Pacific Journal of Mathematics

ON ANTI-AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

ERLING STORMER

Vol. 21, No. 2 December 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 21, No. 2, 1967

ON ANTI-AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

ERLING STØRMER

Two types of $*$-anti-automorphisms of a von Neumann algebra $\mathfrak{A}$ acting on a Hilbert space $\mathscr{H}$ leaving the center of $\mathfrak{A}$ elementwise fixed are discussed, those of order two and those of the form $A \to V^{-1}A^*V$, V being a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^2 \in \mathfrak{A}$. The latter anti-automorphisms are called inner, and are the composition of inner $*$-automorphisms and $*$-anti-automorphisms of the form $A \to JA^*J$, where J is a conjugation, i.e. a conjugate linear isometry of $\mathscr{H}$ onto itself such that $J^2 = I$. The former anti-automorphisms are also closely related to conjugations; they are almost, and in many cases exactly of the form $A \to JA^*J$. Moreover, the existence of $*$-anti-automorphisms of order two leaving the center fixed implies the existence of a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$, and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$.

There are two main problems concerning $*$-anti-automorphisms of von Neumann algebras, namely their existence and their description. In the present paper we shall deal with the latter question. It turns out that anti-automorphisms are closely associated with conjugations, a conjugation being a conjugate linear isometry of a Hilbert space onto itself whose square is the identity. This is not surprising, as such maps induce most of the important anti-isomorphisms of von Neumann algebras, cf. [1]. We shall characterize two classes of anti-automorphisms, namely those of order two leaving the center of the von Neumann algebra elementwise fixed, and the so-called inner anti-automorphisms, both characterizations being in terms of conjugations. In the process of doing so we shall make heavy use of Jordan and real operator algebra theory, as developed in [8], [9], and [10]. The second section is devoted to this theory; we shall generalize some of the results in [8] and [9], and in particular classify all weakly closed self-adjoint real abelian operator algebras.

We refer the reader to [1] for terminology and results concerning von Neumann algebras. If $\mathscr{R}$ is a family of operators on a Hilbert space we denote by $\mathscr{R}_{SA}$ the set of self-adjoint operators in $\mathscr{R}$. We say $\mathscr{R}$ is *self-adjoint* if $A^* \in \mathscr{R}$ whenever $A \in \mathscr{R}$. $\mathscr{R}$ is a *self-adjoint real operator algebra* if $\mathscr{R}$ is a self-adjoint family of operators which form an algebra over the real numbers. By a *JW-algebra* we shall mean a weakly closed real linear family of self-adjoint operators closed under squaring. By a *real $*$-isomorphism* of one self-adjoint

real algebra into another we shall mean a one-to-one real linear map ϕ such that $\phi(A^*) = \phi(A)^*$, and $\phi(AB) = \phi(A)\phi(B)$ for all A, B in the algebra. By a *$*$-anti-automorphism* (or just anti-automorphism) of a von Neumann algebra $\mathfrak{A}$ we shall mean a one-to-one (complex) linear map ϕ of $\mathfrak{A}$ onto itself such that $\phi(A^*) = \phi(A)^*$ and $\phi(AB) = \phi(B)\phi(A)$ for all $A, B \in \mathfrak{A}$. We note that such a map is ultra-weakly continuous [1, Corollaire 1, p. 57]. We shall identify projections and their ranges. If $\mathfrak{A}$ is a family of operators and $\mathscr{M}$ is a set of vectors we write $[\mathfrak{A}\mathscr{M}]$ for the subspace generated by all vectors of the form Ax with $A \in \mathfrak{A}$ and $x \in \mathscr{M}$.

