  $A(F)$  is total over  $N(F)$ ,  $G \in A(F)$  if and only if  $G$  is continuous in the  $A(F)$  topology of  $N(F)$  ([6], V. 3.11). But  $G$  is  $A(F)$ -continuous if and only if it is continuous in the relative  $A(F)$  topology of the ball  $\{S: \|S\|_{PM} \leq a\}$  for every  $a > 0$  ([6], V. 5.6). Therefore it suffices to show that for arbitrary  $a$  and  $\varepsilon$ , there exist  $N$  and  $\eta > 0$  such that:

$$(8.4) \quad \begin{aligned} \|S\|_{PM} \leq a \quad \text{and} \quad |\hat{S}(n)| < \eta \quad \text{for } |n| \leq N \\ \Rightarrow |G(S)| < \varepsilon. \end{aligned}$$

If  $\|S\|_{PM} \leq a$ , then by (8.2) and the definition of  $N_k$ ,

$$\begin{aligned} |(\mu_k, e^{i\varphi})| &= |\hat{\mu}_k^*(1)| \leq \|\mu_k^*\|_{PM} \leq \|\mu_k\|_{PM} \\ &\leq (1 - \alpha)^{-1} \max_{|n| \leq N_k} |\hat{\mu}_k(n)|, \end{aligned}$$

which by (7.3) is

$$\leq (1 - \alpha)^{-1} \left[ \max_{|n| \leq N_k} |\hat{S}(n)| + \varepsilon_k a \right];$$

so by (8.3),

$$|G(S)| \leq \varepsilon/2 + (1 - \alpha)^{-1} \max_{|n| \leq N_k} |\hat{S}(n)|$$

for  $k$  large enough; and if  $N = N_k$  and  $\eta \leq \varepsilon(1 - \alpha)/2$ , then (8.4) follows. The theorem is proved.

REMARK. For the extension of  $\varphi^*$  to a continuous map on  $N(F)$ , it would suffice to have

$$(8.5) \quad \|\mu^*\|_{PM} \leq c \|\mu\|_{PM} \quad \text{for } \mu \in M(F_k), \quad k = 1, 2, \dots,$$

for some  $c \geq 1$ . Condition III seems too strong, since it gives not only (8.5) with  $c = 1$ , but much more, by Lemma 3. But we prefer to state the theorem using Condition III, because it gives an explicit sufficient condition on the selection of the points  $\{r_j\}$  and  $\{s_j\}$ ; and we do not know of any *essentially* weaker condition that will yield (8.5).

9. Examples. To obtain an isomorphism of  $A(E)$  and  $A(F)$ , we apply Theorem 2 twice, requiring that the triple  $E, F, \varphi^{-1}$ , as well as  $F, E, \varphi$ , obey requirements analogous to Conditions I, II, and III, respectively. Then  $\varphi^{-1}$  will induce  $\varphi^{-1}$ , whose norm will not exceed  $(1 - \alpha')^{-1}$ , say. If Condition I holds on  $F$  and  $E$  for every positive  $\alpha$  and  $\alpha'$ , respectively, then  $A(E)$  and  $A(F)$  will be isometrically isomorphic.

Let us point out an example. For  $i = 1$  and 2, let  $G_i$  be the symmetric set  $\{\sum_{k=1}^{\infty} \varepsilon_k \xi_k^{(i)} : \varepsilon_k = 0 \text{ or } 1\}$ , where  $\{\xi_k^{(i)}\}$  is a sequence of numbers independent over the rationals. If  $\xi_k^{(i)} \rightarrow 0$  fast enough, then  $A(G_1)$  and  $A(G_2)$  are isomorphic. For instance  $\{\xi_k^{(i)}\}$  could be a sequence  $\{\eta_i^{p(k)}\}$  of powers of a transcendental number  $\eta_i$ .

The arguments for Theorems 1 and 2 may be modified to deal with many sets not of the simple, convenient type described in § 5. For example, we may allow each  $E^k$  (and  $F^k$ ) to be made up of intervals of various lengths, with  $d'_k$  (and  $d_k$ , respectively) as a bound rather than as the common value.

There exists a set  $E$  with the following properties: (1) except for the variation just mentioned,  $E$  is of the type described in § 5, with  $J(k) = 2^k$ , such that (2)  $E$  satisfies Condition II; (3) the points of  $E$  are linearly independent over the rationals; and (4)  $E$  is a set of multiplicity in the strict sense (and hence not a Helson set—cf. [7], Ch. XI, Theorem V). Rudin ([12]; cf. also [7], p. 103) constructed a set with properties (3) and (4), and (1) and (2) are easily assured in his procedure. Let  $F$  be constructed as in § 5, such that Condition I is satisfied,  $J(k) = 2^k$ , and the sequence  $\{s_1, s_2, \dots\}$  is independent over the rationals. Then since  $B(s_1, \dots, s_{J(k)}) = 1$  for every  $k$ ,  $F$  is a Helson set by Theorem 2, Corollary 4 (and hence a set of uniqueness in the broad sense—cf. [7], Ch. XI, Theorem V). Let  $E$  be the set of Rudin just described, and define  $\varphi: F \rightarrow E$  in the manner of § 5, taking  $\varphi(s_j) = r_j$ . Since both  $\{s_j\}$  and  $\{r_j\}$  are independent, Condition III is satisfied and by Theorem 1,  $\varphi$  is an isomorphism of  $A(E)$  into  $A(F) \cong C(F)$ . The map cannot be surjective, for then  $E$  would be a

Helson set. The map  $\varphi^*$  maps  $N(F) \cong M(F)$  continuously into  $N(E)$ , and *onto*  $M(E)$ . It is notable that  $\varphi^*$  thus must map some measures which are not pseudofunctions into the nonempty class  $M(E) \cap PF$ .

