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Two types of $*$-anti-automorphisms of a von Neumann algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ leaving the center of $\mathfrak{A}$ elementwise fixed are discussed, those of order two and those of the form $A \rightarrow V^{-1}A^*V$, V being a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^2 \in \mathfrak{A}$. The latter anti-automorphisms are called inner, and are the composition of inner $*$-automorphisms and $*$-anti-automorphisms of the form $A \rightarrow JA^*J$, where J is a conjugation, i.e. a conjugate linear isometry of $\mathscr{H}$ onto itself such that $J^2 = I$. The former anti-automorphisms are also closely related to conjugations; they are almost, and in many cases exactly of the form $A \rightarrow JA^*J$. Moreover, the existence of $*$-anti-automorphisms of order two leaving the center fixed implies the existence of a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$, and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$.

There are two main problems concerning $*$-anti-automorphisms of von Neumann algebras, namely their existence and their description. In the present paper we shall deal with the latter question. It turns out that anti-automorphisms are closely associated with conjugations, a conjugation being a conjugate linear isometry of a Hilbert space onto itself whose square is the identity. This is not surprising, as such maps induce most of the important anti-isomorphisms of von Neumann algebras, cf. [1]. We shall characterize two classes of anti-automorphisms, namely those of order two leaving the center of the von Neumann algebra elementwise fixed, and the so-called inner anti-automorphisms, both characterizations being in terms of conjugations. In the process of doing so we shall make heavy use of Jordan and real operator algebra theory, as developed in [8], [9], and [10]. The second section is devoted to this theory; we shall generalize some of the results in [8] and [9], and in particular classify all weakly closed self-adjoint real abelian operator algebras.

We refer the reader to [1] for terminology and results concerning von Neumann algebras. If $\mathscr{R}$ is a family of operators on a Hilbert space we denote by $\mathscr{R}_{SA}$ the set of self-adjoint operators in $\mathscr{R}$. We say $\mathscr{R}$ is *self-adjoint* if $A^* \in \mathscr{R}$ whenever $A \in \mathscr{R}$. $\mathscr{R}$ is a *self-adjoint real operator algebra* if $\mathscr{R}$ is a self-adjoint family of operators which form an algebra over the real numbers. By a *JW-algebra* we shall mean a weakly closed real linear family of self-adjoint operators closed under squaring. By a *real $*$-isomorphism* of one self-adjoint real algebra into another we shall mean a one-to-one real linear map ϕ such that $\phi(A^*) = \phi(A)^*$, and $\phi(AB) = \phi(A)\phi(B)$ for all A, B in the algebra. By a *$*$-anti-automorphism* (or just anti-automorphism) of a von Neumann algebra $\mathfrak{A}$ we shall mean a one-to-one (complex) linear map ϕ of $\mathfrak{A}$ onto itself such that $\phi(A^*) = \phi(A)^*$ and $\phi(AB) = \phi(B)\phi(A)$ for all A, $B \in \mathfrak{A}$. We note that such a map is ultra-weakly continuous [1, Corollaire 1, p. 57]. We shall identify projections and their ranges. If $\mathfrak{A}$ is a family of operators and $\mathscr{M}$ is a set of vectors we write $[\mathfrak{A}\mathscr{M}]$ for the subspace generated by all vectors of the form Ax with $A \in \mathfrak{A}$ and $x \in \mathscr{M}$.

The $*$-anti-automorphisms ϕ studied in this paper will all turn out to be spatial, i.e. there exists a conjugate linear isometry V of the Hilbert space $\mathscr{H}$ such that $\phi(A) = V^{-1}A^*V$. That any such map ϕ is a $*$-anti-automorphism of $\mathscr{B}(\mathscr{H})$—the bounded linear operators on $\mathscr{H}$—is seen as follows. By polarization $(Vx, Vy) = \overline{(x, y)}$ for all $x, y \in \mathscr{H}$. Hence

$$((V^{-1}AV)^*x, y) = (x, V^{-1}AVy) = \overline{(Vx, AVy)} = (V^{-1}A^*Vx, y)$$

for all x, y, and $(V^{-1}AV)^* = V^{-1}A^*V$ for all $A \in \mathscr{B}(\mathscr{H})$. Clearly ϕ is linear and anti-isomorphic. If $(e_\alpha)_{\alpha \in I}$ is an orthonormal basis for $\mathscr{H}$ then the map $J\colon \sum \lambda_\alpha e_\alpha \rightarrow \sum \bar{\lambda}_\alpha e_\alpha$ is a conjugation of $\mathscr{H}$, hence there exist $*$-anti-automorphisms of factors of type I. The problem is open for general nontype I factors; however, it is known to the affirmative in constructed examples, a few examples will show how.

Let G be a countable discrete group such that the set $\{gg_0g^{-1}: g \in G\}$ is infinite for every $g_0 \neq e$. Let $\mathfrak{A}$ be the usual Hilbert algebra of complex functions x on G having finite support, where multiplication is convolution, $x^*(g) = \overline{x(g^{-1})}$, and

$$(x, y) = \sum_g x(g)\bar{y}(g) ,$$

[1, pp. 301–303]. For $x \in l^2(G)$ set $Jx(g) = \bar{x}(g)$. Then J is a conjugation. Let $\mathfrak{A}(G)$ be the II_1 factor of all left multiplications L_x by bounded dements of $l^2(G)$. Then simple calculations show

(i) x bounded implies Jx bounded.

(ii) $JL_xJ = L_{Jx}$ for all bounded x.

Thus $J\mathfrak{A}(G)J = \mathfrak{A}(G)$, and $\phi(A) = JA^*J$ is a $*$-anti-automorphism of $\mathfrak{A}(G)$ of order 2.

By specializing G, one can get $\mathfrak{A}(G)$ to be any one of the three known II_1 factors on a separable $\mathscr{H}$, see [6].

In the notation of [7, p. 112] one can define a conjugation J by

$$JF(\gamma, x) = \bar{F}(\gamma, x) .$$

Then $JU_\gamma J = U_\gamma$, and $JL_\phi J = L_{\bar{\phi}}$. So J induces a $*$-anti-automorphism of order 2 of the type III factor obtained in that construction.

2. Real operator algebras. We begin this section with four lemmas all of which are practically known.

LEMMA 2.1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras with identities I; let ρ be a real $*$-isomorphism of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ such that $\rho(I) = I$. Then there exist two orthogonal central projections E and F in $\mathfrak{A}_2$ with $E + F = I$, such that $E\rho$ is complex linear and $F\rho$ is complex conjugate linear.*

Proof. Let $A = \rho(iI)$. Then $A^2 = \rho(iI)^2 = \rho((iI)^2) = \rho(-I) = -I$. Thus $A = iE - iF$ with E and F as above. Clearly $E\rho$ is linear and $F\rho$ is conjugate linear.

The next lemma is a slight generalization of [9, Theorem 2.4]. The proof is practically the same as that in [9], and is omitted.

LEMMA 2.2. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Then $\mathscr{R} + i\mathscr{R}$ is a von Neumann algebra.*

If $\mathfrak{A}$ is a JW-algebra or a von Neumann algebra and E is a projection in $\mathfrak{A}$ then its central carrier with respect to $\mathfrak{A}$ is the smallest central projection in $\mathfrak{A}$ greater than or equal to E. It is denoted by $C_E(\mathfrak{A})$. The next lemma is a modification of [8, Lemma 8.1].

LEMMA 2.3. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let E be a projection in $\mathscr{R}$. Then $C_E(\mathscr{R}_{SA}) = C_E(\mathscr{R} + i\mathscr{R})$.*

Proof. Let $\mathscr{B}$ denote the von Neumann algebra $\mathscr{R} + i\mathscr{R}$ (Lemma 2.2). In view of [8, Lemma 8.1] it suffices to show $C_E(\mathscr{R}_{SA}) = [\mathscr{R}_{SA}E]$ belongs to $\mathscr{B}'$. Let $x \in E, A \in \mathscr{R}_{SA}, B \in \mathscr{R}$. Then

$$BAx = (BAE + EAB^*)x - EAB^*x \in [\mathscr{R}_{SA}x] \vee E \leqq [\mathscr{R}_{SA}E] \, .$$

Thus B leaves $[\mathscr{R}_{SA}E]$ invariant, hence $\mathscr{B}$ leaves $[\mathscr{R}_{SA}E]$ invariant, hence $[\mathscr{R}_{SA}E] \in \mathscr{B}'$.

The proof of the next lemma is a modification of that of a similar result in the proof of [9, Theorem 6.4].

LEMMA 2.4. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{C}$ denote the center of the von Neumann algebra $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. Assume $\mathscr{C}_{SA} \neq \mathscr{C} \cap \mathscr{R}_{SA}$. Then there exists a projection $E \neq 0$ in $\mathscr{C}$ such that $E\mathscr{B} \cap \mathscr{R} = \{0\}$.*

Proof. Let E_1 be a nonzero projection in $\mathscr{C}$ which is not in $\mathscr{R}_{SA}$. Let F_1 be the smallest central projection in $\mathscr{R}_{SA}$ such that $F_1 \geqq E_1$. Then $F_1 \neq E_1$. $E_1\mathscr{B}$ is an ideal in $\mathscr{B}$, hence $E_1\mathscr{B} \cap \mathscr{R}_{SA}$ is a weakly closed Jordan ideal in the JW-algebra $\mathscr{R}_{SA}$. Hence there exists a central projection F_2 in $\mathscr{R}_{SA}$ such that $E_1\mathscr{B} = \mathscr{R}_{SA} \cap F_2\mathscr{R}_{SA}$ [10]. Then $F_2 \leqq E_1$, hence $F_2 < E_1$. Let $F_3 = F_1 - F_2$. Then $F_3 \neq 0$ and belongs to $\mathscr{C} \cap \mathscr{R}_{SA}$ (Lemma 2.3). Let $E = E_1F_3 = E_1 - F_2$. Then $E \neq 0$ and belongs to $\mathscr{C}$. Moreover $E\mathscr{B}$ is an ideal in $\mathscr{B}$. As before there exists a central projection F_4 in $\mathscr{R}_{SA}$ such that $E\mathscr{B} \cap \mathscr{R}_{SA} = F_4\mathscr{R}_{SA}$. Then $F_4 \leqq E \leqq F_3$. Since $E \leqq E_1$, $E\mathscr{B} \cap \mathscr{R}_{SA} \subset F_2\mathfrak{A}$, hence $F_4 \leqq F_2$. But $F_3F_2 = 0$, so $F_4 = 0$. Thus $E\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. Let $A \in E\mathscr{B} \cap \mathscr{R}$. Then $A^*A \in E\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, so $A = 0$, $E\mathscr{B} \cap \mathscr{R} = \{0\}$.