The $*$-anti-automorphisms ϕ studied in this paper will all turn out to be spatial, i.e. there exists a conjugate linear isometry V of the Hilbert space $\mathscr{H}$ such that $\phi(A) = V^{-1}A^*V$. That any such map ϕ is a $*$-anti-automorphism of $\mathscr{B}(\mathscr{H})$—the bounded linear operators on $\mathscr{H}$—is seen as follows. By polarization $(Vx, Vy) = \overline{(x, y)}$ for all $x, y \in \mathscr{H}$. Hence

$$((V^{-1}AV)^*x, y) = (x, V^{-1}AVy) = \overline{(Vx, AVy)} = (V^{-1}A^*Vx, y)$$

for all x, y, and $(V^{-1}AV)^* = V^{-1}A^*V$ for all $A \in \mathscr{B}(\mathscr{H})$. Clearly ϕ is linear and anti-isomorphic. If $(e_\alpha)_{\alpha \in I}$ is an orthonormal basis for $\mathscr{H}$ then the map $J: \sum \lambda_\alpha e_\alpha \to \sum \bar{\lambda}_\alpha e_\alpha$ is a conjugation of $\mathscr{H}$, hence there exist $*$-anti-automorphisms of factors of type I. The problem is open for general nontype I factors; however, it is known to the affirmative in constructed examples, a few examples will show how.

Let G be a countable discrete group such that the set $\{gg_0g^{-1} : g \in G\}$ is infinite for every $g_0 \neq e$. Let $\mathfrak{A}$ be the usual Hilbert algebra of complex functions x on G having finite support, where multiplication is convolution, $x^*(g) = \overline{x(g^{-1})}$, and

$$(x, y) = \sum_g x(g)\bar{y}(g) ,$$

[1, pp. 301–303]. For $x \in l^2(G)$ set $Jx(g) = \bar{x}(g)$. Then J is a conjugation. Let $\mathfrak{A}(G)$ be the II_1 factor of all left multiplications L_x by bounded dements of $l^2(G)$. Then simple calculations show

(i) x bounded implies Jx bounded.

(ii) $JL_xJ = L_{Jx}$ for all bounded x.

Thus $J\mathfrak{A}(G)J = \mathfrak{A}(G)$, and $\phi(A) = JA^*J$ is a $*$-anti-automorphism of $\mathfrak{A}(G)$ of order 2.

By specializing G, one can get $\mathfrak{A}(G)$ to be any one of the three known II_1 factors on a separable $\mathscr{H}$, see [6].

In the notation of [7, p. 112] one can define a conjugation J by

$$JF(\gamma, x) = \bar{F}(\gamma, x) .$$

Then $JU_\gamma J = U_\gamma$, and $JL_\phi J = L_{\bar{\phi}}$. So J induces a $*$-anti-automorphism of order 2 of the type *III* factor obtained in that construction.

2. Real operator algebras. We begin this section with four lemmas all of which are practically known.

LEMMA 2.1. *Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be C^*-algebras with identities I; let ρ be a real $*$-isomorphism of $\mathfrak{A}_1$ into $\mathfrak{A}_2$ such that $\rho(I) = I$. Then there exist two orthogonal central projections E and F in $\mathfrak{A}_2$ with $E + F = I$, such that $E\rho$ is complex linear and $F\rho$ is complex conjugate linear.*

Proof. Let $A = \rho(iI)$. Then $A^2 = \rho(iI)^2 = \rho((iI)^2) = \rho(-I) = -I$. Thus $A = iE - iF$ with E and F as above. Clearly $E\rho$ is linear and $F\rho$ is conjugate linear.

The next lemma is a slight generalization of [9, Theorem 2.4]. The proof is practically the same as that in [9], and is omitted.

LEMMA 2.2. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Then $\mathscr{R} + i\mathscr{R}$ is a von Neumann algebra.*

If $\mathfrak{A}$ is a *JW*-algebra or a von Neumann algebra and E is a projection in $\mathfrak{A}$ then its central carrier with respect to $\mathfrak{A}$ is the smallest central projection in $\mathfrak{A}$ greater than or equal to E. It is denoted by $C_E(\mathfrak{A})$. The next lemma is a modification of [8, Lemma 8.1].

