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One of the fundamental theorems in the theory of Jordan algebras is that of I. G. Macdonald which says that any identity in three variables x, y, z of degree zero or one in z will be valid in all Jordan algebras if it is valid in the special Jordan algebras.

In this paper we will extend this result to identities which also involve the inverses of x and y.

Following the method and notation of N. Jacobson [3] we have the

THEOREM. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are respectively the free Jordan algebra and free special Jordan algebra on three free generators x, y, z and the inverses x^{-1}, y^{-1}, with $\mathfrak{E}$ and $\mathfrak{E}_s$ the associative algebras of linear transformations in $\mathfrak{J}$ and $\mathfrak{J}_s$ respectively generated by the multiplications by elements of the subalgebra generated by x, y, x^{-1}, y^{-1}, then the canonical homomorphism ν of $\mathfrak{E}$ onto $\mathfrak{E}_s$ is an isomorphism. If $\mathfrak{F}$ is the free associative algebra with free generators $f_{i,j} (i, j \in Z)$ and π the homomorphism of $\mathfrak{F}$ onto $\mathfrak{E}$ determined by $f_{i,j} \rightarrow U_{x^i, y^j}$ then the kernel of π is the ideal $\mathfrak{R}$ generated by the elements*

$$
(1) \qquad
\begin{array}{ll}
\text{(i)} & f_{0,0} - 1 \\
\text{(ii)} & 2f_{i,0}f_{j,k} - (2f_{i,0}^2 - f_{2i,0})f_{j-i,k} - f_{i+j,k} \\
 & 2f_{0,i}f_{k,j} - (2f_{0,i}^2 - f_{0,2i})f_{k,j-i} - f_{k,i+j} \\
\text{(iii)} & 2f_{j,k}f_{i,0} - f_{j-i,k}(2f_{i,0}^2 - f_{2i,0}) - f_{i+j,k} \\
 & 2f_{k,j}f_{0,i} - f_{k,j-i}(2f_{0,i}^2 - f_{0,2i}) - f_{k,i+j} \,.
\end{array}
$$

From this as immediate corollaries we have

MACDONALD'S THEOREM WITH INVERSES [4]. *If $\mathfrak{J}$ and $\mathfrak{J}_s$ are the free Jordan algebra and free special Jordan algebra on three free generators x, y, z and the inverses x^{-1}, y^{-1} then the kernel of the canonical homomorphism ν of $\mathfrak{J}$ onto $\mathfrak{J}_s$ contains no elements of degree zero or one in z.*

SHIRSHOV'S THEOREM WITH INVERSES [6]. *The free Jordan algebra on two free generators x, y and their inverses x^{-1}, y^{-1} is special.*

More generally, we have the

SHIRSHOV-COHN THEOREM WITH INVERSES [1]. *Any Jordan algebra* *generated by two elements and their inverses is special.*

Indeed, such an algebra $\mathfrak{K}$ is a homomorphic image of the free Jordan algebra $\mathfrak{H}$ generated by x, y, x^{-1}, y^{-1}; by Shirshov's Theorem with Inverses $\mathfrak{H} = \mathfrak{H}_s$, the free special Jordan algebra generated by x, y, x^{-1}, y^{-1}; thus for some ideal $\mathfrak{K}_s$ we have $\mathfrak{K} = \mathfrak{H}_s/\mathfrak{K}_s$. By a result of P. M. Cohn, $\mathfrak{K}$ is special if and only if

$$\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s \subset \mathfrak{K}_s$$

where $\mathfrak{H}_s$ is imbedded in the free associative algebra $\mathfrak{A}$ generated by x, y, x^{-1}, y^{-1} (we are following the argument of [1, p. 307]). Noting that $\mathfrak{K}_s \subset \mathfrak{H}_s$ and the elements of $\mathfrak{H}_s$ are symmetric under the reversal involution * of $\mathfrak{A}$, we see $\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s$ is contained in the linear span of the

$$f(x, y, x^{-1}, y^{-1}, k) = akb + b^*ka^*$$

where $a, b \in \mathfrak{A}$, $k \in \mathfrak{K}_s$, and $f(x, y, x^{-1}, y^{-1}, z)$ is a symmetric element of the free associative algebra $\mathfrak{B}$ generated by x, y, z, x^{-1}, y^{-1}. Ordering the generators of $\mathfrak{B}$ by $z < x < x^{-1} < y < y^{-1}$ we see that the tetrads

$$\begin{aligned}
\{xx^{-1}yy^{-1}\} &= 1\\
\{zx^{-1}yy^{-1}\} &= z\cdot x^{-1}\\
\{zxyy^{-1}\} &= z\cdot x\\
\{zxx^{-1}y^{-1}\} &= z\cdot y^{-1}\\
\{zxx^{-1}y\} &= z\cdot y
\end{aligned}$$