LEMMA 2.5. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$ and $\mathscr{C}$ be the center of $\mathscr{B}$. Then there exist three orthogonal projections P, Q, R in $\mathscr{C}$ such that $P + Q + R = I$ and such that,*

(i) $P\mathscr{C}_{SA} = P\mathscr{C} \cap \mathscr{R}_{SA}$.

(ii) $Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\}$.

(iii) $R\mathscr{C}_{SA} = R\mathscr{C} \cap R\mathscr{R}_{SA}$.

Moreover, the map $R\mathscr{R} \rightarrow Q\mathscr{R}$ by $RA \rightarrow QA$ with $A \in \mathscr{R}$ is a real $*$-isomorphism onto.

Proof. We may assume $\mathscr{R} \cap i\mathscr{R} = \{0\}$. Let P be the largest projection in $\mathscr{C}$ such that $P\mathscr{C}_{SA} = P\mathscr{C} \cap \mathscr{R}_{SA}$. Assume $P \neq I$, so $\mathscr{C}_{SA} \neq \mathscr{C} \cap \mathscr{R}_{SA}$. From Lemma 2.4 we can choose a projection $Q \leqq I - P$ in $\mathscr{C}$, maximal with respect to the property $Q\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. Let $R = I - P - Q$. Then $R\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, for if not, let E be a projection in $\mathscr{R}$ with $E \leqq R$. By Lemma 2.3 $C_E(\mathscr{R}_{SA}) \in \mathscr{C}$, and $C_E(\mathscr{R}_{SA}) \leqq R$ since $E \leqq R$. We may assume $E \in \mathscr{C}$. By maximality of $P, E\mathscr{C}_{SA} \neq E\mathscr{C} \cap \mathscr{R}_{SA}$. By Lemma 2.4 there exists $F \neq 0$ in $\mathscr{C}, F \leqq E$, such that $F\mathscr{B} \cap \mathscr{R} = \{0\}$. Then $(Q + F)\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, for if $A \in (Q + F)\mathscr{B} \cap \mathscr{R}_{SA}$ then $A = AQ + AF$. Then $AF = AE \in \mathscr{R}_{SA}$, hence $AF = 0$. Therefore

$$A = AQ \in Q\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}, A = 0, (Q + F)\mathscr{B} \cap \mathscr{R}_{SA} = \{0\} ,$$

contradicting the maximality of Q. Thus $F = 0$, hence $E = 0$, hence $R\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. As in the proof of Lemma 2.4

$$Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\} .$$

Assume $R\mathscr{C} \cap R\mathscr{R}_{SA} \neq R\mathscr{C}_{SA}$. Then Lemma 2.4 yields the existence of a projection $F \neq 0$ in $R\mathscr{C}$ such that $F\mathscr{B} \cap R\mathscr{R} = \{0\}$. Then $(F + Q)\mathscr{B} \cap \mathscr{R} = \{0\}$, for if $A \in (F + Q)\mathscr{B} \cap \mathscr{R}$ then $A =$

$AF + AQ \in \mathscr{R}$. Hence $RA = FA \in F\mathscr{B} \cap R\mathscr{R} = \{0\}$, so $FA = 0$. Thus $A = AQ \in Q\mathscr{B} \cap \mathscr{R} = \{0\}$, $A = 0$. Thus $(F + Q)\mathscr{B} \cap \mathscr{R} = \{0\}$, contradiction the maximality of Q. Thus $R\mathscr{C} \cap R\mathscr{R}_{SA} = R\mathscr{C}_{SA}$.

Finally let ρ denote the map $R\mathscr{R} \to Q\mathscr{R}$ defined by $RA \to QA$, $A \in \mathscr{R}$. Then ρ is a real $*$-isomorphism onto. In fact, $QA = 0$ with $A \in (I - P)\mathscr{R}$ if and only if $A = RA \in R\mathscr{B} \cap \mathscr{R} = \{0\}$ if and only if $A = 0$, and by the same argument, if and only if $RA = 0$. Thus ρ is well defined. It is then clear that ρ is a real $*$-isomorphism onto. The proof is complete.

We are now in the position to classify all self-adjoint weakly closed abelian real operator algebras. If X is a compact Hausdorff space we denote by $C(X)$ (resp. $C_R(X)$) the complex (resp. real) continuous function on X.

THEOREM 2.6. *Let $\mathscr{R}$ be a self-adjoint weakly closed abelian real operator algebra. Let $\mathscr{B}$ denote the (abelian) von Neumann algebra $\mathscr{R} + i\mathscr{R}$. Then there exist three orthogonal projections E, F and G in $\mathscr{R}$ such that $E + F + G = I$, and such that*

(i) $E\mathscr{R} = E\mathscr{B}_{SA}$

(ii) $F\mathscr{R} = F\mathscr{B}$

(iii) $G\mathscr{R} = \{AR + \rho(A)Q : R$ *and* Q *are projections in* $\mathscr{B}$ *such that* $R + Q = G$, $A \in R\mathscr{B}$, ρ *is a real $*$-isomorphism of* $R\mathscr{B}$ *onto* $Q\mathscr{R}\}$.

Proof. Let P, Q, and R be the projections found in Lemma 2.5. We first consider $P\mathscr{R}$. Since $P\mathscr{B}_{SA} = P\mathscr{B} \cap \mathscr{R}_{SA}$, $P \in \mathscr{R}$ and

$$P\mathscr{R}_{SA} + iP\mathscr{R}_{SA} = P\mathscr{B} .$$

Let $\mathscr{T} = P\mathscr{R} \cap iP\mathscr{R}$. Then $\mathscr{T}$ is a weakly closed ideal in $\mathscr{B}$, hence there exists a projection F in $\mathscr{B}$ such that $F\mathscr{B} = \mathscr{T} = F\mathscr{R}$, so $F \in \mathscr{R}$. Let $E = P - F$. Then $E \in \mathscr{R}$, $E\mathscr{R} \cap iE\mathscr{R} = \{0\}$. By spectral theory we may assume $E\mathscr{B} = C(X)$. Since

$$E\mathscr{R}_{SA} + iE\mathscr{R}_{SA} = E\mathscr{B} = C(X) ,$$

an application of the Stone-Weierstrass Theorem shows $E\mathscr{R}_{SA} = C_R(X)$. Since $E\mathscr{R} \cap iE\mathscr{R} = \{0\}$, $E\mathscr{R} = C_R(X) = E\mathscr{B}_{SA}$, (i) and (ii) are taken care of.

Let $G = I - P$. Then $G \in \mathscr{R}$, $G = Q + R$. By Lemma 2.5

$$R\mathscr{R}_{SA} + iR\mathscr{R}_{SA} = R\mathscr{B} .$$

By the argument in the preceding paragraph there exist two projections E_1 and F_1 in $R\mathscr{R}$ such that

$$E_1 + F_1 = R, E_1\mathscr{R} = E_1\mathscr{B}_{SA}, F_1\mathscr{R} = F_1\mathscr{B} .$$

Let ρ be the real $*$-isomorphism of $R\mathscr{R}$ onto $Q\mathscr{R}$ defined in Lemma 2.5. Let $H = E_1 + \rho(E_1)$. Since $E_1 = RE'$ with E' a projection in $G\mathscr{R}$, and $\rho(E_1) = QE'$, $H = E'(R + Q) = E' \in \mathscr{R}$. Since

$$E_1\mathscr{R} = E_1\mathscr{R}_{SA} = E_1\mathscr{B}_{SA},\ H\mathscr{R} = \{E_1A + \rho(E_1)A : A \in \mathscr{R}_{SA}\} = H\mathscr{R}_{SA} .$$

Thus

$$H(\mathscr{R}_{SA} + i\mathscr{R}_{SA}) = H(\mathscr{R} + i\mathscr{R}) = H\mathscr{B} = H(\mathscr{B}_{SA} + i\mathscr{B}_{SA}) .$$

As in the preceding paragraph we conclude $H\mathscr{R} = H\mathscr{B}_{SA}$. By the maximality of P, $H = 0$, hence $E_1 = 0$, and $R\mathscr{R} = R\mathscr{B}$. Another application of Lemma 2.5 completes the proof.

We note that the real $*$-isomorphism ρ in Theorem 2.6 is characterized by Lemma 2.1. Let U be a unitary operator. Let $\mathscr{U}$ denote the (abelian) von Neumann algebra generated by U. Then U has a square root V in $\mathscr{U}$; cf, [2, proof of Lemma 2.6]. Whenever we write $U^{1/2}$ we shall mean a unitary operator V in $\mathscr{U}$ such that $V^2 = U$. Thus $U^{1/2}$ is not necessarily unique. The following application of Theorem 2.6 will be of technical value. The second half of it was pointed out to us by the referee, together with a purely analytic proof not using Theorem 2.6. However, our proof is more in the spirit of our treatment.