LEMMA 2.3. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let E be a projection in $\mathscr{R}$. Then $C_E(\mathscr{R}_{SA}) = C_E(\mathscr{R} + i\mathscr{R})$.*

Proof. Let $\mathscr{B}$ denote the von Neumann algebra $\mathscr{R} + i\mathscr{R}$ (Lemma 2.2). In view of [8, Lemma 8.1] it suffices to show $C_E(\mathscr{R}_{SA}) = [\mathscr{R}_{SA}E]$ belongs to $\mathscr{B}'$. Let $x \in E$, $A \in \mathscr{R}_{SA}$, $B \in \mathscr{R}$. Then

$$BAx = (BAE + EAB^*)x - EAB^*x \in [\mathscr{R}_{SA}x] \vee E \leqq [\mathscr{R}_{SA}E] .$$

Thus B leaves $[\mathscr{R}_{SA}E]$ invariant, hence $\mathscr{B}$ leaves $[\mathscr{R}_{SA}E]$ invariant, hence $[\mathscr{R}_{SA}E] \in \mathscr{B}'$.

The proof of the next lemma is a modification of that of a similar result in the proof of [9, Theorem 6.4].

LEMMA 2.4. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{C}$ denote the center of the von Neumann algebra $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. Assume $\mathscr{C}_{SA} \neq \mathscr{C} \cap \mathscr{R}_{SA}$. Then there exists a projection $E \neq 0$ in $\mathscr{C}$ such that $E\mathscr{B} \cap \mathscr{R} = \{0\}$.*

Proof. Let E_1 be a nonzero projection in $\mathscr{C}$ which is not in $\mathscr{R}_{SA}$. Let F_1 be the smallest central projection in $\mathscr{R}_{SA}$ such that $F_1 \geqq E_1$. Then $F_1 \neq E_1$. $E_1\mathscr{B}$ is an ideal in $\mathscr{B}$, hence $E_1\mathscr{B} \cap \mathscr{R}_{SA}$ is a weakly closed Jordan ideal in the JW-algebra $\mathscr{R}_{SA}$. Hence there exists a central projection F_2 in $\mathscr{R}_{SA}$ such that $E_1\mathscr{B} = \mathscr{R}_{SA} \cap F_2\mathscr{R}_{SA}$ [10]. Then $F_2 \leqq E_1$, hence $F_2 < E_1$. Let $F_3 = F_1 - F_2$. Then $F_3 \neq 0$ and belongs to $\mathscr{C} \cap \mathscr{R}_{SA}$ (Lemma 2.3). Let $E = E_1F_3 = E_1 - F_2$. Then $E \neq 0$ and belongs to $\mathscr{C}$. Moreover $E\mathscr{B}$ is an ideal in $\mathscr{B}$. As before there exists a central projection F_4 in $\mathscr{R}_{SA}$ such that $E\mathscr{B} \cap \mathscr{R}_{SA} = F_4\mathscr{R}_{SA}$. Then $F_4 \leqq E \leqq F_3$. Since $E \leqq E_1$, $E\mathscr{B} \cap \mathscr{R}_{SA} \subset F_2\mathfrak{A}$, hence $F_4 \leqq F_2$. But $F_3F_2 = 0$, so $F_4 = 0$. Thus $E\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. Let $A \in E\mathscr{B} \cap \mathscr{R}$. Then $A^*A \in E\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, so $A = 0$, $E\mathscr{B} \cap \mathscr{R} = \{0\}$.

LEMMA 2.5. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$ and $\mathscr{C}$ be the center of $\mathscr{B}$. Then there exist three orthogonal projections P, Q, R in $\mathscr{C}$ such that $P + Q + R = I$ and such that,*

(i) $P\mathscr{C}_{SA} = P\mathscr{C} \cap \mathscr{R}_{SA}$.

