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The main purpose of this paper is to give the function which maps a bounded domain onto the m-representative domain ($m \geqq 1$) (hereafter called *m-representative function*) without utilizing the minimum problems in the case of several complex variables. One of the results obtained is that a m-representative domain becomes also a $(m+1)$-representative domain.

Let D be a bounded domain and $k_D(z, \bar{t})$ $z, t \in D$ be the Bergman kernel function. Recently M. Maschler [7] made use of the minimum problems to establish the m-representative function in one variable:

$$\text{(I)} \qquad w(z) = \frac{M_D^{010\cdots0}(z, t_0)}{m_D^{10\cdots0}(z, t_0)} \quad \text{fixed } t_0 \in D ,$$

where $M_D^{010\cdots0}(z, t_0)$, $m_D^{10\cdots0}(z, t_0)$ are both Maschler's minimizing functions and represented in a closed from by using $k_D(z, \bar{t})$ and its derivatives, respectively. Moreover this result has been generalized successfully by T. Tsuboi [13] in the case of several complex variables. In this case, however, for example the 2-representative function of a unit circle is nonregular if we choose a fixed point t_0 in $1/2 < |t_0| < 1$.

In this paper we consider the following m-representative function of other type which coincides with the ordinary Bergman representative function when $m = 1$ (§ 4):

$$\text{(II)} \qquad w(z) = \int_{t_0}^{z} N_D^{E_n 0\cdots0}(z, t_0)dz ,$$

$$N_D^{E_n 0\cdots 0}(z, t_0) \equiv (E_n 0 \cdots 0)\begin{pmatrix} T_D(t_0, \bar{t}_0) & \cdots & \dfrac{\partial^{m-1}}{\partial z^{m-1}} T_D(t_0, \bar{t}_0) \\ \vdots & & \vdots \\ \dfrac{\partial^{m-1}}{\partial t^{*m-1}} T_D(t_0, \bar{t}_0) & \cdots & \dfrac{\partial^{2(m-1)}}{\partial t^{*m-1}\partial z^{m-1}} T_D(t_0, \bar{t}_0) \end{pmatrix}^{-1} \cdot \begin{pmatrix} T_D(z, \bar{t}_0) \\ \vdots \\ \dfrac{\partial^{m-1}}{\partial t^{*m-1}} T_D(z, \bar{t}_0) \end{pmatrix},$$

where the matrix function $T_D(z, \bar{t})$(for definition, see § 1) is, as is well known, relatively invariant under any pseudo-conformal mapping. We define in § 1 the relative invariant $T_{2D}(z, \bar{t})$, which plays an important role throughout this paper. In § 2 we have a necessary and sufficient condition that a minimal domain becomes simultaneously a representative domain with the same center. Section 3 is devoted to the canonical domains of other types.

1. Preliminaries. Various properties on the several complex variables can be treated simply in many cases, considering functions as reciprocal mappings of vector spaces. In this paper we assume that each domain D we deal with is a bounded domain in n-dimensional complex Euclidean $z \equiv (z_1, z_2, \cdots, z_n)'$-space and we shall consider $w(z) \equiv (w_1(z), w_2(z), \cdots, w_n(z))'$ which is called a vector function after Bochner-Martin, for each point z of D. We shall call this $w(z)$ an analytic function of z, if $w_j(z) (j = 1, 2, \cdots, n)$ are analytic with respect to z(Ozaki and others [14]).

We start with the definitions of power of vector and differentiation with respect to vector variable. Vectors and matrices marked with the symbol $'$ and $*$ denote the transposed and transposed conjugate vectors or matrices, respectively. We define the k-th power of z as

$$z^k \equiv (z_1^k, \cdots, z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n}, \cdots, z_n^k)' , \tag{1.1}$$

where $(k_1, k_2, \cdots, k_n)$ runs over all the nonnegative integers such that $k_1 + k_2 + \cdots + k_n = k$ and ${}_nH_k$ monomials of degree k in $z_1, z_2, \cdots, z_n$ are arranged in a certain determined way (e.g., in the lexicographical order) to form an ${}_nH_k$-tuple column vector. Next we define the k-th partial differentiation of function $w(z, \bar{t})$ with respect to z and t^* as

$$\frac{\partial^k}{\partial z^k} w(z, \bar{t}) \equiv \left(\frac{\partial^k}{\partial z_1^k}, \cdots, \frac{k!}{k_1! k_2! \cdots k_n!} \frac{\partial^k}{\partial z_1^{k_1} \partial z_2^{k_2} \cdots \partial z_n^{k_n}}, \cdots, \frac{\partial^k}{\partial z_n^k}\right) \times w(z, \bar{t}) \tag{1.2}$$