COROLLARY 2.7. *Let U be a unitary operator, and let $\mathscr{R}$ denote the self-adjoint weakly closed (abelian) real operator algebra generated by U. Let G be as in Theorem 2.6. The $U^{1/2}$ can be chosen so that $GU^{1/2} \in \mathscr{R}$. Moreover, if -1 is not an eigenvalue of $U(\{x : Ux = -x\} = \{0\})$, then $U^{1/2} \in \mathscr{R}$.*

Proof. $GU = VR + \rho(V)Q$ with V a unitary operator in the von Neumann algebra $R\mathscr{B} = R\mathscr{R} + iR\mathscr{R}$. V has a square root $V^{1/2} \in R\mathscr{B}$. Let $GU^{1/2} = V^{1/2}R + \rho(V^{1/2})Q$. Then $GU^{1/2} \in \mathscr{R}$, and

$$(GU^{1/2})^2 = VR + \rho(V^{1/2})^2Q = GU .$$

The first assertion follows. If -1 is not an eigenvalue of U then in the notation of Theorem 2.6, $E = EU = EU^{1/2}$ since EU is self-adjoint. Since $F\mathscr{R}$ is a von Neumann algebra, $FU^{1/2} \in F\mathscr{R}$, by the above remarks. Thus $U^{1/2} \in \mathscr{R}$.

We shall need information on real algebras $\mathscr{R}$ such that $\mathscr{R}_{SA}$ is abelian. The simplest such algebras were characterized in [8, Theorem 2.1]. The general ones are characterized by means of Theorem 2.6 and the next result.

THEOREM 2.8. *Let $\mathscr{R}$ be a self-adjoint weakly closed real oper-*

ator algebra such that $\mathscr{R}_{SA}$ *is abelian. Let* $\mathscr{B}$ *denote the von Neumann algebra* $\mathscr{R} + i\mathscr{R}$. *Then there exist two central projections* P *and* Q *in* $\mathscr{B}$ *such that* $P + Q = I$, $P\mathscr{B}$ *is abelian,* $Q\mathscr{B}$ *is of type* I_2.

Proof. Let P be the central projection on the type I_1 portion of $\mathscr{B}$. Let $Q = I - P$. Assume there exist three orthogonal equivalent nonzero projections E_1, E_2, and E_3 in $Q\mathscr{B}$. Let φ be an irreducible representation of Q not annihilating the E_j. Then $\varphi(\mathscr{R})$ is irreducible, and $\varphi(\mathscr{R})_{SA} = \varphi(\mathscr{R}_{SA})$ is abelian. By [8, Corollary 2.3] φ is a representation on a Hilbert space of dimension 2 or 1, contradicting the existence of the E_j. Thus $Q\mathscr{B}$ is of type I_2.

LEMMA 2.9. *Let* $\mathscr{R}$ *be a self-adjoint weakly closed real operator algebra. Let* $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, *and let* $\mathscr{C}$ *denote the center of* $\mathscr{B}$. *Then*

(i) $\mathscr{C} = \mathscr{C} \cap \mathscr{R} + i\mathscr{C} \cap \mathscr{R}$.

(ii) *If* $Q \neq 0$ *is a projection in* $\mathscr{C}$ *such that* $Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$, *then* $Q\mathscr{B} \cap \mathscr{R} = \{0\}$.

Proof. We may assume $\mathscr{R} \cap i\mathscr{R} = \{0\}$. By Lemma 2.2 every operator in $\mathscr{C}$ is of the form $S + iT$ with S and T in $\mathscr{R}$. Let $A \in \mathscr{R}$; then $AS + iAT = SA + iTA$ since $S + iT \in \mathscr{C}$. By the uniqueness of the sum, $AS = SA$, $TA = AT$, so $S, T \in \mathscr{C} \cap \mathscr{R}$. (i) follows.

In order to show (ii) Let G be a projection in $Q\mathscr{B} \cap \mathscr{R}$. Then $G \leqq Q$, hence $C_G(\mathscr{B}) \leqq Q$ and belongs to $\mathscr{R}$ by Lemma 2.3. Hence, $C_G(\mathscr{B}) \in Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$, $G = 0$, (ii) follows.

We next improve Lemma 2.5.

LEMMA 2.10. *Let* $\mathscr{R}$ *be a self-adjoint weakly closed real operator algebra. Let* $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, *and let* $\mathscr{C}$ *denote the center of* $\mathscr{B}$. *Then there exist three projections* E, F, *and* G *in* $\mathscr{C} \cap \mathscr{R}_{SA}$ *such that* $E + F + G = I$ *and*

(i) $E(\mathscr{C} \cap \mathscr{R}) = E\mathscr{C}_{SA}$.

(ii) $F(\mathscr{C} \cap \mathscr{R}) = F\mathscr{C}$, *hence* $F\mathscr{R} = F\mathscr{B}$.

(iii) *There exist two projections* Q *and* R *in* $\mathscr{C}$ *such that* $Q + R = G$, $Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\}$, $R\mathscr{B} = R\mathscr{R}$, *and there exists a real* $*$*-isomorphism of* $R\mathscr{B}$ *onto* $Q\mathscr{R}$.

Proof. By Lemma 2.9 and Theorem 2.6 there exist three projections E, F, G in $\mathscr{C} \cap \mathscr{R}_{SA}$ such that $E + F + G = I$, $E(\mathscr{C} \cap \mathscr{R}) = E\mathscr{C}_{SA}$, $F(\mathscr{C} \cap \mathscr{R}) = F\mathscr{C}$, $G(\mathscr{C} \cap \mathscr{R}) = \{AR + \rho(A)Q : Q, R$ projections in $\mathscr{C}$, $Q + R = G$, ρ is a real $*$-isomorphism of $R\mathscr{C}$ onto $Q(\mathscr{C} \cap \mathscr{R})\}$.

Moreover, $Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$. By Lemma 2.9 $Q\mathscr{B} \cap \mathscr{R} = \{0\}$, and similarly $R\mathscr{B} \cap \mathscr{R} = \{0\}$. By Theorem 2.6 $R\mathscr{C} = R(\mathscr{C} \cap \mathscr{R})$. In particular, $iR \in R\mathscr{R}$. Hence $R\mathscr{R}$ is a von Neumann algebra, since iR belongs to the ideal $R\mathscr{R} \cap iR\mathscr{R}$ in $R\mathscr{B}$. Thus $R\mathscr{R} = R\mathscr{B}$. The same argument shows $F\mathscr{R} = F\mathscr{B}$. As in Lemma 2.5 there exists a real $*$-isomorphism of $R\mathscr{B} = R\mathscr{R}$ onto $Q\mathscr{R}$.

If $\mathfrak{A}$ is a JW-algebra a projection E in $\mathfrak{A}$ is said to be *abelian* if $E\mathfrak{A}E$ is abelian. $\mathfrak{A}$ is of type I if there exists an abelian projection in $\mathfrak{A}$ with central carrier I. The next result is a generalization of [8, Theorem 8.2].

LEMMA 2.11. *Let $\mathscr{R}$ be a self-adjoint weakly closed real algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. If $\mathscr{R}_{SA}$ is a JW-algebra of type I then $\mathscr{B}$ is a von Neumann algebra of type I.*

Proof. Clearly $E\mathscr{R}_{SA}, F\mathscr{R}_{SA}, Q\mathscr{R}_{SA}, R\mathscr{R}_{SA}$ are all of type I, E, F, Q, R being as in Lemma 2.10. Thus by Lemmas 2.10 and 2.1 we may assume $\mathscr{C} \cap \mathscr{R}_{SA} = \mathscr{C}_{SA}$, and $\mathscr{R} \cap i\mathscr{R} = \{0\}$. By [8, Theorem 8.2] the von Neumann algebra $\mathscr{R}''_{SA}$ is of type I. Since $\mathscr{C} \cap \mathscr{R}_{SA} = \mathscr{C}_{SA}$ we may, cutting down by central projections in $\mathscr{B}$ if necessary, assume $\mathscr{R}''_{SA}$ is homogeneous [1, p. 252]. We assume $\mathscr{R}''_{SA} = \mathscr{C} \otimes \mathscr{B}(\mathscr{H}), \mathscr{C}$ being an abelian von Neumann algebra acting on a Hilbert space $\mathscr{K}$ and $\mathscr{B}(\mathscr{H})$ denoting all bounded operators on the Hilbert space $\mathscr{H}$. Since $\mathscr{R}''_{SA} \subset \mathscr{B}$, $\mathscr{B}' \subset \mathscr{R}'_{SA} = \mathscr{C}' \otimes \boldsymbol{C}$, $\boldsymbol{C}$ denoting the operators of the form λI, $\lambda \in \boldsymbol{C}$, I being the identity operator on $\mathscr{H}$. Thus $\mathscr{B}' = \mathscr{D} \otimes \boldsymbol{C}$, $\mathscr{D}$ being a von Neumann algebra acting on $\mathscr{K}$, $\mathscr{D} \subset \mathscr{C}'$. Since the center of $\mathscr{B}$ equals that of $\mathscr{R}''_{SA}$, the center of $\mathscr{B}'$ equals $\mathscr{C} \otimes \boldsymbol{C}$. Thus $\mathscr{C} \subset \mathscr{D} \subset \mathscr{C}'$. Hence $\mathscr{C} \subset \mathscr{D}' \subset \mathscr{C}'$. By [1, p. 26],

$$\mathscr{B} = \mathscr{B}'' = (\mathscr{D} \otimes \boldsymbol{C})' = \mathscr{D}' \otimes \mathscr{B}(\mathscr{H}) .$$

Hence

$$\begin{aligned} \mathscr{B} \cap \mathscr{R}'_{SA} &= (\mathscr{D}' \otimes \mathscr{B}(\mathscr{H})) \cap (\mathscr{C}' \otimes \boldsymbol{C}) \\ &= \mathscr{D}' \otimes \boldsymbol{C} . \end{aligned}$$

In fact, by [1, p. 26], if $C' \in \mathscr{C}'$ and $C' \otimes I \in \mathscr{D}' \otimes \mathscr{B}(\mathscr{H})$, the matrix representation of $C' \otimes I$ is $(T_{\iota\kappa})$ with $T_{\iota\kappa} = \delta_{\iota\kappa} C'$, $\delta_{\iota\kappa}$ being the Kronecker symbol, and as an operator in $\mathscr{D}' \otimes \mathscr{B}(\mathscr{H})$ its matrix representation is $(S_{\iota\kappa})$ with $S_{\iota\kappa} \in \mathscr{D}'$. Thus $S_{\iota\kappa} = T_{\iota\kappa}$, so $S_{\iota\kappa} = \delta_{\iota\kappa} C'$. Thus $C' \in \mathscr{D}'$, $C' \otimes I \in \mathscr{D}' \otimes \boldsymbol{C}$.