(ii) $Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\}$.

(iii) $R\mathscr{C}_{SA} = R\mathscr{C} \cap R\mathscr{R}_{SA}$.

Moreover, the map $R\mathscr{R} \to Q\mathscr{R}$ by $RA \to QA$ with $A \in \mathscr{R}$ is a real $*$-isomorphism onto.

Proof. We may assume $\mathscr{R} \cap i\mathscr{R} = \{0\}$. Let P be the largest projection in $\mathscr{C}$ such that $P\mathscr{C}_{SA} = P\mathscr{C} \cap \mathscr{R}_{SA}$. Assume $P \neq I$, so $\mathscr{C}_{SA} \neq \mathscr{C} \cap \mathscr{R}_{SA}$. From Lemma 2.4 we can choose a projection $Q \leqq I - P$ in $\mathscr{C}$, maximal with respect to the property $Q\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. Let $R = I - P - Q$. Then $R\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, for if not, let E be a projection in $\mathscr{R}$ with $E \leqq R$. By Lemma 2.3 $C_E(\mathscr{R}_{SA}) \in \mathscr{C}$, and $C_E(\mathscr{R}_{SA}) \leqq R$ since $E \leqq R$. We may assume $E \in \mathscr{C}$. By maximality of P, $E\mathscr{C}_{SA} \neq E\mathscr{C} \cap \mathscr{R}_{SA}$. By Lemma 2.4 there exists $F \neq 0$ in $\mathscr{C}$, $F \leqq E$, such that $F\mathscr{B} \cap \mathscr{R} = \{0\}$. Then $(Q + F)\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$, for if $A \in (Q + F)\mathscr{B} \cap \mathscr{R}_{SA}$ then $A = AQ + AF$. Then $AF = AE \in \mathscr{R}_{SA}$, hence $AF = 0$. Therefore

$$A = AQ \in Q\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}, A = 0, (Q + F)\mathscr{B} \cap \mathscr{R}_{SA} = \{0\} ,$$

contradicting the maximality of Q. Thus $F = 0$, hence $E = 0$, hence $R\mathscr{B} \cap \mathscr{R}_{SA} = \{0\}$. As in the proof of Lemma 2.4

$$Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\} .$$

Assume $R\mathscr{C} \cap R\mathscr{R}_{SA} \neq R\mathscr{C}_{SA}$. Then Lemma 2.4 yields the existence of a projection $F \neq 0$ in $R\mathscr{C}$ such that $F\mathscr{B} \cap R\mathscr{R} = \{0\}$. Then $(F + Q)\mathscr{B} \cap \mathscr{R} = \{0\}$, for if $A \in (F + Q)\mathscr{B} \cap \mathscr{R}$ then $A =$

$AF + AQ \in \mathscr{R}$. Hence $RA = FA \in F\mathscr{B} \cap R\mathscr{R} = \{0\}$, so $FA = 0$. Thus $A = AQ \in Q\mathscr{B} \cap \mathscr{R} = \{0\}$, $A = 0$. Thus $(F + Q)\mathscr{B} \cap \mathscr{R} = \{0\}$, contradiction the maximality of Q. Thus $R\mathscr{C} \cap R\mathscr{R}_{SA} = R\mathscr{C}_{SA}$.

Finally let ρ denote the map $R\mathscr{R} \to Q\mathscr{R}$ defined by $RA \to QA$, $A \in \mathscr{R}$. Then ρ is a real $*$-isomorphism onto. In fact, $QA = 0$ with $A \in (I - P)\mathscr{R}$ if and only if $A = RA \in R\mathscr{B} \cap \mathscr{R} = \{0\}$ if and only if $A = 0$, and by the same argument, if and only if $RA = 0$. Thus ρ is well defined. It is then clear that ρ is a real $*$-isomorphism onto. The proof is complete.