are Jordan elements of $\mathfrak{B}$, hence by Cohn's Theorem [1, p. 306] $f(x, y, x^{-1}, y^{-1}, z)$ is a Jordan element of $\mathfrak{B}$. As a Jordan product of x, y, x^{-1}, y^{-1} and the element k of the Jordan ideal $\mathfrak{K}_s$, the element $f(x, y, x^{-1}, y^{-1}, k) \in \mathfrak{K}_s$. Thus $\mathfrak{A}\mathfrak{K}_s\mathfrak{A} \cap \mathfrak{H}_s \subset \mathfrak{K}_s$ as desired.

1. Preliminaries. By "algebra" we will mean algebra *with identity* over a field Φ *of characteristic* $\neq 2$; associativity and finite-dimensionality are not assumed.

Recall [5, p. 18] that an element a of a Jordan algebra is *invertible* with (Jordan) inverse b if

$$a\cdot b = 1,\ a^2\cdot b = a\ .$$

In this case b is invertible with inverse a, and a, b generate a commutative associative subalgebra; we write $b = a^{-1}$. In a special Jordan algebra the notion of Jordan inverse is equivalent to inverse in the associative sense.

Given a set $\mathfrak{X}$ and a subset $\mathfrak{Y}$ we denote by $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ the *free*

Jordan algebra generated by $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$. If $\mathfrak{Y} \to \mathfrak{Y}^{-1}$ is a bijection of $\mathfrak{Y}$ onto a set $\mathfrak{Y}^{-1}$ disjoint from $\mathfrak{X}$ we may set $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) = \mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})/\mathfrak{K}$ where $\mathfrak{K}$ is the ideal in the free Jordan algebra $\mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})$ generated by all $y \cdot y^{-1} - 1$, $y^2 \cdot y^{-1} - y$ for $y \in \mathfrak{Y}$. Similarly we have the *free special Jordan algebra* $\mathfrak{J}_s(\mathfrak{X}/\mathfrak{Y})$ *generated by* $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$; this may be regarded as the subalgebra of $\mathfrak{F}(\mathfrak{X}/\mathfrak{Y})^+$ generated by $\mathfrak{X} \cup \mathfrak{Y}^{-1}$, where $\mathfrak{F}(\mathfrak{X}/\mathfrak{Y})$ is the *free associative algebra generated by* $\mathfrak{X}$ *and the inverses of* $\mathfrak{Y}$.

If L_a denotes left-multiplication by an element a of a Jordan algebra $\mathfrak{A}$ we have the following operator identities

$$(2)\qquad \begin{gathered}[L_a, L_{b\cdot c}] + [L_b, L_{c\cdot a}] + [L_c, L_{a\cdot b}] = 0\\ L_aL_bL_c + L_cL_bL_a + L_{b\cdot(a\cdot c)} = L_{a\cdot b}L_c + L_{b\cdot c}L_a + L_{c\cdot a}L_b\ .\end{gathered}$$

If we set

$$(3)\qquad U_{a,b} = L_aL_b + L_bL_a - L_{a\cdot b}$$

then we have $U_a = U_{a,a}$, $L_a = U_{a,1} = U_{1,a}$. It is well known [3, p. 243] that if $\mathfrak{X}$ is a set of generators (containing 1) for a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ then the operators $U_{x,y}$ for $x, y \in \mathfrak{X}$ generate the same algebra of linear transformations as the L_b for $b \in \mathfrak{B}$. In particular, it is not hard to see that if $\mathfrak{B}$ is generated by x, y, x^{-1}, y^{-1} then the U_{x^i,y^j} for $i, j \in Z$ generate the same algebra $\mathfrak{E}$ of linear transformations as do the L_b for $b \in \mathfrak{B}$.

2. The presentation π.

The above remarks show that the homomorphism $\pi: \mathfrak{F} \to \mathfrak{E}$ in the Theorem is surjective. We next show that the ideal $\mathfrak{R}$ generated by the elements (1) is contained in the kernel of π, i.e. $\pi(f) = 0$ for f of the form (i), (ii), (iii) in (1). Part (i) is trivial since $U_{x^0,y^0} = I$. Parts (ii) and (iii) follow from the first part of (ii) by symmetry in x and y and symmetry in the operator relations (a consequence of the symmetry in (2); more precisely, this "symmetry" corresponds to the canonical involution in the universal multiplication envelope). The first part of (ii) follows from the following lemma by taking $a = x^i$, $b = x^{j-i}$, $c = y^k$ and noting [5, p. 19] that $[L_{x^n}, L_{x^m}] = 0$ for all $n, m \in Z$.