and $\partial^k w(z, \bar{t})/\partial t^{*k} \equiv (\partial^k/\partial t^k)^* \times w(z, \bar{t})$, where $\partial^k/\partial z_1^k, \cdots, \partial^k/\partial z_n^k$ are arranged in the order corresponding to $z_1^k, \cdots, z_n^k$ in (1.1) and the sign $\times$ denotes the Kronecker product. In particular, for $k = 1$ we have $n \times n$ matrix derivative from (1.2) whose (i, j)-element is $\partial w_i(z, \bar{t})/\partial z_j$ $i, j = (1, 2, \cdots, n)$, where $w(z, \bar{t}) = (w_1(z, \bar{t}), w_2(z, \bar{t}), \cdots, w_n(z, \bar{t}))'$. For a function $w \equiv w(z)$ the k-th partial differentiation will be merely denoted as follows: $\partial^k w(z)/\partial z^k = d^k w(z)/dz^k$. Moreover, we define the k-th power $(dw/dz)^k$ of dw/dz as an ${}_nH_k \times {}_nH_k$ reduced matrix of the $n^k \times n^k$ matrix $(dw/dz) \times (dw/dz) \times \cdots \times (dw/dz) (\equiv A)$ in the following way:

(i) Construct an $n^k \times {}_nH_k$ matrix B by adding up the columns in A of which the first elements are equal to each other, and by arranging the above ${}_nH_k$ columns so that the first elements of them make a following ${}_nH_k$-tuple row vector

$$\left(\left(\frac{\partial w_1}{\partial z_1}\right)^k, \cdots, \frac{k!}{k_1!k_2!\cdots k_n!}\left(\frac{\partial w_1}{\partial z_1}\right)^{k_1}\left(\frac{\partial w_1}{\partial z_2}\right)^{k_2}\cdots\left(\frac{\partial w_1}{\partial z_n}\right)^{k_n}, \cdots, \left(\frac{\partial w_1}{\partial z_n}\right)^k\right),$$

where $(\partial w_1/\partial z_1)^k, \cdots, (\partial w_1/\partial z_n)^k$ are arranged in the order corresponding to $z_1^k, \cdots, z_n^k$ in (1.1).

(ii) Construct ${}_nH_k \times {}_nH_k$ matrix by leaving only one and removing others among the row vectors in B which are identically equal to each other, and by arranging all remained rows so that the first elements of them make a following ${}_nH_k$ column vector

$$\left(\left(\frac{\partial w_1}{\partial z_1}\right)^k, \cdots, \left(\frac{\partial w_1}{\partial z_1}\right)^{k_1}\left(\frac{\partial w_2}{\partial z_1}\right)^{k_2}\cdots\left(\frac{\partial w_n}{\partial z_1}\right)^{k_n}, \cdots, \left(\frac{\partial w_n}{\partial z_1}\right)^k\right)',$$

where $(\partial w_1/\partial z_1)^k, \cdots, (\partial w_n/\partial z_1)^k$ are arranged in the order corresponding to $z_1^k, \cdots, z_n^k$ in (1.1)(Ono [11], Tsuboi [13]).

We shall introduce some differential formulas for convenience after calculations. Let the functions A, B of z be $k \times l, l \times m$ matrices, respectively. The following formulas can be easily calculated: $dAB/dz = A(dB/dz) + (dA/dz)(E_n \times B)$, $dA = (dA/dz)(dz \times E_l)$, $dA/dz = (dA/d\zeta) \cdot ((d\zeta/dz) \times E_l)$, where E_n, E_l denote $n \times n, l \times l$ unit matrices, respectively.

The following lemmas are trivial.

LEMMA 1.1. *If for a vector function $w \equiv w(z)$ we have $dw(z)/dz = A$, where A denotes an $n \times n$ constant matrix, then $w(z) = Az + C$, where C is a constant vector of integration.*

LEMMA 1.2. *For a matrix $P \equiv \begin{pmatrix} K & L \\ M & N \end{pmatrix}$ with the block subdivisions, where K, N are square matrices, it holds that*

$$P^{-1} = \begin{pmatrix} K^{-1} + XZ^{-1}Y & -XZ^{-1} \\ -Z^{-1}Y & Z^{-1} \end{pmatrix},$$

where $X \equiv K^{-1}L$, $Y \equiv MK^{-1}$, $Z \equiv N - MK^{-1}L$.

Let $\mathfrak{L}^2(D)$ be the class of all functions $w(z)$ which are regular and singlevalued in a domain D and for which the Lebesgue integral $\int_D |w(z)|^2 dv_z < \infty$. Then $\mathfrak{L}^2(D)$ is a Hilbert space. Since $\mathfrak{L}^2(D)$ is separable, there exists a closed orthonormal system $\varphi(z) \equiv (\varphi_1(z), \varphi_2(z), \cdots)'$. Using this $\varphi(z)$, the Bergman kernel function $k_D(z, \bar{t})$ may be represented as follows: $k_D(z, \bar{t}) \equiv \varphi^*(t)\varphi(z)$ $z, t \in D$. This function is independent of the choice of a closed orthonormal system $\varphi(z)$, and $k_D(z, \bar{z}) > 0$. It is well known that if a function $\zeta \equiv \zeta(z)$ is a pseudo-conformal mapping of a domain D onto a domain Δ, then we have