In order to show $\mathscr{B}$ is of type I it thus suffices to show $\mathscr{B} \cap \mathscr{R}'_{SA}$ is of type I. Let $B \in \mathscr{B} \cap \mathscr{R}'_{SA}$. By Lemma 2.2 $B = S + iT$ with $S, T \in \mathscr{R}$. As $\mathscr{R} \cap i\mathscr{R} = \{0\}$, the argument of Lemma 2.9 (i) shows $S, T \in \mathscr{R} \cap \mathscr{R}'_{SA}$. In particular

$$\mathscr{B} \cap \mathscr{R}'_{SA} = \mathscr{R} \cap \mathscr{R}'_{SA} + i\mathscr{R} \cap \mathscr{R}'_{SA} .$$

Now $(\mathscr{R} \cap \mathscr{R}'_{SA})_{SA}$ is abelian. By Theorem 2.8 $\mathscr{B} \cap \mathscr{R}'_{SA}$ is of type I; the proof is complete.

LEMMA 2.12. *Let $\mathscr{R}$ be a self-adjoint weakly closed real algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. Assume $\mathscr{B}$ has no type I portion. Then there exists a unitary operator U in $\mathscr{R}$ such that $U^* = -U$.*

Proof. $\mathscr{R}_{SA}$ has no type I portion, for if P is a central projection in $\mathscr{R}_{SA}$ such that $\mathscr{R}_{SA}P$ is of type I, then by Lemma 2.3 P is central in $\mathscr{B}$. Since $\mathscr{R}P + i\mathscr{R}P = \mathscr{B}P$, $\mathscr{B}P$ is of type I by Lemma 2.11. Thus $P = 0$. By the "halving lemma" then, [10, Theorem 17] there exist two orthogonal projections E and F in $\mathscr{R}_{SA}$ such that $E + F = I$, and a self-adjoint unitary operator S in $\mathscr{R}_{SA}$ such that $E = SFS$. Let $U = (E - F)S$. Then U, being the product of two unitary operators in $\mathscr{R}$, is a unitary operator in $\mathscr{R}$, and

$$U^* = ((E - F)S)^* = SE - SF = FS - ES = -(E - F)S = -U .$$

3. Anti-automorphisms of order 2.

We classify all anti-automorphisms of order 2 of von Neumann algebras leaving the centers elementwise fixed. Our first lemma is of general nature.

LEMMA 3.1. *Let V be a conjugate linear isometry of a Hilbert space $\mathscr{H}$ onto itself. Then V^2 is a unitary operator. If $\mathscr{R}$ denotes the self-adjoint weakly closed (abelian) real operator algebra generated by V^2, then $VA = AV$ for all A in $\mathscr{R}$.*

Proof. Since V is a conjugate linear isometry of $\mathscr{H}$ onto itself V^2 is a (complex) linear isometry of $\mathscr{H}$ onto itself, hence is a unitary operator. Clearly $VV^2 = V^2V$ and $VV^{-2} = V^{-2}V$. Since V^{-2} is unitary and $V^{-2}V^2 = I$, $V^{-2} = (V^2)^*$. Since operators in $\mathscr{R}$ are weak limits of real polynomials in V^2 and $(V^2)^*$, V commutes with every operator in $\mathscr{R}$.

It was noted in [9, Lemma 3.2] that if $\mathfrak{A}$ is a von Neumann algebra, $\mathscr{R}$ a self-adjoint weakly closed real subalgebra of $\mathfrak{A}$ such that $\mathscr{R} + i\mathscr{R} = \mathfrak{A}$, $\mathscr{R} \cap i\mathscr{R} = \{0\}$, then the map $A + iB \rightarrow A^* + iB^*$, $A, B \in \mathscr{R}$, is an anti-automorphism of order 2 of $\mathfrak{A}$. The next lemma shows that all anti-automorphisms of order 2 are of this form.

LEMMA 3.2. *Let $\mathfrak{A}$ be a von Neumann algebra, and let ϕ be a $*$-anti-automorphism of order 2 of $\mathfrak{A}$. Let $\mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$. Then $\mathscr{R}$ is a self-adjoint ultra-weakly closed real operator algebra, $\mathscr{R} + i\mathscr{R} = \mathfrak{A}$, $\mathscr{R} \cap i\mathscr{R} = \{0\}$, and $\phi(A + iB) = A^* + iB^*$, $A, B \in \mathscr{R}$.*

Proof. By [1, Théorème 2, p. 56] ϕ is ultra-weakly continuous. Clearly $\mathscr{R}$ is a self-adjoint real algebra, and is ultra-weakly closed. Since every operator A in $\mathfrak{A}$ is of the form

$$A = \frac{1}{2}(A + \phi(A^*)) + i\Big[\frac{1}{2i}(A - \phi(A^*))\Big]$$

with

$$\frac{1}{2}(A + \phi(A^*)) \in \mathscr{R}$$

and

$$\frac{1}{2i}(A - \phi(A^*)) \in \mathscr{R}, \mathfrak{A} = \mathscr{R} + i\mathscr{R} .$$

The rest of the proof is equally simple.

From now on the anti-automorphisms will leave the center elementwise fixed. This is because of the next lemma.

LEMMA 3.3. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, and let ϕ be a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 leaving the center of $\mathfrak{A}$ elementwise fixed. Then*

(i) *If E is a projection in $\mathfrak{A}$ then $E \sim \phi(E)$.*

(ii) *If E' is a projection in $\mathfrak{A}'$ then the map $AE' \rightarrow \phi(A)E'$ is a $*$-anti-automorphism of $\mathfrak{A}E'$ of order 2 leaving the center of $\mathfrak{A}E'$ elementwise fixed. It is denoted by $\phi_{E'}$.*

Proof. Let E be a projection in $\mathfrak{A}$. Let $F = \phi(E)$. Then $E = \phi(F)$. By the Comparison Theorem [1, Théorème 1. p. 228] there exist central projections P and Q in $\mathfrak{A}$ such that $P + Q = I$, $PF \precsim PE$, $QF \succsim QE$. There exists a projection $E_1 \leqq E$ in $\mathfrak{A}$ such that $PF \sim PE_1 \leqq PE$. Hence there exists a partial isometry V in $\mathfrak{A}$ such that $V^*V = PF$, $VV^* = PE_1$. As $P = \phi(P)$,

$$\begin{aligned} PE = \phi(PF) = \phi(V^*V) = \phi(V)\phi(V)^* \sim \phi(V)^*\phi(V) \\ = \phi(VV^*) = \phi(PE_1) \leqq \phi(PE) = PF . \end{aligned}$$

Thus $PE \precsim PF \precsim PE$, so $PE \sim PF$ [1, Proposition 1, p. 226]. Similarly $QE \sim QF$. $E \sim F$, and (i) is proved.

Let E' be a projection in $\mathfrak{A}'$. Let $A \in \mathfrak{A}$. Following [5] we define C_A to be the intersection of all central projections Q with the property $QA = A$. Clearly $C_A = C_{\varphi(A)}$. By [5, Lemma 3.1.1] $AE' = 0$ if and only if $C_{\varphi(A)}C_{E'} = C_A C_{E'} = 0$ if and only if $\phi(A)E' = 0$. (ii) follows.

LEMMA 3.4. *Let $\mathfrak{A}$ and ϕ be as in Lemma 3.3. Let ω_x be a* *vector state on* $\mathfrak{A}$. *Then there exists a unit vector* y *such that* $\omega_x\phi = \omega_y$ *on* $\mathfrak{A}$.

Proof. Let $\omega = \omega_x\phi$. Then ω is a normal state of $\mathfrak{A}$. Let E be the support of ω_x in $\mathfrak{A}$ [1, p. 61]. Let $F = \phi(E)$. By Lemma 3.3 $E \sim F$. Hence there exists a partial isometry V in $\mathfrak{A}$ such that $E = V^*V$, $F = VV^*$. Consider the state ω_{Vx} on $\mathfrak{A}$.

$$\omega_{Vx}(F) = (VV^*Vx, Vx) = (Ex, x) = 1 ,$$

so $Vx \in F$. Moreover, if $\omega_{Vx}(S^*S) = 0$ for $S \in \mathfrak{A}$, then $SVx = 0$. Since E is the support of ω_x in $\mathfrak{A}$, $SVE = 0 = SFV$. Hence $SF = 0$. Thus F is the support of ω_{Vx} in $\mathfrak{A}$, hence Vx is a separating vector for the von Neumann algebra $F\mathfrak{A}F$. Since ω is a normal state of $F\mathfrak{A}F$, there exists by [1, Théorème 4, p. 233] a vector y in F such that $\omega = \omega_y$.