We are now in the position to classify all self-adjoint weakly closed abelian real operator algebras. If X is a compact Hausdorff space we denote by $C(X)$ (resp. $C_R(X)$) the complex (resp. real) continuous function on X.

THEOREM 2.6. *Let $\mathscr{R}$ be a self-adjoint weakly closed abelian real operator algebra. Let $\mathscr{B}$ denote the (abelian) von Neumann algebra $\mathscr{R} + i\mathscr{R}$. Then there exist three orthogonal projections E, F and G in $\mathscr{R}$ such that $E + F + G = I$, and such that*

(i) $E\mathscr{R} = E\mathscr{B}_{SA}$

(ii) $F\mathscr{R} = F\mathscr{B}$

(iii) $G\mathscr{R} = \{AR + \rho(A)Q : R$ *and* Q *are projections in* $\mathscr{B}$ *such that* $R + Q = G$, $A \in R\mathscr{B}$, ρ *is a real* $*$*-isomorphism of* $R\mathscr{B}$ *onto* $Q\mathscr{R}\}$.

Proof. Let P, Q, and R be the projections found in Lemma 2.5. We first consider $P\mathscr{R}$. Since $P\mathscr{B}_{SA} = P\mathscr{B} \cap \mathscr{R}_{SA}$, $P \in \mathscr{R}$ and

$$P\mathscr{R}_{SA} + iP\mathscr{R}_{SA} = P\mathscr{B} .$$

Let $\mathscr{T} = P\mathscr{R} \cap iP\mathscr{R}$. Then $\mathscr{T}$ is a weakly closed ideal in $\mathscr{B}$, hence there exists a projection F in $\mathscr{B}$ such that $F\mathscr{B} = \mathscr{T} = F\mathscr{R}$, so $F \in \mathscr{R}$. Let $E = P - F$. Then $E \in \mathscr{R}$, $E\mathscr{R} \cap iE\mathscr{R} = \{0\}$. By spectral theory we may assume $E\mathscr{B} = C(X)$. Since

$$E\mathscr{R}_{SA} + iE\mathscr{R}_{SA} = E\mathscr{B} = C(X) ,$$

an application of the Stone-Weierstrass Theorem shows $E\mathscr{R}_{SA} = C_R(X)$. Since $E\mathscr{R} \cap iE\mathscr{R} = \{0\}$, $E\mathscr{R} = C_R(X) = E\mathscr{B}_{SA}$, (i) and (ii) are taken care of.

Let $G = I - P$. Then $G \in \mathscr{R}$, $G = Q + R$. By Lemma 2.5

$$R\mathscr{R}_{SA} + iR\mathscr{R}_{SA} = R\mathscr{B} .$$

By the argument in the preceding paragraph there exist two projections E_1 and F_1 in $R\mathscr{R}$ such that

$$E_1 + F_1 = R, E_1\mathscr{R} = E_1\mathscr{B}_{SA}, F_1\mathscr{R} = F_1\mathscr{B} .$$

Let ρ be the real $*$-isomorphism of $R\mathscr{R}$ onto $Q\mathscr{R}$ defined in Lemma 2.5. Let $H = E_1 + \rho(E_1)$. Since $E_1 = RE'$ with E' a projection in $G\mathscr{R}$, and $\rho(E_1) = QE'$, $H = E'(R + Q) = E' \in \mathscr{R}$. Since

$$E_1\mathscr{R} = E_1\mathscr{R}_{SA} = E_1\mathscr{B}_{SA}, H\mathscr{R} = \{E_1A + \rho(E_1)A : A \in \mathscr{R}_{SA}\} = H\mathscr{R}_{SA} .$$

Thus

$$H(\mathscr{R}_{SA} + i\mathscr{R}_{SA}) = H(\mathscr{R} + i\mathscr{R}) = H\mathscr{B} = H(\mathscr{B}_{SA} + i\mathscr{B}_{SA}) .$$

As in the preceding paragraph we conclude $H\mathscr{R} = H\mathscr{B}_{SA}$. By the maximality of P, $H = 0$, hence $E_1 = 0$, and $R\mathscr{R} = R\mathscr{B}$. Another application of Lemma 2.5 completes the proof.