$k_D(z, \bar{t}) = \overline{\det d\tau/dt} \cdot k_\Delta(\zeta, \bar{\tau}) \cdot \det d\zeta/dz$, where $\tau \equiv \zeta(t)$. Also, that if we define $T_D(z, \bar{z}) \equiv \partial^2 \log k_D(z, \bar{z})/\partial z^* \partial z$, then the $n \times n$ Hermitian matrix $T_D(z, \bar{z})$ is transformed under any pseudo-conformal mapping $\zeta \equiv \zeta(z)$ as a relative invariant; that is, $T_D(z, \bar{z}) = (d\zeta/dz)^* T_\Delta(\zeta, \bar{\zeta})(d\zeta/dz)$ and is positive definite. Therefore the Bergman metric

$$ds^2 \equiv dz^* T_D(z, \bar{z}) dz$$

is absolutely invariant under any pseudo-conformal mapping (Bergman [3]).

Now, we shall define $k_{\mu D}(z, \bar{t})$, $T_{\mu D}(z, \bar{t})(\mu = 1, 2)$ as follows, respectively:

$$T_{\mu D}(z, \bar{t}) \equiv \frac{\partial^2}{\partial t^* \partial z} \log k_{\mu D}(z, \bar{t}) \ , \quad k_{2D}(z, \bar{t}) \equiv \det\{k_{1D}^2(z, \bar{t}) T_{1D}(z, \bar{t})\} \ z, t \in D$$

$$(k_{1D}(z, \bar{t}) \equiv k_D(z, \bar{t})) \ ,$$

where we assume that $k_{\mu D}(z, \bar{t}) \neq 0$. Then the $n \times n$ matrix functions $T_D(z, \bar{t})(\equiv T_{1D}(z, \bar{t}))$ and $T_{2D}(z, \bar{t})$ may be calculated as follows, respectively:

$$(1.3) \qquad T_{\mu D}(z, \bar{t}) = k_{\mu D}^{-2}(z, \bar{t}) \Big\{ k_{\mu D}(z, \bar{t}) \frac{\partial^2}{\partial t^* \partial z} k_{\mu D}(z, \bar{t}) - \frac{\partial}{\partial t^*} k_{\mu D}(z, \bar{t}) \frac{\partial}{\partial z} k_{\mu D}(z, \bar{t}) \Big\} \qquad (\mu = 1, 2)$$

$$(1.4) \qquad T_{2D}(z, \bar{t}) = 2n \cdot T_D(z, \bar{t}) + \frac{\partial^2}{\partial t^* \partial z} \log \det T_D(z, \bar{t}) \ .$$

REMARK 1. As we assume $k_{\mu D}(z, \bar{t}) \neq 0 (\mu = 1, 2)$, our method does not apply to all bounded domains.

LEMMA 1.3. *$T_{\mu D}(z, \bar{t})(\mu = 1, 2)$ is relatively invariant under any pseudo-conformal mapping $\zeta \equiv \zeta(z)$, and $T_{2D}(z, \bar{z})$ is a positive definite Hermitian matrix. Thus we may define the 2-nd invariant metric $ds^2 \equiv dz^* T_{2D}(z, \bar{z}) dz$.*

Proof. $T_D(z, \bar{t})$ is relatively invariant (Tsuboi [8]), hence $T_{2D}(z, \bar{t})$ is also relatively invariant from (1.4). Thus, it holds that

$$(1.5) \qquad T_{\mu D}(z, \bar{t}) = \left(\frac{d\tau}{dt}\right)^* T_{\mu \Delta}(\zeta, \bar{\tau}) \frac{d\zeta}{dz} \qquad (\mu = 1, 2) \ ,$$

where $\tau \equiv \zeta(t)$ and $\Delta \equiv \zeta(D)$. Next it has been known that the matrix

$$(1.6) \qquad (n + 1)(g_{\bar{\alpha}\beta}) - (R_{\bar{\alpha}\beta}) \ (\alpha, \beta = 1, 2, \cdots, n)$$

is positive definite, where $g_{\bar{\alpha}\beta}$ and $R_{\bar{\alpha}\beta}$ are covariant metric tensor and the Ricci curvature tensor for bounded complex manifold, respectively (Kobayashi [5]). In our case, we can take $T_D(z, \bar{z})$ as $(g_{\bar{\alpha}\beta})$ and $-\partial^2 \log\det T_D(z, \bar{z})/\partial z^* \partial z$ as $(R_{\bar{\alpha}\beta})$. Hence (1.6) becomes as follows:

$$(1.7) \qquad u^*\left[(n+1)T_D(z, \bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log\det T_D(z, \bar{z})\right]u > 0 ,$$

where u is an arbitrary n-tuple nonzero constant vector. We utilize this relation (1.7). By (1.4) we have

$$\begin{aligned} u^* T_{2D}(z, \bar{z})u &= u^*\left[2n \cdot T_D(z, \bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log\det T_D(z, \bar{z})\right]u \\ &\geqq u^*\left[(n+1)T_D(z, \bar{z}) + \frac{\partial^2}{\partial z^* \partial z} \log\det T_D(z, \bar{z})\right]u > 0 , \end{aligned}$$

the proof is completed.