LEMMA 1. *If elements* a, b, c *of a Jordan algebra satisfy*

$$[L_a, L_b] = [L_{a^2}, L_b] = 0$$

then

$$2L_aU_{a\cdot b,c} = U_aU_{b,c} + U_{a^2\cdot b,c}\ .$$

Proof. By (2), (3) and our hypotheses we have

$$\begin{aligned}
2L_aU_{a\cdot b,c} &= 2L_a\{L_{a\cdot b}L_c + L_cL_{a\cdot b} - L_{c\cdot(a\cdot b)}\} + 2[L_a, L_b]\{L_{a\cdot c} - L_cL_a\} \\
&= \{2L_aL_{a\cdot b}\}L_c + 2L_a\{L_cL_{a\cdot b} - L_{c\cdot(a\cdot b)} + L_bL_{a\cdot c} - L_bL_cL_a\} \\
&\quad + L_b\{2L_aL_cL_a - 2L_aL_{a\cdot c}\} \\
&= \{2L_aL_bL_a + L_{a^2\cdot b} - L_bL_{a^2}\}L_c + 2L_a\{L_aL_cL_b - L_aL_{b\cdot c}\} \\
&\quad + L_b\{L_cL_{a^2} - L_{c\cdot a^2}\} \\
&= \{2L_a^2L_b + L_{a^2\cdot b} - L_{a^2}L_b\}L_c + 2L_a^2\{L_cL_b - L_{b\cdot c}\} \\
&\quad + \{L_cL_{a^2\cdot b} + L_{a^2}L_{b\cdot c} - L_{a^2}L_cL_b - L_{c\cdot(a^2\cdot b)}\} \\
&= \{2L_a^2 - L_{a^2}\}\{L_bL_c + L_cL_b - L_{b\cdot c}\} \\
&\quad + \{L_{a^2\cdot b}L_c + L_cL_{a^2\cdot b} - L_{c\cdot(a^2\cdot b)}\} \\
&= U_aU_{b,c} + U_{a^2\cdot b,c} .
\end{aligned}$$

Thus π induces a homomorphism σ of $\mathfrak{A} = \mathfrak{F}/\mathfrak{R}$ onto $\mathfrak{E}$.

LEMMA 2. *If $e_{i,j} \in \mathfrak{A} = \mathfrak{F}/\mathfrak{R}$ is the image of $f_{i,j} \in \mathfrak{F}$ and we set $a_i = e_{i,0}$, $b_i = 2a_i^2 - a_{2i}$, $c_i = e_{0,i}$, $d_i = 2c_i^2 - c_{2i}$ then we have the following identities:*

$$\begin{array}{lll}
 & (\mathrm{i}) & a_0 = b_0 = c_0 = d_0 = e_{0,0} = 1 \\
 & (\mathrm{ii}) & 2a_ie_{j,k} = b_ie_{j-i,k} + e_{i+j,k},\ 2c_ie_{k,j} = d_ie_{k,j-i} + e_{k,i+j} \\
 & (\mathrm{iii}) & 2e_{j,k}a_i = e_{j-i,k}b_i + e_{i+j,k},\ 2e_{k,j}c_i = e_{k,j-i}d_i + e_{k,i+j} \\
 & (\mathrm{iv}) & 2a_ia_j = b_ia_{j-i} + a_{i+j},\ 2c_ic_j = d_ic_{j-i} + c_{i+j} \\
(4) & (\mathrm{v}) & 2a_ja_i = a_{j-i}b_i + a_{i+j},\ 2c_jc_i = c_{j-i}d_i + c_{i+j} \\
 & (\mathrm{vi}) & a_i = a_{-i}b_i = b_ia_{-i},\ c_i = c_{-i}d_i = d_ic_{-i} \\
 & (\mathrm{vii}) & b_ib_{-i} = b_{-i}b_i = 1,\ d_id_{-i} = d_{-i}d_i = 1 \\
 & (\mathrm{viii}) & [a_i, a_j] = [a_i, b_j] = 0,\ [c_i, c_j] = [c_i, d_j] = 0 \\
 & (\mathrm{ix}) & b_ib_j = b_{i+j},\ d_id_j = d_{i+j} .
\end{array}$$

