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**ON ASYMPTOTIC ESTIMATES FOR KERNELS OF
CONVOLUTION TRANSFORMS**

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In this paper we shall try to answer two open questions posed by Dauns and Widder in their paper "Convolution transforms whose inversion functions have complex roots" (*Pacific Journal of Mathematics*, 1965, Volume 15(2), pp. 427-442) on page 441.

We shall be interested in the function $G_{2m}(t)$ defined by

$$(1.1) \quad G_{2m}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{st} ds}{E_{2m}(s)}$$

where

$$(1.2) \quad E_{2m}(s) = \prod_{k=m+1}^{\infty} \left(1 - \frac{s^2}{a_k^2}\right)$$

where $\{a_k\}$ is a sequence of complex numbers such that

$$|\arg a_k| < \frac{\pi}{4} - \eta$$

for a fixed η , $0 < \eta < \pi/4$,

$$\sum_k |a_k|^{-2} < \infty, \quad 0 < \operatorname{Re} a_i \leq \operatorname{Re} a_{i+1} \text{ for all } i$$

and

$$(1.3) \quad \lim_{m \rightarrow \infty} |a_{m+1}|^2 \sum_{k=m+1}^{\infty} |a_k|^{-2} = \infty.$$

If a sequence $\{a_k\}$ satisfies all the above assumptions, we shall denote it by $\{a_k\} \in \text{class } C$. We obtain condition B , defined in [1, p. 436], if we replace (1.3) by (1.4)

$$(1.4) \quad \lim_{n \rightarrow \infty} |a_{n+1}|^{4/3} \sum_{k=1}^{\infty} |a_k|^{-2} = \infty.$$

If we take $a_k = k^\lambda 1/2 < \lambda < \infty$ then $\{a_k\} \in \text{class } C$, but of these sequences only those for which $1/2 < \lambda < 3/2$ satisfy condition B .

We define as in [1]

$$(1.5) \quad V_m = \sum_{k=m+1}^{\infty} a_k^{-2} \quad \text{and} \quad S_m = \sum_{k=m+1}^{\infty} |a_k|^{-2}$$

and whenever $\{a_k\} \in \text{class } C$ we prove

$$(1.6) \quad \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} |tG'_{2m}(t)| dt = (\cos^2 \varphi_m - \sin^2 \varphi_m)^{-3/2}$$

where

$$\varphi_m = \frac{1}{2} \arg V_m \left(-\frac{\pi}{2} < \arg V_m < \frac{\pi}{2} \right)$$

which answers the question posed in remark (3) [1, p. 441].

We shall also prove under the restriction $\{a_k\} \in \text{class } C$ Corollary 4.3 and an analogous theorem to Theorem 4.1.

As a by product we shall have

$$(1.7) \quad \lim_{m \rightarrow \infty} S_m^{1/2} \frac{d^n}{dt^n} G_{2m}(S_m^{1/2}t) = \frac{1}{\sqrt{4\pi}} \left(\frac{S_m}{V_m} \right)^{1/2} \frac{d^n}{dt^n} \exp \left(-t^2 \frac{S_m}{4V_m} \right)$$

which is more than necessary for proving other results and is an interesting estimate of $G_{2m}^{(n)}(t)$ by itself.

2. Some lemmas. In the author's thesis [2] and in a paper in collaboration with A. Jakimovski [3; Lemma 2.1.] the following lemma was proved:

LEMMA 2.1. *Suppose $\sum_{k=1}^{\infty} |a_k|^{-2} < \infty$ then the assumptions*

$$(2.1) \quad \sum_{k=m+1}^{\infty} |a_k|^{-(2+\alpha)} = o \left(\left(\sum_{k=m+1}^{\infty} |a_k|^{-2} \right)^{1+(\alpha/2)} \right) \quad m \rightarrow \infty$$

for some fixed $\alpha > 0$ and

$$(2.2) \quad \lim_{m \rightarrow \infty} \left(\max_{k > m} |a_k|^{-2} \right) \left(\sum_{k=m+1}^{\infty} |a_k|^{-2} \right)^{-1} = 0$$

are equivalent, and therefore the assumptions (2.1) for all positive α are equivalent.