If D and Δ are domains which are pseudo-conformally equivalent to each other by $\zeta \equiv \zeta(z)$, then for the scalar function $k_{2D}(z, \bar{t})$ we have

$$(1.8) \qquad k_{2D}(z, \bar{t}) = \overline{\left(\det \frac{d\tau}{dt}\right)^m} k_{2\Delta}(\zeta, \bar{\tau})\left(\det \frac{d\zeta}{dz}\right)^m , \qquad m = 1 + 2n$$

and it is evident that $k_{2D}(z, \bar{z})$ is positive from its definition.

2. μ-th Representative domains and μ-th quasiminimal domains ($\mu = 1, 2$). Generalization of the Riemann mapping theorem to the case of domains in the spase of several complex variables leads to various other types of canonical domains.

Firstly we shall introduce the generalized representative domain. We define the vector function $M^{0E_n}_{\mu D}(z, t_0)$ and the scalar function $m^1_{\mu D}(z, t_0)$ in D for a fixed point $t_0(\in D)$, which is not a point on a branch manifold, as

$$(2.1) \qquad M^{0E_n}_{\mu D}(z, t_0) \equiv (0 E_n)\begin{pmatrix} k_{\mu D}(t_0, \bar{t}_0) & \dfrac{\partial}{\partial z} k_{\mu D}(t_0, \bar{t}_0) \\ \dfrac{\partial}{\partial t^*} k_{\mu D}(t_0, \bar{t}_0) & \dfrac{\partial^2}{\partial t^* \partial z} k_{\mu D}(t_0, \bar{t}_0) \end{pmatrix}^{-1} \begin{pmatrix} k_{\mu D}(z, \bar{t}_0) \\ \dfrac{\partial}{\partial t^*} k_{\mu D}(z, \bar{t}_0) \end{pmatrix}$$

and $m^1_{\mu D}(z, t_0) \equiv k_{\mu D}(z, \bar{t}_0)/k_{\mu D}(t_0, \bar{t}_0)$, respectively, where 0 denotes an n-tuple column zero-vector. Hereafter we assume that $\det T_{\mu D}(z, \bar{t}_0) \neq 0$ for all $z \in D$.

Remark 2. Let P^{-1} be the second matrix of right side in (2.1), then by a simple calculation of determinants we have

$$\det P = k_{\mu D}^{n+1}(t_0, \bar{t}_0) \det T_{\mu D}(t_0, \bar{t}_0) > 0 .$$

Hence there exists the inverse of P. Further from the rules of matrices we see that $M_{\mu D}^{0E_n}(t_0, t_0) = 0$.

LEMMA 2.1. *The matrix derivative* $d(M_{\mu D}^{0E_n}/m_{\mu D}^1)/dz$ *is given by the matrix product*

$$\frac{d}{dz} \frac{M_{\mu D}^{0E_n}(z, t_0)}{m_{\mu D}^1(z, t_0)} = T_{\mu D}^{-1}(t_0, \bar{t}_0) T_{\mu D}(z, \bar{t}_0) . \tag{2.2}$$

Proof. Using Lemma 1.2 and the relation (1.3), further noting the definition of $T_{\mu D}(z, \bar{t})$ the left side of (2.2) is calculated as follows:

$$k_{\mu D}(t_0, \bar{t}_0)(0 E_n)\begin{pmatrix} * & * \\ * & T_{\mu D}^{-1}(t_0, \bar{t}_0)/k_{\mu D}(t_0, \bar{t}_0)\end{pmatrix}(0\, T'_{\mu D}(z, \bar{t}_0))' .$$

From this we obtain the desired relation $T_{\mu D}^{-1}(t_0, \bar{t}_0) T_{\mu D}(z, \bar{t}_0)$.

THEOREM 2.1. *If a domain D is mapped onto a domain Δ by a pseudo-conformal mapping $\zeta = \zeta(z)$ which satisfies $\zeta(t_0) = \tau_0$, $d\zeta(t_0)/dz = E_n$ at a fixed point t_0 $(\in D)$, then we have*

$$\frac{M_{\mu D}^{0E_n}(z, t_0)}{m_{\mu D}^1(z, t_0)} = \frac{M_{\mu \Delta}^{0E_n}(\zeta(z), \tau_0)}{m_{\mu \Delta}^1(\zeta(z), \tau_0)} . \tag{2.3}$$

Thus by the function $(M_{\mu}^{0E_n}/m_{\mu}^1) + v_0$ D and Δ generate the same domain $B(\ni v_0$(not located on a branch manifold)) of certain kind.