Proof. (i)–(vi) follow immediately from the relations (1). (vii) follows from

$$b_ib_{-i} = b_i\{2a_{-i}^2 - a_{-2i}\} = 2a_ia_{-i} - b_ia_{-2i}$$

(by vi) $= a_0$ (by iv) $= 1$ (by i). For (viii) it suffices to show $[a_i,a_j] = 0$, and this only for $i, j \geqq 0$ since $b_i = 2a_i^2 - a_{2i}$ and $a_{-i} = b_i^{-1}a_i$ by (vi), and finally only for $i = 1$, $j = 2$ since (iv) shows by induction that the a_i for $i \geqq 0$ are generated by a_1, b_1 (hence a_1, a_2). But (iv), (v) show $2[a_1, a_2] = [b_1, a_1] = -[a_2, a_1] = [a_1, a_2]$, so $[a_1, a_2] = 0$ as desired. For (ix) it suffices to show $b_i = b_1^i$, and this only for $i \geqq 0$ by (vii); this follows by induction from (i) and

$$
\begin{aligned}
b_{i+1} &= 2a_{i+1}^2 - a_{2i+2} \\
&= 2a_{i+1}\{2a_i a_1 - a_{i-1}b_1\} - a_{2i+2} \quad \text{(by v)} \\
&= 2\{a_1 b_i + a_{2i+1}\}a_1 - \{a_2 b_{i-1} + a_{2i}\}b_1 - \{2a_{2i+1}a_1 - a_{2i}b_1\} \quad \text{(by v)} \\
&= 2a_1 b_i a_1 - a_2 b_{i-1} b_1 \\
&= \{2a_1^2 - a_2\}b_i \quad \text{(by viii and induction)} \\
&= b_1 b_i \,.
\end{aligned}
$$

3. The idea of the proof. We have surjective homomorphisms $\sigma: \mathfrak{A} \to \mathfrak{E}$ and $\nu: \mathfrak{E} \to \mathfrak{E}_s$, and a linear mapping $\tau: \mathfrak{E}_s \to \mathfrak{F}_s$ by $L \to L(z)$. The theorem will be proven if we show $\mu = \tau \circ \nu \circ \sigma$ is injective, for then σ and ν will be isomorphisms. This will be the case if we find a spanning set in $\mathfrak{A}$ whose image under μ is independent in $\mathfrak{F}_s$. A hint is provided by Cohn's Theorem [1, p. 307] which says that $\mu(\mathfrak{A})$ is precisely the set of all elements of the free associative algebra $\mathfrak{F}(x, y, z/x, y)$ which are linear in z and symmetric under the reversal involution $*$. A basis for this set consists of the distinct

$$f_s(p, q) = \tfrac{1}{2}\{pzq^* + qzp^*\} = f_s(q, p)$$

for monomials $p, q \in \mathfrak{F}(x, y/x, y)$. The idea of the proof [3, p. 249] is to construct pre-images

$$f(p, q) = f(q, p)$$

in $\mathfrak{A}$ satisfying

$$\mu(f(p, q)) = f_s(p, q) \,. \tag{5}$$

By definition the images in $\mathfrak{F}_s$ will be independent, and the only question is whether these elements span $\mathfrak{A}$. Since $\mathfrak{A}$ is generated by 1 and the elements $b_k, d_k, e_{k,l}, a_k, c_k$ it suffices to show the set of $f(p, q)$ contains 1

$$f(1, 1) = 1 \tag{6}$$

and is invariant under left multiplication by the generators

$$
\begin{aligned}
&\text{(i)} && b_k f(p, q) = f(x^k p, x^k q) \\
&\text{(ii)} && d_k f(p, q) = f(y^k p, y^k q) \\
(7)\quad &\text{(iii)} && e_{k,l} f(p, q) = \tfrac{1}{2}\{f(x^k p, y^l q) + f(y^l p, x^k q)\} \quad (k, l \neq 0) \\
&\text{(iv)} && a_k f(p, q) = \tfrac{1}{2}\{f(x^k p, q) + f(p, x^k q)\} \\
&\text{(v)} && c_k f(p, q) = \tfrac{1}{2}\{f(y^k p, q) + f(p, y^k q)\} \,.
\end{aligned}
$$

To this end we define $f(p, q)$ by induction as follows. First we inductively define sets $\mathfrak{X}_n, \mathfrak{Y}_n$ $(n \geqq 0)$ of monomials in $\mathfrak{F}(x, y/x, y)$ by

$$\mathfrak{X}_0 = \mathfrak{Y}_0 = \{1\}$$
$$\mathfrak{X}_{n+1} = \{x^k p \mid k \neq 0, p \in \mathfrak{Y}_n\} \qquad \mathfrak{Y}_{n+1} = \{y^k p \mid k \neq 0, p \in \mathfrak{X}_n\} .$$