We note that the real $*$-isomorphism ρ in Theorem 2.6 is characterized by Lemma 2.1. Let U be a unitary operator. Let $\mathscr{U}$ denote the (abelian) von Neumann algebra generated by U. Then U has a square root V in $\mathscr{U}$; cf, [2, proof of Lemma 2.6]. Whenever we write $U^{1/2}$ we shall mean a unitary operator V in $\mathscr{U}$ such that $V^2 = U$. Thus $U^{1/2}$ is not necessarily unique. The following application of Theorem 2.6 will be of technical value. The second half of it was pointed out to us by the referee, together with a purely analytic proof not using Theorem 2.6. However, our proof is more in the spirit of our treatment.

COROLLARY 2.7. *Let U be a unitary operator, and let $\mathscr{R}$ denote the self-adjoint weakly closed (abelian) real operator algebra generated by U. Let G be as in Theorem 2.6. The $U^{1/2}$ can be chosen so that $GU^{1/2} \in \mathscr{R}$. Moreover, if -1 is not an eigenvalue of $U(\{x : Ux = -x\} = \{0\})$, then $U^{1/2} \in \mathscr{R}$.*

Proof. $GU = VR + \rho(V)Q$ with V a unitary operator in the von Neumann algebra $R\mathscr{B} = R\mathscr{R} + iR\mathscr{R}$. V has a square root $V^{1/2} \in R\mathscr{B}$. Let $GU^{1/2} = V^{1/2}R + \rho(V^{1/2})Q$. Then $GU^{1/2} \in \mathscr{R}$, and

$$(GU^{1/2})^2 = VR + \rho(V^{1/2})^2Q = GU .$$

The first assertion follows. If -1 is not an eigenvalue of U then in the notation of Theorem 2.6, $E = EU = EU^{1/2}$ since EU is self-adjoint. Since $F\mathscr{R}$ is a von Neumann algebra, $FU^{1/2} \in F\mathscr{R}$, by the above remarks. Thus $U^{1/2} \in \mathscr{R}$.

We shall need information on real algebras $\mathscr{R}$ such that $\mathscr{R}_{SA}$ is abelian. The simplest such algebras were characterized in [8, Theorem 2.1]. The general ones are characterized by means of Theorem 2.6 and the next result.

THEOREM 2.8. *Let $\mathscr{R}$ be a self-adjoint weakly closed real oper-*

ator algebra such that $\mathscr{R}_{SA}$ is abelian. Let $\mathscr{B}$ denote the von Neumann algebra $\mathscr{R} + i\mathscr{R}$. Then there exist two central projections P and Q in $\mathscr{B}$ such that $P + Q = I$, $P\mathscr{B}$ is abelian, $Q\mathscr{B}$ is of type I_2.

Proof. Let P be the central projection on the type I_1 portion of $\mathscr{B}$. Let $Q = I - P$. Assume there exist three orthogonal equivalent nonzero projections E_1, E_2, and E_3 in $Q\mathscr{B}$. Let φ be an irreducible representation of Q not annihilating the E_j. Then $\varphi(\mathscr{R})$ is irreducible, and $\varphi(\mathscr{R})_{SA} = \varphi(\mathscr{R}_{SA})$ is abelian. By [8, Corollary 2.3] φ is a representation on a Hilbert space of dimension 2 or 1, contradicting the existence of the E_j. Thus $Q\mathscr{B}$ is of type I_2.

LEMMA 2.9. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, and let $\mathscr{C}$ denote the center of $\mathscr{B}$. Then*

(i) $\mathscr{C} = \mathscr{C} \cap \mathscr{R} + i\mathscr{C} \cap \mathscr{R}$.