Proof. Integrating from t_0 to z both sides of (2.2), and noting $M_{\mu D}^{0E_n}(t_0, t_0)/m_{\mu D}^1(t_0, t_0) = 0$ the vector function $M_{\mu D}/m_{\mu D}$ is represented by

$$\frac{M_{\mu D}^{0E_n}(z, t_0)}{m_{\mu D}^1(z, t_0)} = T_{\mu D}^{-1}(t_0, \bar{t}_0) \int_{t_0}^{z} T_{\mu D}(z, \bar{t}_0) dz . \tag{2.4}$$

Using the relations (2.4) and (1.5), the invariant (2.3) is easily obtained.

We call this unique domain B(in Theorem 2.1) a *μ-th representative domain of D with respect to t_0 with center at a point v_0.* Further we shall name $w(z) = (M_{\mu D}^{0E_n}(z, t_0)/m_{\mu D}^1(z, t_0)) + v_0$ a *μ-th representative function of the domain D.*

If a domain D is homogeneous, then we have $\det T_D(z, \bar{t}) = c \cdot k_D(z, \bar{t})$ in D. Hence $T_{2D}(z, \bar{t})$ is equal to $(2n+1) T_D(z, \bar{t})$. Thus in this case 2-nd representative function coincides with the usual representative function.

We shall consider the mapping of D onto B by means of the function $w = (M_{\mu D}^{0E_n}(z, t_0)/m_{\mu D}^1(z, t_0)) + v_0$. Then the domain B becomes a μ-th representative domain with center at v_0, and further it holds that $(M_{\mu D}^{0E_n}(z, t_0)/m_{\mu D}^1(z, t_0)) + v_0 = (M_{\mu B}^{0E_n}(w, v_0)/m_{\mu B}^1(w, v_0)) + v_0$ by Theorem 2.1 because $w(z)$ satisfies the normalized conditions $w(t_0) = v_0$, $dw(t_0)/dz = E_n$. Therefore we have

COROLLARY 2.1. *μ-th Representative function of a μ-th representative domain B is the identity function.*

COROLLARY 2.2. *A domain B is a μ-th representative domain with center at v_0 if and only if*

$$(2.5) \qquad T_{\mu B}(w, \bar{v}_0) = T_{\mu B}(v_0, \bar{v}_0) = \text{const. } for \ w \in B .$$

Proof. Suppose that B is a μ-th representative domain with center at v_0. Then from the property of μ-th representative domain, it holds that $w - v_0 = M_{\mu B}^{0E_n}(w, v_0)/m_{\mu B}^1(w, v_0)$. By differentiating this equation with respect to w, we obtain $E_n = T_{\mu B}^{-1}(v_0, \bar{v}_0) T_{\mu B}(w, \bar{v}_0)$ from (2.2). Suppose conversely that the relation (2.5) holds. Let B_1 be the μ-th representative domain of B with respect to v_0, that is, the image domain by the μ-th representative function $w_1(w)$ of B. Then it suffices to show that $w_1(w)$ is equal to w. Since

$$dw_1(w)/dw = T_{\mu B}^{-1}(v_0, \bar{v}_0) T_{\mu B}(w, \bar{v}_0)$$

by Lemma 2.1, we have $dw_1(w)/dw = E_n$ from our assumption (2.5). Therefore, $w_1(w) = w$ because $w_1(v_0) = v_0$ (See Lemma 1.1).

A domain B is minimal with center v_0 if and only if $k_B(w, \bar{v}_0)$ is constant for $w \in B$. In view of this fact we shall define the μ-th quasiminimal domain which is the generalization of minimal domain, and secondly we shall state some results.

A domain B is called a *μ-th quasiminimal domain with center at a point $v_0(\in B)$* if and only if $k_{\mu B}(w, \bar{v}_0) = k_{\mu B}(v_0, \bar{v}_0) = \text{const.}$ for $w \in B$. Therefore the 1-st quasiminimal domain coincides with the ordinary minimal domain.

Let a domain B be minimal and representative with the same center at v_0, then $k_B(w, \bar{v}_0) = \text{const.}$ and $T_B(w, \bar{v}_0) = \text{const.}$ for $w \in B$, respectively (Maschler [12], Tsuboi [8]). Hence we have $k_{2B}(w, \bar{v}_0) = \text{const.}$ Thus there exists the 2-nd quasiminimal domain.

THEOREM 2.2. *A necessary and sufficient condition that a μ-th representative domain B of a domain D with respect to t_0 with center at v_0 becomes simultaneously a μ-th quasiminimal domain with the same center v_0 is*

$$(\det T_{\mu D}(z, \bar{t}_0))^m / k_{\mu D}(z, \bar{t}_0) = (\det T_{\mu D}(t_0, \bar{t}_0))^m / k_{\mu D}(t_0, \bar{t}_0) = \text{const}\,.,$$

where we take $m = 1, 1 + 2n$ *for* $\mu = 1, 2$, *respectively.*

Proof. According to the invariant relations (1.5) and (1.8), we have $(\det T_{\mu D}(z, \bar{t}_0))^m / k_{\mu D}(z, \bar{t}_0) = (\det T_{\mu B}(w, \bar{v}_0))^m / k_{\mu B}(w, \bar{v}_0)$,

$$(\det T_{\mu D}(t_0, \bar{t}_0))^m / k_{\mu D}(t_0, \bar{t}_0) = (\det T_{\mu B}(v_0, \bar{v}_0))^m / k_{\mu B}(v_0, \bar{v}_0)$$

for a μ-th representative function $w = w(z)$ of D. From this formulas our required result is directly obtained.

