

# Pacific Journal of Mathematics

**ON OPERATORS WHOSE FREDHOLM SET IS THE COMPLEX  
PLANE**

MARINUS A. KAASHOEK AND DAVID CLARK LAY

## ON OPERATORS WHOSE FREDHOLM SET IS THE COMPLEX PLANE

M. A. KAASHOEK AND D. C. LAY

Let  $T$  be a closed linear operator with domain and range in a complex Banach space  $X$ . The Fredholm set  $\Phi(T)$  of  $T$  is the set of complex numbers  $\lambda$  such that  $\lambda - T$  is a Fredholm operator. If the space  $X$  is of finite dimension then, obviously, the domain of  $T$  is closed and  $\Phi(T)$  is the whole complex plane  $C$ . In this paper it is shown that the converse is also true. When  $T$  is defined on all of  $X$  this is a well-known result due to Gohberg and Krein.

Examples of nontrivial closed operators with  $\Phi(T) = C$  are the operators whose resolvent operator is compact. A characterization of the class of closed linear operators with a nonempty resolvent set and a Fredholm set equal to the complex plane will be given.

Throughout the present paper  $X$  and  $Y$  will denote complex Banach spaces. Let  $T$  be an arbitrary closed linear operator with domain  $\mathcal{D}(T)$  in  $X$  and range  $\mathcal{R}(T)$  in  $Y$ . The nullity  $n(T)$  of  $T$  is the dimension of the null space  $\mathcal{N}(T)$  of  $T$ . The defect  $d(T)$  of  $T$  is the dimension of the quotient space  $Y/\mathcal{R}(T)$ . No distinction is made between infinite dimensions, so that  $n(T)$  and  $d(T)$  may be nonnegative integers or  $+\infty$ . We say that  $T$  is Fredholm if  $n(T)$  and  $d(T)$  are both finite. Note that  $d(T) < \infty$  implies  $\mathcal{R}(T)$  is closed (cf. [5], Lemma 332).

In 1957 Gohberg and Krein [3] showed that if  $A$  is a bounded linear operator on  $X$  with  $\Phi(A) = C$ , then the dimension of  $X$  (denoted by  $\dim X$ ) is finite. The following theorem extends this result.

**THEOREM 1.** *Let  $T$  and  $S$  be bounded linear operators from  $X$  into  $Y$ . Suppose that  $S$  is a homeomorphism, and that  $T + \lambda S$  is Fredholm for each  $\lambda \in C$ . Then*

$$\dim X \leq \dim Y < \infty .$$

*Proof.* Since  $S$  is a homeomorphism,  $\mathcal{R}(S)$  is closed and  $n(S) = 0$ . By a well-known stability theorem (cf. [5], Theorem 1), this implies the existence of a positive constant  $\rho$  such that for  $0 < |\mu| < \rho$

$$d(S) = d(S) - n(S) = d(S + \mu T) - n(S + \mu T) .$$

The right-hand side is finite because  $S + \mu T$  is Fredholm for  $\mu \neq 0$ . Hence  $d(S) < \infty$ , and so  $S$  has a bounded left inverse, say  $R$ . Then  $n(R) \leq d(S) < \infty$  and  $d(R) = 0$ , so  $R$  is Fredholm. Define  $A = RT$ .

Then  $A$  is a bounded linear operator on  $X$  and

$$\lambda - A = \lambda RS - RT = R(\lambda S - T).$$

For each complex value of  $\lambda$ ,  $\lambda - A$  is the product of two bounded Fredholm operators and hence is Fredholm. But  $\Phi(A) = \mathbf{C}$  implies that  $\dim X < \infty$  by the result of Gohberg and Krein ([3], Theorem 3.2). Then  $\dim Y = \dim X + d(S) < \infty$ , concluding the proof.

**COROLLARY.** *Let  $T$  be a closed linear operator with domain  $\mathcal{D}(T)$  and range in  $X$ . Then  $\dim X < \infty$  if and only if  $\mathcal{D}(T)$  is closed and  $\Phi(T) = \mathbf{C}$ .*

In [1] Caradus has proved that if  $T$  is a closed linear operator with domain and range in  $X$  such that  $\dim X/\mathcal{D}(T) < \infty$ ,  $\Phi(T) = \mathbf{C}$  and such that the resolvent set of  $T$  is neither empty nor the whole complex plane, then  $\dim X < \infty$ . The following lemma shows that Caradus' result is contained in the Corollary.