(ii) *If $Q \neq 0$ is a projection in $\mathscr{C}$ such that $Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$, then $Q\mathscr{B} \cap \mathscr{R} = \{0\}$.*

Proof. We may assume $\mathscr{R} \cap i\mathscr{R} = \{0\}$. By Lemma 2.2 every operator in $\mathscr{C}$ is of the form $S + iT$ with S and T in $\mathscr{R}$. Let $A \in \mathscr{R}$; then $AS + iAT = SA + iTA$ since $S + iT \in \mathscr{C}$. By the uniqueness of the sum, $AS = SA$, $TA = AT$, so $S, T \in \mathscr{C} \cap \mathscr{R}$. (i) follows.

In order to show (ii) Let G be a projection in $Q\mathscr{B} \cap \mathscr{R}$. Then $G \leqq Q$, hence $C_G(\mathscr{B}) \leqq Q$ and belongs to $\mathscr{R}$ by Lemma 2.3. Hence, $C_G(\mathscr{B}) \in Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$, $G = 0$, (ii) follows.

We next improve Lemma 2.5.

LEMMA 2.10. *Let $\mathscr{R}$ be a self-adjoint weakly closed real operator algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, and let $\mathscr{C}$ denote the center of $\mathscr{B}$. Then there exist three projections E, F, and G in $\mathscr{C} \cap \mathscr{R}_{SA}$ such that $E + F + G = I$ and*

(i) $E(\mathscr{C} \cap \mathscr{R}) = E\mathscr{C}_{SA}$.

(ii) $F(\mathscr{C} \cap \mathscr{R}) = F\mathscr{C}$, *hence* $F\mathscr{R} = F\mathscr{B}$.

(iii) *There exist two projections Q and R in $\mathscr{C}$ such that $Q + R = G$, $Q\mathscr{B} \cap \mathscr{R} = R\mathscr{B} \cap \mathscr{R} = \{0\}$, $R\mathscr{B} = R\mathscr{R}$, and there exists a real $*$-isomorphism of $R\mathscr{B}$ onto $Q\mathscr{R}$.*

Proof. By Lemma 2.9 and Theorem 2.6 there exist three projections E, F, G in $\mathscr{C} \cap \mathscr{R}_{SA}$ such that $E + F + G = I$, $E(\mathscr{C} \cap \mathscr{R}) = E\mathscr{C}_{SA}$, $F(\mathscr{C} \cap \mathscr{R}) = F\mathscr{C}$, $G(\mathscr{C} \cap \mathscr{R}) = \{AR + \rho(A)Q : Q, R$ projections in $\mathscr{C}$, $Q + R = G$, ρ is a real $*$-isomorphism of $R\mathscr{C}$ onto $Q(\mathscr{C} \cap \mathscr{R})\}$.

Moreover, $Q\mathscr{C} \cap (\mathscr{C} \cap \mathscr{R}) = \{0\}$. By Lemma 2.9 $Q\mathscr{B} \cap \mathscr{R} = \{0\}$, and similarly $R\mathscr{B} \cap \mathscr{R} = \{0\}$. By Theorem 2.6 $R\mathscr{C} = R(\mathscr{C} \cap \mathscr{R})$. In particular, $iR \in R\mathscr{R}$. Hence $R\mathscr{R}$ is a von Neumann algebra, since iR belongs to the ideal $R\mathscr{R} \cap iR\mathscr{R}$ in $R\mathscr{B}$. Thus $R\mathscr{R} = R\mathscr{B}$. The same argument shows $F\mathscr{R} = F\mathscr{B}$. As in Lemma 2.5 there exists a real $*$-isomorphism of $R\mathscr{B} = R\mathscr{R}$ onto $Q\mathscr{R}$.