Next we define sets of pairs of monomials by

$$\begin{aligned} \mathfrak{X}_{n,m} &= \mathfrak{X}_n \times \mathfrak{X}_m \cup \mathfrak{X}_m \times \mathfrak{X}_n = \mathfrak{X}_{m,n} \\ \mathfrak{Y}_{n,m} &= \mathfrak{Y}_n \times \mathfrak{Y}_m \cup \mathfrak{Y}_m \times \mathfrak{Y}_n = \mathfrak{Y}_{m,n} \\ \mathfrak{Z}_{n,m} &= \mathfrak{X}_n \times \mathfrak{Y}_m \cup \mathfrak{Y}_m \times \mathfrak{X}_n . \end{aligned}$$

Finally, f is defined recursively on the sets $\mathfrak{X}_{n,m}, \mathfrak{Y}_{n,m}, \mathfrak{Z}_{n,m}$ by

(D.0) On $\mathfrak{X}_{0,0} = \mathfrak{Y}_{0,0} = \mathfrak{Z}_{0,0} = \{(1, 1)\}$:

$$f(1, 1) = 1 .$$

(D.1) On $\mathfrak{X}_{n+1,m+1}$: for $i, j \neq 0$, $i \geqq j$, $(r, s) \in \mathfrak{Y}_{n,m}$

$$f(x^i r, x^j s) = f(x^j s, x^i r) = b_j f(x^{i-j} r, s) .$$

(D.2) On $\mathfrak{Y}_{n+1,m+1}$: for $i, j \neq 0$, $i \geqq j$, $(r, s) \in \mathfrak{X}_{n,m}$

$$f(y^i r, y^j s) = f(y^j s, y^i r) = d_j f(y^{i-j} r, s) .$$

(D.3) On $\mathfrak{Z}_{n+1,m+1}$: for $i, j \neq 0$, $r \in \mathfrak{Y}_n$, $s \in \mathfrak{X}_m$

$$f(x^i r, y^j s) = f(y^j s, x^i r) = 2e_{i,j} f(r, s) - f(y^j r, x^i s)$$

which is defined by induction unless $n = m = 0$, $r = s = 1$, and on $\mathfrak{Z}_{1,1}$:

$$f(x^i, y^j) = f(y^j, x^i) = e_{i,j} .$$

(D.4) On $\mathfrak{X}_{n+1,0} = \mathfrak{X}_{0,n+1} = \mathfrak{Z}_{n+1,0}$: for $i \neq 0$, $r \in \mathfrak{Y}_n$

$$f(x^i r, 1) = f(1, x^i r) = 2a_i f(r, 1) - f(r, x^i)$$

which is defined by induction if $n \neq 0$, and on $\mathfrak{X}_{1,0} = \mathfrak{X}_{0,1} = \mathfrak{Z}_{1,0}$:

$$f(x^i, 1) = f(1, x^i) = a_i .$$

(D.5) Similarly, on $\mathfrak{Y}_{n+1,0} = \mathfrak{Y}_{0,n+1} = \mathfrak{Z}_{0,n+1}$:

$$f(y^i r, 1) = f(1, y^i r) = 2c_i f(r, 1) - f(r, y^i)$$

and on $\mathfrak{Y}_{1,0} = \mathfrak{Y}_{0,1} = \mathfrak{Z}_{0,1}$:

$$f(y^i, 1) = f(1, y^i) = c_i .$$

It is easy to verify that $f(p, q) = f(q, p)$ is a well-defined element of $\mathfrak{A}$ for all monomials p, q in $\mathfrak{F}(x, y/x, y)$.

4. The main lemma. The previous considerations have reduced the proof of the theorem to the following.

LEMMA 3. *The elements* $f(p, q) = f(q, p) \in \mathfrak{A}$ *defined by* (D.0)—(D.5) *satisfy* (5), (6), (7).

Proof. (5) can be verified at each step of the inductive definition, and (6) is just (D.0). We will prove (7.i)-(7.v) for (p, q) in $\mathfrak{X}_{n,m}, \mathfrak{Y}_{n,m}, \mathfrak{Z}_{n,m}$ by induction on the *weight* $n + m$; the case $n + m = 0$ follows immediately from the definitions (D.1)-(D.5), and we assume the result proven for all weights less than $n + m$. We claim that if $i, j \neq 0$ (but $k, l = 0$ are allowed) then