Proof. Let us assume (2.1) for some $\alpha > 0$. If (2.3) is not valid then a subsequence $\{m(r)\}$ of $m + 1, m + 2, \dots$ exists such that for some $\beta > 0$

$$\left(\max_{k \geq m(r)+1} |a_k|^{-2} \right) S_{m(r)}^{-1} \geq \beta > 0$$

for all $r \geq 1$. Therefore

$$\sum_{k=m(r)+1}^{\infty} |a_k|^{-2-\alpha} \geq \left(\max_{k \geq m(r)+1} |a_k|^{-2} \right)^{1+(\alpha/2)} \geq \beta^{1+(\alpha/2)} S_{m(r)}^{1+(\alpha/2)}$$

which contradicts (2.1).

Assuming (2.2) then

$$\begin{aligned} \sum_{k=m+1}^{\infty} |a_k|^{-2-\alpha} &= \sum_{k=m+1}^{\infty} |a_k|^{-2} \leq \left(\max_{k \geq m+1} |a_k|^{-\alpha} \right) S_m \\ &= \left(\left(\max_{k \geq m+1} |a_k|^{-2} \right) S_m^{-1} \right)^{\alpha/2} S_m^{1+(\alpha/2)} \\ &= o(S_m^{1+(\alpha/2)}) \end{aligned} \quad (m \rightarrow \infty).$$

The following two lemmas are easy to verify.

LEMMA 2.2. *If $\{a_k\} \in \text{class } C$ then $\{a_k\}$ satisfies assumption (2.2). If $|\arg a_k| < \pi/4 - \eta$, $0 < \operatorname{Re} a_i \leq \operatorname{Re} a_{i+1}$ and $\{a_k\}$ satisfies assumption (2.2) then $\{a_k\} \in \text{class } C$.*

LEMMA 2.3. *If $|\arg a_k| < (\pi/4) - \eta$ and $\sum |a_k|^{-2} < \infty$, then*

$$(2.3) \quad \cos\left(\frac{\pi}{2} - 2\eta\right) S_n \leq |V_n| \leq S_n.$$

We define now $F_m(z)$ by

$$(2.4) \quad F_m(z) = E_m(z \cdot S_m^{-1/2}) = \sum_{k=m+1}^{\infty} \left(1 - \frac{z^2}{a_k^2 S_m} \right).$$

LEMMA 2.4. *Suppose $\{a_k\} \in \text{class } C$ then there exist constants $k(p) > 0$ independent of m so that for all real y*

$$(2.6) \quad |F_m(iy)| > 1 + k(p)y^{2p} \quad \text{for } m > m_0(p).$$

Proof. Define $a_k = |a_k| e^{i\beta_k}$, $-(\pi/4) + \eta < \beta_k < (\pi/4) - \eta$

$$\begin{aligned} |F_m(iy)| &= \left| \prod_{k=m+1}^{\infty} \left(1 - \frac{(iy)^2}{a_k^2 S_m} \right) \right| \\ &\geq \prod_{k=m+1}^{\infty} \left(1 + \frac{y^2}{|a_k|^2 S_m} \cos 2\beta_k \right) \\ &\geq \prod_{k=m+1}^{\infty} \left(1 + \frac{y^2 \cos\left(\frac{\pi}{2} - 2\eta\right)}{|a_k|^2 S_m} \right) \\ &= 1 + \sum_{p=1}^{\infty} \frac{y^{2p} \cos^p\left(\frac{\pi}{2} - 2\eta\right)}{S_m^p p!} \sum_{\substack{k(i) > m \\ i \neq j \\ k(i) \neq k(j)}} |a_{k(1)} \cdots a_{k(p)}|^{-2}. \end{aligned}$$