THEOREM 2.3. *A necessary and sufficient condition that a μ-th quasiminimal domain B with center at v_0 becomes simultaneously a μ-th representative domain with the same center v_0 is*

$$\partial^2 k_{\mu B}(w, \bar{v}_0)/\partial v^* \partial w = k_{\mu B}(v_0, \bar{v}_0) T_{\mu B}(v_0, \bar{v}_0) = \text{const}\,.$$

Proof. Since B is a μ-th quasiminimal domain, we have

$$T_{\mu B}(w, \bar{v}_0) = k_{\mu B}^{-1}(v_0, \bar{v}_0) \frac{\partial^2}{\partial v^* \partial w} k_{\mu B}(w, \bar{v}_0) \tag{2.6}$$

from (1.3) and the equation $\partial k_{\mu B}(w, \bar{v}_0)/\partial w = 0$ which is equivalent to $k_{\mu B}(w, \bar{v}_0) = \text{const}$. From (2.6) and Corollary 2.2, our desired conclusion is at once obtained.

For $\mu = 1$ we have the following corollary.

COROLLARY 2.3. *A necessary and sufficient condition that a minimal domain B with center at v_0 becomes simultaneously a representative domain with the same center v_0 is*

$$\frac{\partial^2}{\partial v^* \partial w} k_B(w, \bar{v}_0) = k_B(v_0, \bar{v}_0) T_B(v_0, \bar{v}_0) = \text{const}\,. \tag{2.7}$$

If we denote $A \equiv k_B(v_0, \bar{v}_0) T_B(v_0, \bar{v}_0)$ in Corollary 2.3, then A is a positive definite constant Hermitian matrix. Using this A the condition (2.7) may be described as $\partial^2 k_B(w, \bar{v}_0)/\partial v^* \partial w = A$. This is equivalent to $\partial k_B(w, \bar{v}_0)/\partial v^* = A(w - v_0)$ because $\partial k_B(v_0, \bar{v}_0)/\partial v^* = 0$ (See Maschler [12]).

EXAMPLE 1. Let D be a bounded domain in z-plane. Then there exists a unique function $w \equiv w(z)$ which maps D onto the 2-nd quasiminimal domain B with center at $v_0 (= w(t_0))$ and it is

(2.8) $$w(z) = k_{2D}^{-1/3}(t_0, \bar{t}_0)\int_{t_0}^{z} k_{2D}^{1/3}(z, \bar{t}_0)dz + v_0 .$$

In fact, using the invariant (1.8)($n = 1$) we have

(2.9) $$k_{2D}^{-1/3}(t_0, \bar{t}_0)\int_{t_0}^{z} k_{2D}^{1/3}(z, \bar{t}_0)dz + v_0 = k_{2B}^{-1/3}(v_0, \bar{v}_0)\int_{v_0}^{w} k_{2B}^{1/3}(w, \bar{v}_0)dw + v_0 .$$

From this relation, (2.8) is easily obtained.

REMARK 3. We see from (2.9) that the function (2.8) is also a kind of representative function of the domain D. In general, the function $w(z) \equiv \int_{t_0}^{z} \{m_{\mu D}^{10\cdots 0}(z, t_0)\}^{1/\nu} dz (\nu = 1, 3 \text{ for } \mu = 1, 2)$ in the case of one variable may become a μ-th m-representative function (See § 4).

3. μ-th Normal domains ($\mu = 1, 2$). We denote $N_{\mu D}^{E_n}(z, t_0) \equiv T_{\mu D}^{-1}(t_0, \bar{t}_0)T_{\mu D}(z, \bar{t}_0)$, $\tilde{M}_{\mu D}^{0E_n}(z, t_0) \equiv \int_{t_0}^{z} N_{\mu D}^{E_n}(z, t_0)dz$ (See § 4), and we shall consider the function

(3.1) $$w(z) = T_{\mu D}^{1/2}(t_0, \bar{t}_0)\tilde{M}_{\mu D}^{0E_n}(z, t_0) + v_0 .$$

Then from (1.5) we have

$$\begin{aligned} dw^*dw &= dz^* N_{\mu D}^{E_n}(z, t_0)^* T_{\mu D}(t_0, \bar{t}_0) N_{\mu D}^{E_n}(z, t_0)dz \\ &= d\zeta^* N_{\mu\Delta}^{E_n}(\zeta(z), \tau_0)^* T_{\mu\Delta}(\tau_0, \bar{\tau}_0) N_{\mu\Delta}^{E_n}(\zeta(z), \tau_0)d\zeta \end{aligned}$$

for any pseudo-conformal mapping $\zeta \equiv \zeta(z)$. Hence we obtain

(3.2) $$(dw \equiv) T_{\mu D}^{1/2}(t_0, \bar{t}_0) N_{\mu D}^{E_n}(z, t_0)dz = U \cdot T_{\mu\Delta}^{1/2}(\tau_0, \bar{\tau}_0) N_{\mu\Delta}^{E_n}(\zeta(z), \tau_0)d\zeta ,$$

where U is a constant unitary matrix depending upon the points t_0 and τ_0. Thus we have