$$
(8)\quad
\begin{aligned}
&(\mathrm{i})\quad (r, s) \in \mathfrak{Y}_{n-1,m-1} \Rightarrow b_k f(x^i r, x^j s) = f(x^{k+i} r, x^{k+j} s)\\
&(\mathrm{ii})\quad (r, s) \in \mathfrak{Y}_{n-1} \times \mathfrak{X}_{m-1} \Rightarrow b_k f(x^i r, y^j s) = f(x^{k+i} r, x^k y^j s)\\
&(\mathrm{iii})\quad r \in \mathfrak{Y}_{n-1} \Rightarrow b_k f(x^i r, 1) = f(x^{k+i} r, x^k)\\
&(\mathrm{iv})\quad (r, s) \in \mathfrak{Y}_{n-1,m-1} \Rightarrow 2e_{k,l} f(x^i r, x^j s)\\
&\qquad = f(x^{k+i} r, y^l x^j s) + f(y^l x^i r, x^{k+j} s)\\
&(\mathrm{v})\quad (r, s) \in \mathfrak{Y}_{n-1} \times \mathfrak{X}_{m-1} \Rightarrow 2a_k f(x^i r, y^j s)\\
&\qquad = f(x^{k+i} r, y^j s) + f(x^i r, x^k y^j s)\\
&(\mathrm{vi})\quad r \in \mathfrak{Y}_{n-1} \Rightarrow 2a_k f(x^i r, 1) = f(x^{k+i} r, 1) + f(x^i r, x^k)\ .
\end{aligned}
$$

These suffice to establish the various cases of (7) according to the following table:

	7.i	7.iv	7.iii	7.v	7.ii
$\mathfrak{X}_{n,m}$	8.i	8.iv	8.iv	8.iv	def
$\mathfrak{X}_{n,0} = \mathfrak{Z}_{n,0}$	8.iii	8.vi	def	def	def
$\mathfrak{Z}_{n,m}$	8.ii	8.v	def	8.v*	8.ii*
$\mathfrak{Y}_{n,0} = \mathfrak{Z}_{0,n}$	def	def	def	8.vi*	8.iii*
$\mathfrak{Y}_{n,m}$	def	8.iv*	8.iv*	8.iv*	8.i*

Here the columns indicate the particular cases of (7) and the rows the particular possibilities for (p, q), with $n, m > 0$; "def" means the result follows directly from the definitions, and * denotes the dual formula obtained by everywhere interchanging x and y. The proof of (8.i)-(8.vi) will be broken into corresponding Cases I-VI.

Case I. (a) If $k + i, k + j \neq 0$, say $i \geqq j$, then

$$
\begin{aligned}
b_k f(x^i r, x^j s) &= b_k b_j f(x^{i-j} r, s) && \text{(D.1)}\\
&= b_{k+j} f(x^{i-j} r, s) && \text{(4.ix)}\\
&= f(x^{k+i} r, x^{k+j} s)\ . && \text{(D.1)}
\end{aligned}
$$

(b) If, say, $k + j = 0$ then

$$\begin{aligned} b_k f(x^i r, x^j s) &= b_k b_j f(x^{i-j} r, s) && \text{(induction 7.i)} \\ &= f(x^{i+k} r, x^{j+k} s)\ . && \text{(4.vii)} \end{aligned}$$

Case II. (a) If $0 \neq i + k \geqq k$ the result follows from (D.1).
(b) If $0 = i + k$, $m = n = 0$, $r = s = 1$ we have

$$\begin{aligned} b_k f(x^i, y^j) &= b_k e_{-k,j} && \text{(D.3)} \\ &= 2a_k e_{0,j} - e_{k,j} && \text{(4.ii)} \\ &= f(1, x^k y^j) && \text{(D.4)} \\ &= f(x^{k+i}, x^k y^j)\ . \end{aligned}$$

(c) If $0 = i + k$ but $r \neq 1$ or $s \neq 1$ then

$$\begin{aligned} b_k f(x^i r, y^j s) &= b_k\{2e_{i,j} f(r, s) - f(y^j r, x^i s)\} && \text{(D.3)} \\ &= 2\{2a_k e_{0,j} - e_{k,j}\} f(r, s) - f(x^k y^j r, s) \end{aligned}$$

by 4.ii and induction 7.i—which is applicable since by our assumptions on r and s $(y^j r, x^i s)$ has weight less than $n + m$)

$$\begin{aligned} &= 4a_k c_j f(r, s) - \{f(x^k r, y^j s) + f(y^j r, x^k s)\} - f(x^k y^j r, s) \\ & && \text{(induction 7.iii)} \\ &= 4a_k c_j f(r, s) - f(x^k r, y^j s) - 2a_k f(y^j r, s) && \text{(induction 7.iv)} \\ &= 2a_k f(r, y^j s) - f(x^k r, y^j s) && \text{(induction 7.v)} \\ &= f(r, x^k y^j s) && \text{(induction 7.iv)} \\ &= f(x^{k+i} r, x^k y^j s)\ . \end{aligned}$$

(d) If $0 \neq i + k < k$ then

$$\begin{aligned} b_k f(x^i r, y^j s) &= b_k\{2a_i f(r, y^j s) - f(r, x^i y^j s)\} && \text{(induction 7.iv)} \\ &= b_k b_i\{2a_{-i} f(r, y^j s) - f(x^{-i} r, y^j s)\} && \text{(4.vi, Case IIb, c above)} \\ &= b_{k+i} f(r, x^{-i} y^j s) && \text{(4.ix, induction 7.iv)} \\ &= f(x^{k+i} r, x^k y^j s)\ . && \text{(D.1)} \end{aligned}$$

Case III. The proof is obtained from that of Case II by setting $j = 0$, $s = 1$; the second line of the proof of (c) is justified by Case II rather than by induction 7.i.