LEMMA 3.5. *Let* $\mathfrak{A}$ *and* ϕ *be as in Lemma* 3.3. *Let* x *be a unit vector in* $\mathscr{H}$. *Assume* $[\mathfrak{A}x] = I$. *Let* y *be the unit vector constructed in Lemma* 3.4. *Then the mapping*

$$(S + iT)x \rightarrow (S - iT)y$$

where $S, T \in \mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$, *is isometric, and extends to a conjugate linear isometry* V *of* $\mathscr{H}$ *onto* $[\mathfrak{A}y]$, *such that for* $A \in \mathfrak{A}$,

$$\phi(A) = V^{-1}A^*V .$$

Moreover, if $\mathfrak{A}'$ *is finite then* V *maps* $\mathscr{H}$ *onto* $\mathscr{H}$.

Proof. By Lemma 3.2 $\mathfrak{A} = \mathscr{R} + i\mathscr{R}$. Let $S, T \in \mathscr{R}$. Then $\phi(S + iT) = S^* + iT^*$, hence

$$\begin{aligned} \|(S + iT)x\|^2 &= ((S + iT)^*(S + iT)x, x) \\ &= ((S^*S + T^*T)x, x) + i((S^*T - T^*S)x, x) \\ &= \overline{((S^*S + T^*T)y, y)} + i\overline{((S^*T - T^*S)y, y)} \\ &= ((S^*S + T^*T)y, y) - i((S^*T - T^*S)y, y) \\ &= \|(S - iT)y\|^2 . \end{aligned}$$

Since vectors of the form $(S + iT)x$ are dense in $\mathscr{H}$, the mapping $(S + iT)x \rightarrow (S - iT)y$ extends by continuity to an isometry V of $\mathscr{H}$ onto $[\mathfrak{A}y]$. Clearly V is real linear, and

$$V(i(S + iT))x = V(iS - T)x = (-T - iS)y = -iV(S + iT)x ,$$

so V is conjugate linear. If $A \in \mathscr{R}$, $S, T \in \mathscr{R}$, then

$$\begin{aligned} V^{-1}AV(S+iT)x &= V^{-1}A(S-iT)y \\ &= V^{-1}(AS-iAT)y \\ &= (AS+iAT)x \\ &= A(S+iT)x \,. \end{aligned}$$

By continuity and density, $V^{-1}AV = A$ for all $A \in \mathscr{R}$, i.e. $\phi(A) = A^* = V^{-1}A^*V$ for all $A \in \mathscr{R}$. Thus $\phi(A) = V^{-1}A^*V$ for all $A \in \mathfrak{A}$.

Since ϕ is of order 2, $A = V^{-2}AV^2$ for all $A \in \mathfrak{A}$, hence $V^2A = AV^2$; and $V^2 \in \mathfrak{A}'$. Moreover, V^2 is an isometry of $\mathscr{H}$ onto E, the range of V^2. Thus E, being a projection in $\mathfrak{A}'$, is equivalent to I. Clearly $E \leqq V(\mathscr{H}) = [\mathfrak{A}y]$. Since $[\mathfrak{A}y] \in \mathfrak{A}'$, $[\mathfrak{A}y] \sim I$, as projections in $\mathfrak{A}'$. Consequently, if $\mathfrak{A}'$ is finite $[\mathfrak{A}y] = I$. The proof is complete.

LEMMA 3.6. *Let $\mathfrak{A}$ and ϕ be as in Lemma 3.3. Suppose $\mathfrak{A}$ has no portion of type III. Then there exists a conjugate linear isometry V of $\mathscr{H}$ onto itself such that*

$$\phi(A) = V^{-1}A^*V$$

for all $A \in \mathfrak{A}$.

Proof. Since $\mathfrak{A}$ has no portion of type *III*, neither does $\mathfrak{A}'$ [1, Corollaire 3, p. 102]. Since every projection in $\mathfrak{A}'$ is a sum of finite projections, [1, Corollaire 1, p. 244] and every projection is a sum of cyclic projections, we may choose a family $\{x_\alpha\}_{\alpha \in J}$ of unit vectors in $\mathscr{H}$ such that $\sum_\alpha [\mathfrak{A}x_\alpha] = I$, and $[\mathfrak{A}x_\alpha]\mathfrak{A}'[\mathfrak{A}x_\alpha]$ is finite. Let $\phi[\mathfrak{A}x_\alpha]$ be the anti-automorphism of $[\mathfrak{A}x_\alpha]\mathfrak{A}$ constructed in Lemma 3.3. Since $([\mathfrak{A}x_\alpha]\mathfrak{A})' = [\mathfrak{A}x_\alpha]\mathfrak{A}'[\mathfrak{A}x_\alpha]$, [1, Proposition 1, p. 18] is finite, there exists by Lemma 3.5 a conjugate linear isometry V_α of $[\mathfrak{A}x_\alpha]$ onto itself such that

$$\phi[\mathfrak{A}x_\alpha](A) = V_\alpha^{-1}A^*V_\alpha$$

for each $A \in [\mathfrak{A}x_\alpha]\mathfrak{A}$. Let $V = \sum_\alpha V_\alpha$. Then V is a conjugate linear isometry of $\mathscr{H}$ onto itself, and

$$\begin{aligned} \phi(A) &= \sum_\alpha \phi[\mathfrak{A}x_\alpha](A[\mathfrak{A}x_\alpha]) \\ &= \sum_\alpha V_\alpha^{-1}A^*[\mathfrak{A}x_\alpha]V_\alpha \\ &= \Big(\sum_\alpha V_\alpha^{-1}\Big)A^*\sum_\beta V_\beta \\ &= V^{-1}A^*V \,. \end{aligned}$$

The proof is complete.

THEOREM 3.7. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a*

Hilbert space $\mathscr{H}$. *Let* ϕ *be a* $*$*-anti-automorphism of order* 2 *of* $\mathfrak{A}$ *leaving the center elementwise fixed. Then there exist two orthogonal projections* P' *and* Q' *in* $\mathfrak{A}'$ *with* $P' + Q' = I$, *a conjugation* J *of the Hilbert space* P', *a conjugate linear isometry* J' *of the Hilbert space* Q' *such that* $J'^2 = -Q'$, *such that*

$$\phi(A) = JA^*J - J'A^*J' .$$

for all A *in* $\mathfrak{A}$. *Moreover, if* $\mathfrak{A}$ *is of type III we may assume* $Q' = 0$.

Proof. The two cases when $\mathfrak{A}$ is of type *III* and $\mathfrak{A}$ has no type *III* portion, may be treated separately. First assume $\mathfrak{A}$ has no type *III* portion. By Lemma 3.6 there exists a conjugate linear isometry V of $\mathscr{H}$ onto itself such that $\phi(A) = V^{-1}A^*V$ for $A \in \mathfrak{A}$. Since ϕ is of order 2, V^2 is a unitary operator in $\mathfrak{A}'$. Let $\mathscr{R}$ denote the weakly closed self-adjoint real algebra generated by V^2. Let

$$Q' = \{x \in \mathscr{H} : V^2x = -x\} .$$

Then Q' is a spectral projection of V^2, and by routine calculations $VQ' = Q'V$, a fact which also follows from Theorem 2.6 and Lemma 3.1. Let $J' = VQ'$. Then J' is a conjugate linear isometry of Q' onto itself such that $J'^2 = V^2Q' = -Q'$. Let $P' = I - Q'$. Then $P' \in \mathfrak{A}'$. By Corollary 2.7 $V^{-2}P'$ has a square root W in $\mathscr{R}P'$. Put $J = WVP'$.

Then since W, V, and P' all commute, simple calculations give

$$J^2 = P' ,$$
$$V = J'Q' + W^*JP' = J'Q' + JW^*P' ,$$

and

$$V^{-1} = -J'Q' + JWP' .$$

Hence, $V^{-1}A^*V = -J'A^*J' + JA^*J$. This completes the proof when $\mathfrak{A}$ has no portion of type *III*.

Assume $\mathfrak{A}$ is of type *III*, hence $\mathfrak{A}'$ is of type *III* [1, Corollaire 3, p. 102]. Thus for every projection E' in $\mathfrak{A}'$, $E'\mathfrak{A}$ and $E'\mathfrak{A}'E'$ are of type *III*. Let E' be a maximal projection in $\mathfrak{A}'$ such that $\phi_{E'}$ is induced by a conjugation. If $E' \neq I$ there exists a unit vector $x \in I - E'$. By Lemma 3.4 there exists a unit vector y in $[\mathfrak{A}x]$ such that $\omega_x + \omega_y : \mathscr{R} \to \boldsymbol{R}$, $\mathscr{R}$ denoting the real algebra $\{A \in \mathfrak{A} : \phi(A^*) = A\}$. Since $\omega_x + \omega_y$ is normal, and every normal state of $(I - E')\mathfrak{A}$ is a vector state [1, Corollaire 9, p. 322], there exists a vector $z \in [\mathfrak{A}x]$ such that $\omega_x + \omega_y = \omega_z$. Thus $\omega_z : \mathscr{R} \to \boldsymbol{R}$. Define J by $J(S + iT)z = (S - iT)z$. As in Lemma 3.5 J is a conjugation of $[\mathfrak{A}z]$ such that

$$JA^*[\mathfrak{A}z]J = \phi(A)[\mathfrak{A}z] .$$

Since $z \neq 0$, $[\mathfrak{A}z] \neq 0$, and the maximality of E' is contradicted. Thus $E' = I$, the proof is complete.