THEOREM 3.1. *The function* (3.1) *and the domain* $B \equiv w(D)$ *are both invariant under any pseudo-conformal mapping* $\zeta \equiv \zeta(z)$, *up to the constant unitary matrices.*

The above uniquely determined domain B up to the constant unitary matrix is called a *μ-th normal domain of D with center at a point v_0.*

COROLLARY 3.1. *A domain B is a μ-th normal domain with center at v_0 if and only if*

(3.3) $$N_{\mu B}^{E_n}(w, v_0) = T_{\mu B}^{-1/2}(v_0, \bar{v}_0) \cdot U^* = \text{const. } for \ w \in B .$$

Proof. Let the domain B be a μ-th normal domain of a domain

D. Then the function $w(z) = T^{1/2}_{\mu D}(t_0, \bar{t}_0)\widetilde{M}^{0E_n}_{\mu D}(z, t_0) + v_0$ maps D onto B. For this $w(z)$ it holds from (3.2) that

$$T^{1/2}_{\mu D}(t_0, \bar{t}_0)\widetilde{M}^{0E_n}_{\mu D}(z, t_0) + v_0 = U\cdot T^{1/2}_{\mu B}(v_0, \bar{v}_0)\widetilde{M}^{0E_n}_{\mu B}(w, v_0) + v_0 .$$

Therefore we have $w = U\cdot T^{1/2}_{\mu B}(v_0, \bar{v}_0)\widetilde{M}^{0E_n}_{\mu B}(w, v_0) + v_0$. Differentiating this equation concerning w, we obtain $E_n = U\cdot T^{1/2}_{\mu B}(v_0, \bar{v}_0)N^{E_n}_{\mu B}(w, v_0)$. Conversely if (3.3) holds, then

$$U\cdot T^{1/2}_{\mu B}(v_0, \bar{v}_0)\widetilde{M}^{0E_n}_{\mu B}(w, v_0) = \int_{v_0}^{w} E_n dw = w - v_0 .$$

COROLLARY 3.2. *If a domain B is a μ-th normal domain with center at v_0, then we have $T_{\mu B}(w, \bar{v}_0) = E_n$.*

The proof is obvious.

4. μ-th m-Representative domains ($\mu = 1, 2$). We shall investigate the generalized m-representative domains using the invariant of certain kind which is constructed by $T_{\mu D}(z, \bar{t})$ and its derivatives.

We define the matrix function $N^{E_n 0\cdots 0}_{\mu D}(z, t_0)$ of z for a fixed point $t_0(\in D)$ as follows:

$$N^{E_n 0\cdots 0}_{\mu D}(z, t_0)$$

$$\equiv (E_n 0 \cdots 0)\begin{pmatrix} T_{\mu D}(t_0, \bar{t}_0) & \cdots & \dfrac{\partial^{m-1}}{\partial z^{m-1}}T_{\mu D}(t_0, \bar{t}_0) \\ \vdots & & \vdots \\ \dfrac{\partial^{m-1}}{\partial t^{*m-1}}T_{\mu D}(t_0, \bar{t}_0) & \cdots & \dfrac{\partial^{2(m-1)}}{\partial t^{*m-1}\partial z^{m-1}}T_{\mu D}(t_0, \bar{t}_0) \end{pmatrix}^{-1} \tag{4.1}$$

$$\cdot \begin{pmatrix} T_{\mu D}(z, \bar{t}_0) \\ \vdots \\ \dfrac{\partial^{m-1}}{\partial t^{*m-1}}T_{\mu D}(z, \bar{t}_0) \end{pmatrix},$$

where several zeros denote $n \times {}_nH_2, \cdots, n \times {}_nH_m$ zero-matrices, respectively. And we denote the vector function

$$\widetilde{M}^{0E_n 0\cdots 0}_{\mu D}(z, t_0) \equiv \int_{t_0}^{z} N^{E_n 0\cdots 0}_{\mu D}(z, t_0)dz .$$

Here we assume that there exists the inverse of the second matrix of right side in (4.1), and that $\det N^{E_n 0\cdots 0}_{\mu D}(z, t_0) \neq 0$. Then we have

THEOREM 4.1. *The function $w(z) = \widetilde{M}^{0E_n 0\cdots 0}_{\mu D}(z, t_0)$ satisfies the normalized conditions*

$$w(t_0) = 0 , \qquad \frac{dw(t_0)}{dz} = E_n , \qquad \frac{d^2w(t_0)}{dz^2} = \cdots = \frac{d^m w(t_0)}{dz^m} = 0 . \tag{4.2}$$

And if a domain D is mapped onto a domain Δ by a pseudo-conformal mapping $\zeta \equiv \zeta(z)$ which satisfies

$$\zeta(t_0) = \tau_0 , \quad d\zeta(t_0)/dz = E_n , \quad d^2\zeta(t_0)/dz^2 = \cdots = d^m\zeta(t_0)/dz^m = 0$$

at a fixed point t_0, then we have

$$\widetilde{M}^{0E_n0\cdots0}_{\mu D}(z, t_0) = \widetilde{M}^{0E_n0\cdots0}_{\mu\Delta}(\zeta(z), \tau_0) . \tag{4.3}$$

Thus by the function $\widetilde{M}^{0E_n0\cdots0}_{\mu} + v_0$ D and Δ generate the same domain B.