Case IV. We allow k or l to be zero, and we induct on $|i| + |j|$; the result follows from the induction hypothesis if i or j is zero.

(a) If i, j have the same sign, say $|i| \geqq |j| > 0$, then $|i - j| < |i|$ so

$$
\begin{aligned}
2e_{k,l}f(x^i r,\, x^j s) &= 2e_{k,l}b_j f(x^{i-j}r,\, s) \\
&\qquad \text{(induction 7.i)} \\
&= 2\{2e_{k+j,l}a_j - e_{k+2j,l}\}f(x^{i-j}r,\, s) \\
&\qquad \text{(4.iii)} \\
&= 2e_{k+j,l}\{f(x^i r,\, s) + f(x^{i-j}r,\, x^j s)\} \\
&\quad - \{f(x^{i+j+k}r,\, y^l s) + f(y^l x^{i-j}r,\, x^{k+2j}s)\} \\
&\qquad \text{(induction 7.iii–iv)} \\
&= \{2e_{k+j,l}f(x^{i-j}r,\, x^j s) - f(y^l x^{i-j}r,\, x^{k+2j}s)\} \\
&\quad + \{2e_{k+j,l}f(x^i r,\, s) - f(x^{i+j+k}r,\, y^l s)\} \\
&= f(x^{i+k}r,\, y^l x^j s) + f(y^l x^i r,\, x^{k+j}s)\,. \\
&\qquad \text{(induction, } |i-j| + |j| < |i| + |j|\text{)}
\end{aligned}
$$

(b) If i, j have opposite signs, say $|i| \geqq |j| > 0$, then $|i+j| < |i|$ so

$$
\begin{aligned}
2e_{k,l}f(x^i r,\, x^j s) &= 2e_{k,l}\{2a_j f(x^i r,\, s) - f(x^{i+j}r,\, s)\} \\
&\qquad \text{(induction 7.iv)} \\
&= 2\{e_{k-j,l}b_j + e_{k+j,l}\}f(x^i r,\, s) - 2e_{k,l}f(x^{i+j}r,\, s) \\
&\qquad \text{(4.iii)} \\
&= 2e_{k-j,l}f(x^{i+j}r,\, x^j s) - f(y^l x^{i+j}r,\, x^k s) - f(x^{i+j+k}r,\, y^l s) \\
&\quad + f(x^{i+j+k}r,\, y^l s) + f(y^l x^i r,\, x^{k+j}s) \\
&\qquad \text{(induction 7.i, 7.iii)} \\
&= f(x^{k+i}r,\, y^l x^j s) + f(y^l x^i r,\, x^{k+j}s)\,. \\
&\qquad (|i+j| + |j| < |i| + |j|,\ \text{induction})
\end{aligned}
$$

Case V. (a) If $m = n = 1$, $r = s = 1$ we have

$$
\begin{aligned}
2a_k f(x^i,\, y^j) &= 2a_k e_{i,j} = e_{i+k,j} + b_k e_{i-k,j} && \text{(D.3, 4.ii)} \\
&= f(x^{k+i},\, y^j) + b_k f(x^{i-k},\, y^j) && \text{(D.3)} \\
&= f(x^{k+i},\, y^j) + f(x^i,\, x^k y^j)\,. && \text{(Case II–III above)}
\end{aligned}
$$

(b) If $r \neq 1$ or $s \neq 1$ then

$$
\begin{aligned}
2a_k f(x^i r,\, y^j s) &= 2a_k\{2e_{i,j}f(r,\, s) - f(y^j r,\, x^i s)\} \qquad \text{(D.3)} \\
&= 2\{e_{i+k,j} + b_k e_{i-k,j}\}f(r,\, s) \\
&\quad - \{f(x^k y^j r,\, x^i s) + f(y^j r,\, x^{i+k}s)\}
\end{aligned}
$$