We are indebted to the referee for the proof of the nontype *III* part of Theorem 3.7. Together with the remarks preceding Corollary 2.7 this proof shows that the theorem can be proved without the use of the structure theory in § 2. In addition to the type *III* algebras a great many finite von Neumann algebras have every anti-automorphism like ϕ in Theorem 3.7 induced by a conjugation.

THEOREM 3.8. *Let $\mathfrak{A}$ be a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and having a separating and cyclic vector x. If ϕ is a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 leaving the center of $\mathfrak{A}$ elementwise fixed, then there exists a conjugation J of $\mathscr{H}$ such that*

$$\phi(A) = JA^*J$$

for all $A \in \mathfrak{A}$.

Proof. As in Lemma 3.4 there exists a vector y in $\mathscr{H}$ such that $\omega_x + \omega_y : \mathscr{R} \to \boldsymbol{R}$, $\mathscr{R}$ denoting the real algebra $\{A \in \mathfrak{A} : \phi(A^*) = A\}$. Since x is separating there exists a vector $z \neq 0$ such that $\omega_x + \omega_y = \omega_z$ on $\mathfrak{A}$ [1, Théorème 4, p. 233]. If $A \in \mathfrak{A}$ and $Az = 0$ then

$$0 = \omega_z(A^*A) \geqslant \omega_x(A^*A) \geqslant 0 \ ,$$

so $Ax = 0$, hence $A = 0$. Thus z is separating for A. By [1, Corollaire, p. 235] z is cyclic for $\mathfrak{A}$. Define J by $J(S + iT)z = (S - iT)z$, $S, T \in \mathscr{R}$. As in Lemma 3.5 J is a conjugation such that $\phi(A) = JA^*J$ for all A in $\mathfrak{A}$.

We next show that not every $*$-anti-automorphism of order 2 leaving the center elementwise fixed is induced by a conjugation. For this purpose the next lemma is helpful.

LEMMA 3.9. *If J' is a conjugate linear isometry of a Hilbert space $\mathscr{H}$ such that $J'^2 = -I$, then there exists no conjugation J of $\mathscr{H}$ such that $-J'AJ' = JAJ$ for all operators A.*

Proof. Assume J exists. Then $-J'AJ' = JAJ$, hence

$$A = -J'JAJJ' = (iJ'J)A(iJJ') \ .$$

Note that iJJ' is a unitary operator with inverse $iJ'J$. Thus

$$iJ'J = e^{i\theta}I,\ 0 \leqq \theta < 2\pi \ ,$$

and

$$J' = e^{i\mu}J,\ 0 \leqq \mu < 2\pi\ .$$

Thus

$$J'^2 = e^{i\mu}Je^{i\mu}J = e^{i\mu}e^{-i\mu}J^2 = I\ ,$$

contrary to assumption.

EXAMPLE 3.10. Let M_2 denote the 2×2 complex matrices considered as all bounded operators on $\boldsymbol{C}^2$. Let

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then ϕ is a $*$-anti-automorphism of M_2 of order 2 leaving the center fixed. Note that $\mathscr{R} = \{A \in M_2 : \phi(A^*) = A\}$ is the quaternions. Let J' be the conjugate linear isometry of $\boldsymbol{C}^2$ defined by

$$J'\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}.$$

Then $J'^2 = -I$, and $\phi(A) = -J'A^*J'$ for all $A \in M_2$. By Lemma 3.9 ϕ is not induced by a conjugation.

We are interested in knowing whether there exists a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$ for a von Neumann algebra $\mathfrak{A}$. An affirmative solution of this problem would reduce the study of $*$-anti-automorphisms of $\mathfrak{A}$ to that of $*$-automorphisms, since then a $*$-anti-automorphism can be written in the form $\phi(A) = \rho(JA^*J)$, where ρ is the $*$-automorphism $\rho(B) = \phi(JB^*J)$. For type I algebras the solution is a simple consequence of the structure theory for such algebras.

LEMMA 3.11. *Let $\mathfrak{A}$ be a von Neumann algebra of type I acting on a Hilbert space $\mathscr{H}$. Then there exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$.*

Proof. We first assume $\mathfrak{A}$ is a maximal abelian von Neumann algebra, i.e. $\mathfrak{A} = \mathfrak{A}'$. If E is a projection in $\mathfrak{A}$ then $(E\mathfrak{A})' = E\mathfrak{A}' = E\mathfrak{A}$ when considered as acting on the Hilbert space E, hence $E\mathfrak{A}$ is maximal abelian. By [1, Proposition 9, p. 98] there exists an orthogonal family E_α of projections in $\mathfrak{A}$ such that $\sum E_\alpha = I$ and $E_\alpha\mathfrak{A}$ is countably decomposable. If we can find a conjugation J_α of E_α such that $J_\alpha E_\alpha \mathfrak{A} J_\alpha = E_\alpha\mathfrak{A}$, and $J_\alpha E_\alpha A^* J_\alpha = E_\alpha A$, then $J = \sum J_\alpha$ has all the required properties. We assume therefore that $\mathfrak{A}$ is countably decomposable. By [1, Corollaire, p. 233] $\mathfrak{A}$ has a separating, and hence cyclic, vector x. The identity map of $\mathfrak{A}$ onto itself is a $*$-anti-automorphism of order 2 leaving the center elementwise fixed. Hence an application of Theorem 3.8 completes the proof when $\mathfrak{A}$ is a maximal abelian von

Neumann algebra.

We next assume $\mathfrak{A}$ is an abelian von Neumann algebra. Then $\mathfrak{A}'$ is of type I. Hence by [1, Proposition 2, p. 252] there exist central orthogonal projections P_n in $\mathfrak{A}'$ for each cardinal n, so $P_n \in \mathfrak{A}$, such that $P_n\mathfrak{A}'$ is homogeneous of type I_n or $P_n = 0$, and $\sum_{n\geq 1} P_n = I$. As remarked above we can restrict our attention to the case when $\mathfrak{A}'$ is homogeneous. We assume therefore $\mathfrak{A}' = \mathscr{C} \otimes \mathscr{B}(\mathscr{H}_2)$, where $\mathscr{C}$ is an abelian von Neumann algebra acting on a Hilbert space $\mathscr{H}_1$, $\mathscr{B}(\mathscr{H}_2)$ denoting all bounded operators on the Hilbert space $\mathscr{H}_2$. Since $\mathfrak{A} = \mathfrak{A}'' = \mathscr{C}' \otimes C$ is abelian, $\mathfrak{A} \subset \mathfrak{A}'$, hence $\mathscr{C}' \subset \mathscr{C}$. Thus $\mathscr{C}$ is maximal abelian, and $\mathfrak{A} = \mathscr{C} \otimes C$. By the above paragraph there exists a conjugation J_1 of $\mathscr{H}_1$ such that $A = J_1A^*J_1$ for all $A \in \mathscr{C}$. Let J_2 be any conjugation of $\mathscr{H}_2$. Then $J = J_1 \otimes J_2$ is a conjugation of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ such that $JB^*J = B$ for all B in $\mathfrak{A}$.

In the general case we may by the same argument as above assume $\mathfrak{A}$ is homogeneous, so of the form $\mathfrak{A} = \mathfrak{F} \otimes \mathscr{B}(\mathscr{K}_2)$ with $\mathfrak{F}$ an abelian von Neumann algebra acting on the Hilbert space $\mathscr{K}_1$. By the above paragraph there exists a conjugation J_1 of $\mathscr{K}_1$ such that $J_1A^*J_1 = A$ for all $A \in \mathfrak{F}$. Let J_2 be any conjugation of $\mathscr{K}_2$. Since the center of $\mathfrak{A}$ equals $\mathfrak{F} \otimes C$ the conjugation $J = J_1 \otimes J_2$ has all the required properties. The proof is complete.

The truth of the above lemma without the type I assumption is a deep open problem. We can show that the existence of an anti-automorphism as in Theorem 3.7 implies an affirmative solution.

THEOREM 3.12. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose there exists a $*$-anti-automorphism ϕ of $\mathfrak{A}$ of order 2 leaving the center elementwise fixed. Then there exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ has no type I portion, and $\mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$ then $J\mathscr{R}J = \mathscr{R}$.*

Proof. By Theorem 3.7 we may assume there exists a conjugate linear isometry J' of $\mathscr{H}$ such that $\phi(A) = -J'A^*J'$, and $J'^2 = -I$. By Lemma 3.11 we may assume $\mathfrak{A}$ has no portion of type I. By Lemma 2.12 there exists a unitary operator U in $\mathscr{R}$ such that $U^* = -U$. Let $J = UJ'$. Then J is a conjugate linear isometry of $\mathscr{H}$ onto itself, and since

$$J'U = J'\phi(U^*) = -J'\phi(U) = -J'(-J'U^*J') = UJ',\ J^2 = I\,,$$

hence J is a conjugation. If $A \in \mathscr{R}$ then

$$JAJ = UJ'AJ'U = U^*\phi(A^*)U = U^*AU \in \mathscr{R}\,,$$

so J leaves $\mathscr{R}$ invariant, hence $\mathfrak{A}$ invariant. Finally, if A belongs to the center of $\mathfrak{A}$, then $JA^*J = U^*AU = A$.

4. Inner anti-automorphisms. In the last section anti-automorphisms of order 2 leaving the center elementwise fixed were analysed. One obviously wants to delete the assumption that anti-automorphisms be of order 2. In the present section we shall do this for the anti-automorphisms which are the analogue of inner automorphisms, and show these anti-automorphisms are compositions of inner automorphisms and anti-automorphisms induced by conjugations.