**10. Some questions.** We say that  $\varphi \in A(F)$  is *trivial* if near each point of  $F$ ,  $\varphi(x) = rx + x_0$  for some real  $r$  and  $x_0$ . No example is known of a nontrivial  $\varphi \in A(F)$ , taking  $F$  into the circle, with  $\sup_n \|e^{in\varphi}\|_{A(F)} < \infty$ , where  $F$  is a set of multiplicity.

Consider the sets

$$(10.1) \quad E\{t_j\} = \left\{ \sum_{j=1}^{\infty} x_j t_j : x_j = 0 \text{ or } 1 \right\},$$

where  $t_j \rightarrow 0$  as  $j \rightarrow \infty$ . Perhaps it is the case that whenever  $t_j \rightarrow 0$  and  $t'_j \rightarrow 0$  fast enough (in some sense that disregards arithmetic properties of the sequences), then the sets  $E\{t_j\}$  and  $E\{t'_j\}$  have isomorphic restriction algebras.

Consider the compact group  $X$  which is the complete direct sum of a countably infinite number of copies of the group  $\{0, 1\}$  under addition modulo 2. The elements of  $X$  are the sequences

$$\{(x_1, x_2, \dots) : x_j = 0 \text{ or } 1\}.$$

Let  $Y$  be the dual group of  $X$ , and let  $A(X)$  be the Gel'fand representation of  $L^1(Y)$ . When, if ever, is the restriction algebra of a set (10.1) isomorphic to  $A(X)$ ?

*Added in proof:* H. P. Rosenthal (cf. § 1, *Projections onto translation-invariant subspaces of  $L^p(G)$* , *Memoirs of the A. M. S.* No. 63, 1966) has shown that such an isomorphism never occurs.

Consider the quantity

$$|||f||| = \sup \left\{ \frac{|(f, \mu)|}{\|\mu\|_{PM}} : \mu \in M(E), \mu \neq 0 \right\}.$$

If  $f \in A(E)$ , of course,  $|||f||| \leq \|f\|_{A(E)}$ . How can we characterize the sets  $E$  which have the property that

$$(10.2) \quad |||f||| < \infty \implies f \in A(E)$$

whenever  $f \in C(E)$ ? Only recently, Katznelson constructed a set for which this implication fails. We shall here establish a sufficient condition for (10.2) to hold. The ideas are essentially those of the de Leeuw and Katznelson [4] and Kreĭn ([1], § 77); Kreĭn proved (10.2) in the case when  $E$  is an interval. Let  $f \in C(E)$  and suppose  $|||f|||$  is finite. Then  $f$  provides a bounded linear functional on  $M(E)$  taken as

a subspace of  $PM$ . Let  $g \in PM^*$  be an extension of  $f$  with norm  $\|g\| = \|f\|$ . Then  $g$  may be decomposed,  $g = g_1 + g_2$  where  $g_1 \in A$ ,  $g_2 \in PF^\perp$ , and  $\|f\| = \|g\| = \|g_1\| + \|g_2\|$ . Since  $g_2 \in PF^\perp$ , it has the property that

$$|(g_2, S)| \leq \|g_2\| \cdot \limsup_{|n| \rightarrow \infty} |\hat{S}(n)| \quad \text{for all } S \in PM.$$

Since clearly  $\|g_i\| \leq \|g_i\|$  for  $i = 1$  and  $2$ , and  $\|f\| = \|g_1\| + \|g_2\|$ , it follows that  $\|g_2\| = \|g_2\| = \|f - g_1\|$ . To establish the implication (10.2), it suffices to show that always  $g_2 = 0$ . The situation is as follows:

$$(10.3) \quad \begin{aligned} (f - g_1 - g_2, \mu) &= 0 \quad \text{for } \mu \in M(E); \\ \|f - g_1\| &= \|g_2\|; \\ |(f - g_1, \mu)| &\leq \|f - g_1\| \cdot \limsup_{|n| \rightarrow \infty} |\hat{\mu}(n)| \quad \text{for all } \mu \in M(E). \end{aligned}$$

It follows that if every portion of the set  $E$  is a set of multiplicity in the strict sense, and thus supports a nonzero, positive measure  $\mu \in PF$ , then (10.2) holds. For if  $f - g_1 \neq 0$ , then  $(f - g_1, \mu)$  would have to be nonzero for some  $\mu \in M(E) \cap PF$ —impossible, by (10.3). More generally, if for some  $\eta > 0$ ,  $M(E)$  contains enough measures  $\mu$  with

$$\|\mu\|_{PM} = 1 \quad \text{and} \quad \limsup_{|n| \rightarrow \infty} |\hat{\mu}(n)| \leq 1 - \eta$$

to insure that  $\|f\|$  equals the supremum of  $|(f, \mu)|$  over such  $\mu$ , then (10.3) gives a contradiction unless  $g_2 = 0$ , so that (10.2) must hold. It can be shown that this more general hypothesis is satisfied by the Cantor set.

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