(by 4.ii and induction 7.iv—which is applicable since by our assumptions on r and s $(y^j r,\, x^i s)$ has weight less than $n + m$)

$$
\begin{aligned}
&= \{2e_{i+k,j}f(r,\, s) - f(y^j r,\, x^{i+k}s)\} \\
&\quad + b_k\{2e_{i-k,j}f(r,\, s) - f(y^j r,\, x^{i-k}s)\}
\end{aligned}
$$

(induction 7.i applicable to $(y^j r, x^{i-k} s)$—or use Case II above)

$$= f(x^{i+k} r, y^j s) + b_k f(x^{i-k} r, y^j s) \qquad \text{(induction 7.iii)}$$
$$= f(x^{k+i} r, y^j s) + f(x^i r, x^k y^j s) \ . \qquad \text{(Case 2 above)}$$

Case VI. This follows from Case V by setting $j = 0$, $s = 1$ throughout the proof; the second line in the proof of (b) is justified by Case V rather than by induction 7.iv.

This completes the proof of (8), the Lemma, and all the Theorems.

5. Remarks and conjectures. We will now indicate how the above proof can be modified to prove Macdonald's original theorem without inverses; in a similar manner we obtain a one-inverse form of the theorem.

We require that all indices i, j, k, l etc. be nonnegative; this modifies the free algebra $\mathfrak{F}$ of the theorem, so we add to the relations (1) the further elements of $\mathfrak{R}$

$$\text{(1.iv)} \quad \begin{aligned} & f_{i,0} f_{j,k} + f_{j,0} f_{i,k} - (2f_{j,0}^2 - f_{2j,0}) f_{i-j,0} f_{0,k} - f_{i+j,k} \\ & f_{0,i} f_{k,j} + f_{0,j} f_{k,i} - (2f_{0,j}^2 - f_{0,2j}) f_{0,i-j} f_{k,0} - f_{k,i+j} \end{aligned}$$

for $i \geqq j$ corresponding to the relations

$$\text{(4.x)} \quad \begin{aligned} a_i e_{j,k} + a_j e_{i,k} &= b_j a_{i-j} c_k + e_{i+j,k} \\ c_i e_{k,j} + c_j e_{k,i} &= d_j c_{i-j} a_k + e_{k,i+j} \end{aligned}$$

in the algebra $\mathfrak{A}$. It suffices to establish the first relation in (4.x), and this follows by putting $a = x^j$, $b = x^{i-j}$, $c = y^k$ in the following addition to Lemma 1: if a, b, c are elements of a Jordan algebra satisfying

$$[L_a, L_b] = [L_a, L_{a\cdot b}] = 0$$

then

$$L_a U_{a\cdot b, c} + L_{a\cdot b} U_{a,c} = U_a L_b L_c + U_{a^2 \cdot b, c} \ .$$

The only other thing to be changed is the proof of (8). Cases Ib, IIb-c-d, and IVb are unnecessary, but the proof of Case V works only for $i \geqq k$; for $i < k$ we must use the relations (4.x).

It would be nice if the inverse-less and one-inverse theorems could be obtained directly from the two-inverse form, which leads to a general

Conjecture. If $\mathfrak{X}_0 \subset \mathfrak{X}$, $\mathfrak{Y}_0 \subset \mathfrak{Y}$ then the canonical homomorphism $\mathfrak{J}(\mathfrak{X}_0/\mathfrak{Y}_0) \to \mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ is injective.

If we represent $\mathfrak{J}(\mathfrak{X}_0/\mathfrak{Y}_0)$ by $\mathfrak{J}(\mathfrak{X}_0 \cup \mathfrak{Y}_0^{-1})/\mathfrak{K}_0$ and $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y})$ by $\mathfrak{J}(\mathfrak{X} \cup \mathfrak{Y}^{-1})/\mathfrak{K}$ as in the first section of the paper then the conjecture amounts to

$$\mathfrak{J}(\mathfrak{X}_0 \cup \mathfrak{Y}_0^{-1}) \cap \mathfrak{K} = \mathfrak{K}_0 \, .$$

It is also sufficient to consider only the case $\mathfrak{X} = \mathfrak{X}_0$.

More generally, we have a

Conjecture. $\mathfrak{J}(\mathfrak{X})$ can be imbedded in a universal Jordan division algebra $\mathfrak{D}(\mathfrak{X})$ such that the canonical homomorphisms $\mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) \to \mathfrak{D}(\mathfrak{X})$ are all injective.

It is easy to see that this implies the first conjecture by considering the commutative diagram

$$\begin{tikzcd}
\mathfrak{J}(\mathfrak{X}/\mathfrak{Y}_0) \arrow[rr] \arrow[dr] & & \mathfrak{J}(\mathfrak{X}/\mathfrak{Y}) \arrow[dl] \\
& \mathfrak{D}(\mathfrak{X}) &
\end{tikzcd}$$

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