Since we have $\lim_{m \rightarrow \infty} \max_{k > m} |a_k|^{-2} S_m^{-1} = 0$ we can find $m_0(p)$ so that for $m > m_0(p)$ $\max_{k > m} |a_k^{-2}| < (1/2p) S_m$. Therefore we have

$$\begin{aligned} & \sum_{\substack{k(i) > m \\ k(i) \neq k(j), i \neq j}} |a_{k(1)} \cdots a_{k(p)}|^{-2} \\ &= \sum_{\substack{k(i) > m \\ k(i) \neq k(j), i \neq j}} \left(S_m - \sum_{i=1}^{p-1} |a_{k(i)}|^{-2} \right) |a_{k(1)} \cdots a_{k(p-1)}|^{-2} \\ &\geq \frac{1}{2} S_m \sum_{\substack{k(i) > m \\ k(i) \neq k(j), j \neq i}} |a_{k(1)} \cdots a_{k(p-1)}|^{-2} \geq \left(\frac{1}{2}\right)^p S_m^p. \end{aligned}$$

Hence

$$|F_m(iy)| \geq 1 + y^{2p} \frac{\cos^p\left(\frac{\pi}{2} - 2\eta\right) S_m^p}{S_m^p p! 2^p} = 1 + k(p)y^{2p}.$$

3. The asymptotic estimates for $G_{2m}^{(k)}(t)$.

THEOREM 3.1. *Let $\{a_k\} \in$ class C ; then for all $n = 0, 1, \dots$*

$$(3.1) \quad \lim_{m \rightarrow \infty} S_m^{1/2} \frac{d^n}{dt^n} G_{2m}(S_m^{1/2}t) = \frac{1}{\sqrt{4\pi}} \left(\frac{S_m}{V_m}\right)^{1/2} \frac{d^n}{dt^n} \exp\left(-t^2 \frac{S_m}{4V_m}\right)$$

uniformly in $-\infty < t < \infty$ (we choose $\arg V_m^{1/2} = (1/2) \arg V_m$).

Proof. Following the proof of the special case $n = 0$ and $\arg_1 a_k = 0$ by Hirschman—Widder [4; pp. 140–1] we have

$$S_m^{1/2} G_{2m}(S_m^{1/2}t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{e^{zt} dz}{F_{2m}(s)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iyt} dy}{F_{2m}(iy)}.$$

By an estimate of [5; p. 246] we have for $|z| < R$ and

$$R \cdot [|a_k| S_m^{1/2}]^{-1} \leq \frac{1}{2}$$

$$\left| \log \left\{ \left(1 - \frac{z^2}{a_k^2 S_m} \right) \exp \left(z^2 / a_k^2 S_m \right) \right\} \right| \leq 4R^3 \frac{1}{|a_k|^3 S_m^{3/2}}.$$

Recalling that $\sum_{k=m+1}^{\infty} 1/a_k^2 S_m = V_m/S_m$ and since by Lemma 2.1

$$\sum_{k=m+1}^{\infty} \frac{1}{|a_k|^3 S_m^{3/2}} = o(1) \quad m \rightarrow \infty,$$

we have for $|z| < R$ and $m > m_0(R)$

$$\left| F_m(z) - \exp\left(-\frac{V_m z^2}{S_m}\right) \right| < \varepsilon_1.$$

Since by Lemma 2.4 $R > R_0(\varepsilon_2, \eta)$ implies

$$\int_R^{\infty} \frac{|y|^n dy}{|F_{2m}(iy)|} < \varepsilon_2 \quad \text{and} \quad \int_{-\infty}^{-R} \frac{|y|^n dy}{|F_{2m}(iy)|} < \varepsilon_2$$

we have

$$\begin{aligned}
 & S_m^{1/2} \frac{d^n}{dt^n} G_{2m}(S_m^{1/2}t) \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(iy)^n e^{iyt}}{F_m(iy)} dy \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (iy)^n \exp\left(-\frac{V_m}{S_m}y^2 + iyt\right) dy + o(1) \\
 &= \frac{d^n}{dt^n} \left\{ \exp\left(-\frac{t^2 S_m}{4V_m}\right) \right. \\
 &\quad \times \left. \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[-(V_m^{1/2}S_m^{-1/2}y - itS_m^{1/2}(4V_m)^{-1/2})^2] dy \right\} + o(1) \\
 &= \left(\frac{S_m}{V_m}\right)^{1/2} \frac{d^n}{dt^n} \left\{ \exp\left(-\frac{t^2 S_m}{4V_m}\right) \frac{1}{2\pi} \int_{\Gamma} e^{-z^2} dz \right\} + o(1) \\
 &= \frac{1}{\sqrt{4\pi}} \left(\frac{S_m}{V_m}\right)^{1/2} \frac{d^n}{dt^n} \exp\left(-\frac{t^2 S_m}{4V_m}\right) + o(1) \quad (m \uparrow \infty)
 \end{aligned}$$