Proof. The relation (4.2) is easily obtained from (4.1) by $(N, dN/dz, d^2N/dz^2, \cdots, d^{m-1}N/dz^{m-1}) = (E_n, 0, \cdots, 0)$, where

$$N \equiv N^{E_n0\cdots0}_{\mu D}(t_0, t_0) .$$

Next, noting $\partial T/\partial z = \partial^3 \log k/\partial t^*\partial z^2, \cdots$, etc., we have the following relations for a pseudo-conformal mapping $\zeta \equiv \zeta(z)$ satisfying

$$\zeta(t_0) \equiv \tau_0, \qquad d^2\zeta(t_0)/dz^2 = \cdots = d^m\zeta(t_0)/dz^m = 0$$

at t_0:

$$\partial^{p+q}T_{\mu D}(z, \bar{t}_0)/\partial t^{*p}\partial z^q = (d\tau(t_0)/dt)^{*p+1}\cdot\partial^{p+q}T_{\mu\Delta}(\zeta, \bar{\tau}_0)/\partial\tau^{*p}\partial\zeta^q\cdot(d\zeta(z)/dz)^{q+1} ,$$

where $p, q (0 \leqq p, q \leqq m-1)$ are both integers. Hence we obtain

$$N^{E_n0\cdots0}_{\mu D}(z, t_0) = \left(\frac{d\zeta(t_0)}{dz}\right)^{-1} N^{E_n0\cdots0}_{\mu\Delta}(\zeta(z), \tau_0)\frac{d\zeta(z)}{dz} , \tag{4.4}$$

which is led by a simple calculation (This relation (4.4) is important for study of μ-th m-normal domains). Integrating (4.4) from t_0 to z and noting our added assumption $d\zeta(t_0)/dz = E_n$, we have the required result (4.3).

We call, similarly in § 2, the unique domain B (in Theorem 4.1) a *μ-th m-representative domain ($m \geqq 1$) of D with respect to t_0 with center at a point v_0.* Further we shall name $w(z) = \widetilde{M}^{0E_n0\cdots0}_{\mu D}(z, t_0) + v_0$ a *μ-th m-representative function of the domain D.*

Example 2. For a unit circle in the case of one variable, the 1-st 2-representative function with respect to $t_0 (v_0 = 0)$ is

$$w(z) = (1 - |t_0|^2)(1 - \bar{t}_0 u)u ,$$

where $u \equiv (z - t_0)/(1 - \bar{t}_0 z)$. The other side Maschler's (1-st) 2-representative function of a unit circle is calculated as follows:

$$w(z) = (1 - |t_0|^2)(1 - 3\bar{t}_0 u)u/(1 - 2\bar{t}_0 u) .$$

In this case if we choose a fixed point t_0 in $1/2 < |t_0| < 1$, then $w(z)$ is nonregular.

COROLLARY 4.1. *μ-th m-Representative function of a μ-th m-representative domain B is the indentity function.*

This corollary is proved similarly as the proof of Corollary 2.1. Furthermore, as well as Corollary 2.2, we have

COROLLARY 4.2. *A domain B is a μ-th m-representative domain with center at v_0 if and only if $N_{\mu B}^{E_n 0\cdots 0}(w, v_0) = E_n(=N_{\mu B}^{E_n 0\cdots 0}(v_0, v_0))$ for $w \in B$.*

Let B be a μ-th m-representative domain with center at v_0, then, in the relation

$$N_{\mu B}^{E_n 0\cdots 0}(w, v_0) = N_{\mu B}^{E_n 0\cdots 0}(w, v_0) + (\partial^m N_{\mu B}^{E_n 0\cdots 0}(v_0, v_0)/\partial w^m)(*)$$

which is obtained by a simple calculation using Lemma 1.2, we have $N_{\mu B}^{E_n 0\cdots 0}(w, v_0) = E_n$ from Corollary 4.2 so that

$$\partial^m N_{\mu B}^{E_n 0\cdots 0}(w, v_0)/\partial w^m$$

vanishes at v_0. Thus we conclude $N_{\mu B}^{E_n 0\cdots 0}(w, v_0) = E_n$. Summing up the above argument we have arrived at

THEOREM 4.2. *If a domain B is a μ-th m-representative domain with center at v_0, then B is also a μ-th $(m+1)$-representative domain with the same center v_0.*

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