**LEMMA.** *Let  $T$  be a closed linear operator with domain in  $X$  and range in  $Y$ . Suppose there exists a closed subspace  $M$  of  $X$  such that  $X = \mathcal{D}(T) \oplus M$ . Then  $\mathcal{D}(T)$  is closed.*

*Proof.* Let  $Y_1$  be the Banach space  $Y \times M$ , with the norm

$$\|(y, m)\| = \|y\| + \|m\|.$$

Define the linear operator  $J$  from  $X$  into  $Y_1$  by setting

$$J(x + m) = (Tx, m)$$

for each  $x \in \mathcal{D}(T)$  and  $m \in M$ . It is easily verified that  $J$  is a well-defined closed linear operator. Since the domain of  $J$  is the Banach space  $X$ , the closed graph theorem implies that  $J$  is bounded. Hence

$$(\|Tx\| + \|m\|) \leq \|J\| \cdot \|x + m\|$$

for each  $x \in \mathcal{D}(T)$  and  $m \in M$ . In particular,

$$\|Tx\| \leq \|J\| \cdot \|x\|$$

for each  $x \in \mathcal{D}(T)$ . Thus  $T$  is both closed and bounded, implying that  $\mathcal{D}(T)$  is closed.

We have learned recently that similar statements for the range of a closed linear operator are proved by S. Goldberg in [4]. That this can be done follows easily from the observation that the range of a closed linear operator is always the domain of some other closed linear operator, and conversely (cf. [6], Chapter IV).

The Corollary states that the closed linear operators  $T$  with closed domain and  $\Phi(T) = \mathbf{C}$  are trivial. Examples of nontrivial closed operators whose Fredholm set is the complex plane are the operators with compact resolvent (cf. [7], § 2). The following theorem shows that each closed operator  $T$  with a nonempty resolvent set  $\rho(T)$  and with  $\Phi(T) = \mathbf{C}$  is characterized by the fact that for each  $\mu \in \rho(T)$  the resolvent  $(\mu - T)^{-1}$  is a Riesz operator. For the definition of Riesz operators and one of their characterizations we refer to Dieudonné ([2], XI. 4, problem 5).

**THEOREM 2.** *Let  $T$  be a closed linear operator with domain and range in  $X$ . If  $\Phi(T) = \mathbf{C}$ , then  $(\mu - T)^{-1}$  is a Riesz operator for all  $\mu \in \rho(T)$ . Conversely, if  $(\mu - T)^{-1}$  is a Riesz operator for some  $\mu \in \rho(T)$ , then  $\Phi(T) = \mathbf{C}$ .*

*Proof.* We may assume that  $\dim X = \infty$  and that  $\rho(T)$  is not empty. Take  $\mu$  in  $\rho(T)$  and let  $A = (\mu - T)^{-1}$ . Then for  $\lambda \neq \mu$ ,

$$(\lambda - T)(\mu - T)^{-1} = (\mu - \lambda)(\zeta - A) ,$$

where  $\zeta = (\mu - \lambda)^{-1}$ . This implies that  $\Phi(T) = \mathbf{C}$  if and only if  $\Phi(A) = \mathbf{C} \setminus \{0\}$ . Hence it is enough to show that  $A$  is a Riesz operator if and only if  $\Phi(A) = \mathbf{C} \setminus \{0\}$ . In order to do this, let  $\mathcal{K}$  be the ideal of all compact linear operators in the Banach algebra  $\mathcal{L}(X)$  of all bounded linear operators on  $X$ , and let  $\pi$  denote the canonical homomorphism from  $\mathcal{L}(X)$  onto the quotient algebra  $\mathcal{L}(X)/\mathcal{K}$ . Then it follows from Atkinson's characterization of the class of all Fredholm operators in  $\mathcal{L}(X)$  that  $\zeta - A$  is Fredholm if and only if  $\zeta - \pi(A)$  has an inverse in  $\mathcal{L}(X)/\mathcal{K}$ . So  $\Phi(A) = \mathbf{C} \setminus \{0\}$  if and only if the spectrum of  $\pi(A)$  in  $\mathcal{L}(X)/\mathcal{K}$  is  $\{0\}$ , i.e., the spectral radius  $r(\pi(A))$  of  $\pi(A)$  is zero. But

$$\begin{aligned} r(\pi(A)) &= \lim_{n \rightarrow \infty} \| [\pi(A)]^n \|^{1/n} \\ &= \lim_{n \rightarrow \infty} \| \pi(A^n) \|^{1/n} = \lim_{n \rightarrow \infty} [d(A^n, \mathcal{K})]^{1/n} , \end{aligned}$$

where  $d(A^n, \mathcal{K})$  is the infimum of  $\| A^n - K \|$  for  $K \in \mathcal{K}$ . Thus  $\Phi(A) = \mathbf{C} \setminus \{0\}$  if and only if

$$\lim_{n \rightarrow \infty} [d(A^n, \mathcal{K})]^{1/n} = 0 ,$$

which is equivalent to the statement that  $A$  is a Riesz operator (cf. [2], XI. 4, problem 5).

When  $T$  is a self-adjoint closed linear operator in a Hilbert space Theorem 2 can be strengthened. This is because  $(\mu - T)^{-1}$  is normal for  $\mu \in \rho(T)$ , and a normal operator is Riesz if and only if it is compact.

Hence, in this special case,  $\Phi(T) = C$  if and only if  $(\mu - T)^{-1}$  is compact for each  $\mu$  in  $\rho(T)$ .

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