using the residue theorem, the fact that e^{-z^2} is entire and that

$$|\arg V_m^{1/2}| < \frac{\pi}{4} \quad \text{for all } m .$$

As a corollary we derive

THEOREM 3.2. *If $\{a_k\}$ satisfies assumption C then*

$$(3.2) \quad \int_{-\infty}^{\infty} |G_{2m}(t)| dt = (\cos^2\varphi_m - \sin^2\varphi_m)^{-1/2} + o(1) \quad m \rightarrow \infty$$

and

$$(3.3) \quad \int_{-\infty}^{\infty} |tG'_{2m}(t)| dt = (\cos^2\varphi_m - \sin^2\varphi_m)^{-3/2} + o(1) \quad m \rightarrow \infty$$

where $2\varphi_m = \arg V_m$.

Proof. Since by Lemma 2.4 of [1].

$$(3.4) \quad |G_{2m}(t)| < MS_m^{-1/2} \exp(-KS_m^{-1/2}|t|)$$

we have

$$\begin{aligned}
 \int_{-\infty}^{\infty} |G_{2m}(t)| dt &= \int_{-\infty}^{\infty} |S_m^{1/2}G_{2m}(S_m^{1/2}t)| dt = \int_{-R}^R |S_m^{1/2}G_{2m}(S_m^{1/2}t)| dt \\
 &+ o(1) \quad (R \uparrow \infty) .
 \end{aligned}$$

This combined with (3.1) and a simple integration yield (3.2).

To prove (3.3) we use Lemma 3.2 case A (since for $\{a_k\} \in$ class C $S_m \cong 4r_{m+1}^{-2} \equiv 4|a_{m+1}|^{-2}$ for $m > m_0$) which is

$$(3.5) \quad |G'_{2m}(t)| \leq M_1 S_m^{-1} \exp(-K_1 S_m^{-1/2} |t|).$$

Therefore we have

$$\int_R^\infty |S_m t G'_{2m}(S_m^{1/2} |t|)| dt \leq M_1 \frac{1}{(K_1)^2} e^{-K_1 R} = o(1) \quad R \rightarrow \infty.$$

This implies

$$\begin{aligned} \int_{-\infty}^\infty |t G'_{2m}(t)| dt &= \int_{-\infty}^\infty |S_m t G'_{2m}(S_m^{1/2} t)| dt \\ &= \frac{1}{2\sqrt{4\pi}} \left| \frac{S_m}{V_m} \right|^{3/2} \int_{-\infty}^\infty t^2 \exp\left(-\frac{t^2}{4} S_m \operatorname{Re} \frac{1}{V_m}\right) dt + o(1) \\ &= \left(\frac{|V_m|^{-1}}{(\operatorname{Re}(V_m^{-1}))} \right)^{3/2} + o(1) = (\cos^2 \varphi_m - \sin^2 \varphi_m)^{-3/2} + o(1) \quad (m \uparrow \infty). \end{aligned}$$

4. **Remarks.** I. For the theorems and the lemmas proved in this paper $0 < \operatorname{Re} a_i \leq \operatorname{Re} a_{i+1}$ is not essential and the condition (2.2) can replace it and (1.3).

II. Theorem 3.1 which replaces Theorem 4.1 yields for the case $n = 0$ only the following

$$(4.6) \quad G_{2m}(t) = (4\pi V_m)^{-1/2} \exp(-t^2/4V_m) + o(S_m^{-1/2}) \quad m \rightarrow \infty$$

but if one follows the proof of Theorem 4.1 of [1] and Lemma 4.2 of [1] almost literally one obtains for $\{a_k\} \in \text{class } C$

$$(4.7) \quad G_{2m}(t) = (4\pi V_m)^{-1/2} \exp(-t^2/4V_m) + o(|a_{m+1}|^{-2} S_m^{-3/2}) \quad m \rightarrow \infty$$

which is somewhat more general.

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