LEMMA 4.1. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose V is a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^{-1}\mathfrak{A}V = \mathfrak{A}$. Let $U = V^2$, and assume $X^{-1}\mathfrak{A}X = \mathfrak{A}$ for all square roots X of U in the von Neumann algebra $\mathscr{B}$ generated by U. Then there exists a square root $U^{1/2}$ of U in $\mathscr{B}$ such that if $W = VU^{-1/2}$ then $W^4 = I$ and $W^{-1}\mathfrak{A}W = \mathfrak{A}$.*

Proof. Let $\mathscr{R}$ denote the self-adjoint weakly closed real algebra generated by U. By Lemma 3.1 $AV = VA$ for all A in $\mathscr{R}$. By Theorem 2.6 there exist three orthogonal projections E, F, and G in $\mathscr{R}$ such that $E\mathscr{R} = E\mathscr{B}_{SA}$, $F\mathscr{R} = F\mathscr{B}$, note $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, and $G\mathscr{R} = \{AR + \rho(A)Q : A \in \mathscr{B}, \rho$ being a real *-isomorphism of $\mathscr{B}R$ onto $\mathscr{B}Q$, R and Q are orthogonal projections in $\mathscr{B}$ such that $R + Q = G\}$. Now $iF \in F\mathscr{R}$, hence

$$(iF)V = V(iF) = -iVF = -iFV ,$$

so $F = 0$. By Corollary 2.7 we can choose a square root $U^{1/2}$ of U in $\mathscr{B}$ such that $GU^{1/2} \in G\mathscr{R}$, so commutes with V. EU is self-adjoint so equal to $P_1 - Q_1$, where P_1 and Q_1 are orthogonal projections in $\mathscr{R}$ with sum E. Since we may assume

$$EU^{1/2} = E(P_1 + iQ_1), EVU^{1/2} = E(P_1 - iQ_1)V = EU^{-1/2}V .$$

Let $W = VU^{-1/2}$. Then by hypothesis $W^{-1}\mathfrak{A}W = \mathfrak{A}$, and

$$\begin{aligned} W^2 &= VU^{-1/2}VU^{-1/2} \\ &= V(EU^{-1/2}V + GU^{-1/2}V)U^{-1/2} \\ &= V(VEU^{1/2} + VGU^{-1/2})U^{-1/2} \\ &= V^2(E + GU^{-1}) \\ &= UE + G . \end{aligned}$$

Therefore, $W^4 = (UE + G)^2 = (P_1 - Q_1)^2 + G = P_1 + Q_1 + G = I$. The proof is complete.

LEMMA 4.2. *Let $\mathfrak{A}$ be a von Neumann algebra with no type I portion acting on a Hilbert space $\mathscr{H}$. Let V be a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^{-1}\mathfrak{A}V = \mathfrak{A}$ and $V^2 \in \mathfrak{A}$. Then there exists a unitary operator U in $\mathfrak{A}$ and a conjugation J of $\mathscr{H}$ such that $V = JU$ and such that $J\mathfrak{A}J = \mathfrak{A}$.*

Proof. V satisfies the conditions of Lemma 4.1, hence $V = WU_1^{1/2}$ where $U_1 = V^2 \in \mathfrak{A}$, $W^4 = I$, and $W^{-1}\mathfrak{A}W = \mathfrak{A}$. Let S denote the self-adjoint unitary operator W^2. From the proof of Lemma 4.1 $S \in \mathfrak{A}$. Let E and F be projections in $\mathfrak{A}$ such that $E + F = I$, $E - F = S$. Let $\mathscr{B} = \{A \in \mathfrak{A} : SAS = A\}$. Then $\mathscr{B} = E\mathfrak{A}E + F\mathfrak{A}F$. Moreover, the anti-automorphism ϕ defined by $\phi(A) = W^{-1}A^*W$ leaves $\mathscr{B}$ invariant. In fact, if $A \in \mathscr{B}$ then $S(W^{-1}AW)S = W^{-1}(W^{-2}AW^2)W = W^{-1}AW$, hence $W^{-1}AW \in \mathscr{B}$. Since $W^{-2}AW^2 = SAS = A$ for $A \in \mathscr{B}$, ϕ induces an anti-automorphism of order 2 of $\mathscr{B}$. By Lemma 3.2 $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, where $\mathscr{R} = \{A \in \mathscr{B} : W^{-1}AW = A\} = \{A \in \mathscr{B} : AW = WA\}$ is a self-adjoint weakly closed real algebra satisfying $\mathscr{R} \cap i\mathscr{R} = \{0\}$. Since $\mathscr{B} = E\mathfrak{A}E + F\mathfrak{A}F$ with E and F in $\mathfrak{A}$, $\mathscr{B}$ has no type I portion. Hence by Lemma 2.12 there exists a unitary operator U_2 in $\mathscr{R}$ such that $U_2^* = -U_2$. Then $U_2^{1/2} = 2^{-1/2}(I + U_2) \in \mathscr{R}$, and both U_2 and $U_2^{1/2}$ commute with W. Let $W_1 = WU_2^{1/2}$. Then $W_1^2 = WU_2^{1/2}WU_2^{1/2} = SU_2 \in \mathfrak{A}$, and $W_1^{-2} = SU_2^* = -SU_2 = -W_1^2$. As for U_2, $(W_1^2)^{1/2}$ belongs to the self-adjoint real algebra generated by W_1^2. Moreover, $\mathfrak{A} = W_1^{-1}\mathfrak{A}W_1$. Let $J = W_1(W_1^2)^{-1/2}$. Then $\mathfrak{A} = J^{-1}\mathfrak{A}J$, and

$$J^2 = (W_1(W_1^2)^{-1/2})^2 = W_1^2(W_1^2)^{-1} = I ,$$

since W_1 commutes with $(W_1^2)^{-1/2}$. Thus J is a conjugation, $J = J^{-1}$, and $J\mathfrak{A}J = \mathfrak{A}$.

Finally, if $U_3 = JW$ then a straightforward computation shows $U_3 = (I + U_2)(S - U_2) \in \mathfrak{A}$. Let $U = U_3U_1^{1/2}$. Then $U \in \mathfrak{A}$, and $V = WU_1^{1/2} = JU_3U_1^{1/2} = JU$. The proof is complete.

Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Then an inner $*$-automorphism of $\mathfrak{A}$ is one of the form $A \to U^{-1}AU$, where U is a unitary operator in $\mathfrak{A}$. Clearly such an automorphism leaves the center elementwise fixed. If ϕ is a $*$-anti-automorphism of $\mathfrak{A}$ we say ϕ is *inner* if ϕ leaves the center of $\mathfrak{A}$ elementwise fixed and if there exists a conjugate linear isometry V of $\mathfrak{H}$ onto itself such that $V^2 \in \mathfrak{A}$ and $\phi(A) = V^{-1}A^*V$ for all $A \in \mathfrak{A}$. If U is a unitary operator in $\mathfrak{A}$, and J is a conjugation of $\mathscr{H}$ such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$ and $J\mathfrak{A}J = \mathfrak{A}$, then clearly the $*$-anti-automorphism $A \to U^{-1}JA^*JU$ of $\mathfrak{A}$ is inner. We shall see that every inner $*$-anti-automorphism is of this form. In the type I case every

$*$-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed is inner. The analogous result holds for $*$-anti-automorphisms.

LEMMA 4.3. *Let $\mathfrak{A}$ be a von Neumann algebra of type I acting on a Hilbert space $\mathscr{H}$. Let ϕ be a $*$-anti-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed. Then there exist a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$, and a unitary operator U in $\mathfrak{A}$, such that*

$$\phi(A) = U^{-1}JA^*JU$$

for all A in $\mathfrak{A}$. In particular, ϕ is inner.

Proof. By Lemma 3.11 there exists a conjugation J of $\mathscr{H}$ with the stated properties. The map $A \to \phi(JA^*J)$ is a $*$-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed, hence is inner [1, Corollaire, p. 256]. Let U be a unitary operator in $\mathfrak{A}$ such that $\phi(JA^*J) = U^{-1}AU$ for $A \in \mathfrak{A}$. Then $\phi(A) = \phi(J(JAJ)J) = U^{-1}(JAJ)^*U = U^{-1}JA^*JU$.

THEOREM 4.4. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let ϕ be an inner $*$-anti-automorphism of $\mathfrak{A}$. Then there exist a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$, and a unitary operator U in $\mathfrak{A}$, such that*

$$\phi(A) = U^{-1}JA^*JU$$

for all A in $\mathfrak{A}$.

Proof. The type I portion is taken care of by Lemma 4.3. We may thus assume $\mathfrak{A}$ has no type I portion. By assumption $\phi(A) = V^{-1}A^*V$ for all A in $\mathfrak{A}$, where V is a conjugate linear isometry of $\mathscr{H}$ such that $V^2 \in \mathfrak{A}$. By Lemma 4.2 there exists a unitary operator U in $\mathfrak{A}$ and a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$, and $V = JU$. Thus $\phi(A) = U^{-1}JA^*JU$. If A is in the center of $\mathfrak{A}$ then $A = UAU^{-1} = U\phi(A)U^{-1} = JA^*J$. The proof is complete.

An examination of the proof of Theorem 4.4 shows that in order to find a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$, we used the innerness of ϕ mainly because we cannot in general conclude that if U is a unitary operator such that $U^{-1}\mathfrak{A}U = \mathfrak{A}$, then $U^{-1/2}\mathfrak{A}U^{1/2} = \mathfrak{A}$ for some square root of U in the von Neumann algebra generated by U. This is a bit surprising, for if T is a positive invertible operator such that $T^{-1}\mathfrak{A}T = \mathfrak{A}$, then by a theorem of Gardner [3, Theorem 3.5] $T^{-1/2}\mathfrak{A}T^{1/2} = \mathfrak{A}$. In fact, let M_2 be the complex 2×2 matrices acting on $\boldsymbol{C}^2$, and let $\boldsymbol{C}_2$ be the scalar operators in M_2. Let $\mathfrak{A} = M_2 \otimes \boldsymbol{C}_2$. Let E_1, E_2, F_1, and F_2 be 1-dimensional projections in M_2 such that $E_1 + E_2 = F_1 + F_2 = I$. Let $S_1 = E_1 - E_2, S_2 = F_1 - F_2$ be self-adjoint unitary operators in M_2. Let $S = S_1 \otimes S_2$. Then S is a self-adjoint unitary operator in $M_2 \otimes M_2$, and the map

$$A \otimes I \rightarrow S(A \otimes I)S = S_1AS_1 \otimes I$$

is an automorphism of order 2 of $\mathfrak{A}$. We show $S^{-1/2}\mathfrak{A}S^{1/2} \neq \mathfrak{A}$. Indeed $S = E - F$, where $E = E_1 \otimes F_1 + E_2 \otimes F_2$, $F = E_1 \otimes F_2 + E_2 \otimes F_1$. S has two square roots, namely $E \pm iF$. A straightforward computation yields $S^{-1/2}(A \otimes I)S^{1/2} = (E_1AE_1 + E_2AE_2) \otimes I \pm i(E_1AE_2 - E_2AE_1) \otimes S_2$. Since the second term need not be in $\mathfrak{A}$, $S^{-1/2}\mathfrak{A}S^{1/2} \neq \mathfrak{A}$.

We conclude this section with a result which combines the results in § 3 with Theorem 4.4. For simplicity we state the theorem for factors.

THEOREM 4.5. *Let $\mathfrak{A}$ be a factor acting on a Hilbert space $\mathscr{H}$. Then the following four conditions are equivalent.*

(i) *There exists an inner $*$-anti-automorphism of $\mathfrak{A}$.*

(ii) *There exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$.*

(iii) *There exists a self-adjoint weakly closed real algebra $\mathscr{R}$ such that $\mathscr{R} \cap i\mathscr{R} = \{0\}$, and $\mathfrak{A} = \mathscr{R} + i\mathscr{R}$.*

(iv) *There exists a $*$-anti-automorphism of order 2 of $\mathfrak{A}$.*

Proof. By Theorem 4.4 (i) and (ii) are equivalent. By Lemma 3.2 (ii) implies (iii). Assume (iii). Then the mapping $A + iB \rightarrow A^* + iB^*$ with $A, B \in \mathfrak{A}$ is a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 [9, Lemma 3.2], By Theorem 3.12 (iv) implies (ii).

5. Automorphisms of order 2.

One of the key points of the proof of Theorem 4.4 was that $\mathscr{B}$ had no type I portion if $\mathfrak{A}$ had none. In the proof we used that the self-adjoint unitary operator S, for which $\mathscr{B}$ was the fixed point set, belonged to $\mathfrak{A}$. In general it is unnecessary to assume $S \in \mathfrak{A}$. As this result is closely related to Lemma 2.11 we include a proof.

LEMMA 5.1. *Let $\mathfrak{A}$ be a C^*-algebra. Let ψ be a $*$-automorphism of order two of $\mathfrak{A}$. Let $\mathscr{B} = \{A \in \mathfrak{A} : \psi(A) = A\}$. Then $\mathscr{B}$ is a C^*-algebra. If $\mathscr{B}$ is abelian then every irreducible representation of $\mathfrak{A}$ is on a Hilbert space of dimension at most 2.*

Proof. Clearly $\mathscr{B}$ is a C^*-algebra. Let $\mathscr{C} = \{A \in \mathfrak{A} : -A = \psi(A)\}$. Then $\mathscr{B} \cap \mathscr{C} = \{0\}$, and $\mathfrak{A} = \mathscr{B} + \mathscr{C}$. In fact, the latter equality follows since if $A \in \mathfrak{A}$ then

$$A = \frac{1}{2}(A + \psi(A)) + \frac{1}{2}(A - \psi(A)) ,$$

where the first term is in $\mathscr{B}$ and the second in $\mathscr{C}$. Note that if $B, C \in \mathscr{C}$ then $BC \in \mathscr{B}$ since $\psi(BC) = \psi(B)\psi(C) = (-B)(-C) = BC$. By hypothesis $\mathscr{B}$ is abelian. Let φ be an irreducible representation of $\mathfrak{A}$. Then $\varphi(\mathscr{B})$ is an abelian C^*-algebra, hence isomorphic to some $C(X)$. Assume X contains more than two points. Then there exist three positive operators F_1, F_2, and F_3 in $\varphi(\mathscr{B})$ and orthogonal unit vectors x_1, x_2, and x_3 in $\mathscr{K}$- the Hilbert space on which φ represents $\mathfrak{A}$- such that $F_j x_k = \delta_{jk} x_k$. By [4, Theorem 1 and Lemma 5] there exists a unitary operator U in $\mathfrak{A}$ such that $\varphi(U)x_1 = x_2$, $\varphi(U)x_2 = x_3$. By the above $U = A + B$ with $A \in \mathscr{B}$, $B \in \mathscr{C}$. As

$$I = U^*U = (A^*A + B^*B) + (A^*B + B^*A) ,$$

and the first term is in $\mathscr{B}$ and the second in $\mathscr{C}$, $I = A^*A + B^*B$. In particular, $\|B\| \leqq 1$, hence $\|\varphi(B)x_1\| \leqq 1$. Now

$$\begin{aligned}(\varphi(B)x_1, x_2) &= (\varphi(U)x_1, x_2) - (\varphi(A)x_1, x_2)\\ &= (x_2, x_2) - (\varphi(A)F_1x_1, x_2)\\ &= 1 - (F_1\varphi(A)x_1, x_2)\\ &= 1 .\end{aligned}$$

Thus $1 = (\varphi(B)x_1, x_2) \leqq \|\varphi(B)x_1\| \, \|x_2\| \leqq 1$, so that $\varphi(B)x_1 = x_2$. Similarly $\varphi(B)x_2 = x_3$. Thus

$$\varphi(B^2)x_1 = \varphi(B)\varphi(B)x_1 = \varphi(B)x_2 = x_3 .$$

But $B^2 \in \mathscr{B}$, hence

$$\varphi(B^2)x_1 = \varphi(B^2)F_1x_1 = F_1\varphi(B^2)x_1 = F_1x_3 = 0 ,$$

a contradiction. Thus X contains at most two points. Assume $\dim \mathscr{K} \geqq 3$. Let x_1, x_2, x_3 be three orthogonal unit vectors in $\mathscr{K}$. If $\varphi(\mathscr{B}) = \boldsymbol{C}I$, we can find as above B in $\mathscr{C}$ such that $\varphi(B)x_1 = x_2$, $\varphi(B)x_2 = x_3$, hence $\varphi(B^2)x_1 = x_3$. But $B^2 = aI$ with $a \in \boldsymbol{C}$, hence $\varphi(B^2)x_1 = ax_1$, a contradiction. If X is a two point space $\varphi(\mathscr{B}) = \{aE + bF : a, b \in \boldsymbol{C},\ E$ and F orthogonal projections in $\varphi(\mathscr{B})$ with $E + F = I\}$. We may assume $\dim F \geqq 2$, $x_1 \in E$, $x_2x_3 \in F$. Then B can be chosen as above, hence $x_3 = \varphi(B^2)x_1 = \varphi(B^2)Ex_1 = E\,\varphi(B^2)x_1 = Ex_3 = 0$, a contradiction. Thus $\dim \mathscr{K} \leqq 2$.

THEOREM 5.2. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let ψ be a $*$-automorphism of order two of $\mathfrak{A}$*

Let $\mathscr{B} = \{A \in \mathfrak{A} : \psi(A) = A\}$. *If* $\mathscr{B}$ *is a von Neumann algebra of type I then so is* $\mathfrak{A}$.

Proof. Clearly $\mathscr{B}$ is a von Neumann algebra. Let P be the central projection on the type I portion of $\mathfrak{A}$. Then P is invariant under ψ, hence $P \in \mathscr{B}$. Assume $P \neq I$. Then $\mathfrak{A}(I - P)$ has no type I portion while $\mathscr{B}(I - P)$ is of type I. Let E be a nonzero abelian projection in $\mathscr{B}(I - P)$. Then $A \to E\psi(A)E$ is an automorphism of $E\mathfrak{A}E$ leaving operators in $E\mathscr{B}E$ elementwise fixed. Moreover $E\mathscr{B}E$ is abelian. By Lemma 5.1 every irreducible representation of $E\mathfrak{A}E$ is on a Hilbert space of dimension at most 2. Thus $E\mathfrak{A}E$ is of type I (cf. argument in proof of Theorem 2.8), contradicting the fact that $\mathfrak{A}(I - P)$ has no type I portion. Thus $P = I$, $\mathfrak{A}$ is of type I.

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References

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
2. E. Effros, *Global structure in von Neumann algebras*, Trans. Amer. Math. Soc. **121** (1966), 434-454.
3. L. T. Gardner, *On isomorphisms of C*-algebras*, Amer. J. Math. **87** (1965), 384-396.
4. J. Glimm and R. V. Kadison, *Unitary operators in C*-algebras*, Pacific J. Math. **10** (1960), 547-556.
5. R. V. Kadison, *Unitary invariants for representations of operator algebras*, Ann. of Math. **66** (1957), 304-379.
6. J. Schwartz, *Two finite, non hyperfinite, non-isomorphic factors*, Comm. Pure Appl. Math. **16** (1963), 19-26.
7. ———, *Non-isomorphism of a pair of factors of type III*, Comm. Pure Appl. Math. **16** (1963), 111-120.
8. E. Størmer, *Jordan algebras of type I*, Acta math. **115** (1966), 165-184.
9. ———, *Irreducible Jordan algebras of self-adjoint operators* (to appear).
10. D. M. Topping, *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc. **53** (1965).

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