If $\mathfrak{A}$ is a JW-algebra a projection E in $\mathfrak{A}$ is said to be *abelian* if $E\mathfrak{A}E$ is abelian. $\mathfrak{A}$ is of type I if there exists an abelian projection in $\mathfrak{A}$ with central carrier I. The next result is a generalization of [8, Theorem 8.2].

LEMMA 2.11. *Let $\mathscr{R}$ be a self-adjoint weakly closed real algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. If $\mathscr{R}_{SA}$ is a JW-algebra of type I then $\mathscr{B}$ is a von Neumann algebra of type I.*

Proof. Clearly $E\mathscr{R}_{SA}$, $F\mathscr{R}_{SA}$, $Q\mathscr{R}_{SA}$, $R\mathscr{R}_{SA}$ are all of type I, E, F, Q, R being as in Lemma 2.10. Thus by Lemmas 2.10 and 2.1 we may assume $\mathscr{C} \cap \mathscr{R}_{SA} = \mathscr{C}_{SA}$, and $\mathscr{R} \cap i\mathscr{R} = \{0\}$. By [8, Theorem 8.2] the von Neumann algebra $\mathscr{R}''_{SA}$ is of type I. Since $\mathscr{C} \cap \mathscr{R}_{SA} = \mathscr{C}_{SA}$ we may, cutting down by central projections in $\mathscr{B}$ if necessary, assume $\mathscr{R}''_{SA}$ is homogeneous [1, p. 252]. We assume $\mathscr{R}''_{SA} = \mathscr{C} \otimes \mathscr{B}(\mathscr{H})$, $\mathscr{C}$ being an abelian von Neumann algebra acting on a Hilbert space $\mathscr{K}$ and $\mathscr{B}(\mathscr{H})$ denoting all bounded operators on the Hilbert space $\mathscr{H}$. Since $\mathscr{R}''_{SA} \subset \mathscr{B}$, $\mathscr{B}' \subset \mathscr{R}'_{SA} = \mathscr{C}' \otimes \boldsymbol{C}$, $\boldsymbol{C}$ denoting the operators of the form λI, $\lambda \in \boldsymbol{C}$, I being the identity operator on $\mathscr{H}$. Thus $\mathscr{B}' = \mathscr{D} \otimes \boldsymbol{C}$, $\mathscr{D}$ being a von Neumann algebra acting on $\mathscr{K}$, $\mathscr{D} \subset \mathscr{C}'$. Since the center of $\mathscr{B}$ equals that of $\mathscr{R}''_{SA}$, the center of $\mathscr{B}'$ equals $\mathscr{C} \otimes \boldsymbol{C}$. Thus $\mathscr{C} \subset \mathscr{D} \subset \mathscr{C}'$. Hence $\mathscr{C} \subset \mathscr{D}' \subset \mathscr{C}'$. By [1, p. 26],

$$\mathscr{B} = \mathscr{B}'' = (\mathscr{D} \otimes \boldsymbol{C})' = \mathscr{D}' \otimes \mathscr{B}(\mathscr{H}) .$$

Hence

$$\begin{aligned} \mathscr{B} \cap \mathscr{R}'_{SA} &= (\mathscr{D}' \otimes \mathscr{B}(\mathscr{H})) \cap (\mathscr{C}' \otimes \boldsymbol{C}) \\ &= \mathscr{D}' \otimes \boldsymbol{C} . \end{aligned}$$

In fact, by [1, p. 26], if $C' \in \mathscr{C}'$ and $C' \otimes I \in \mathscr{D}' \otimes \mathscr{B}(\mathscr{H})$, the matrix representation of $C' \otimes I$ is (T_{ix}) with $T_{ix} = \delta_{ix} C'$, δ_{ix} being the Kronecker symbol, and as an operator in $\mathscr{D}' \otimes \mathscr{B}(\mathscr{H})$ its matrix representation is (S_{ix}) with $S_{ix} \in \mathscr{D}'$. Thus $S_{ix} = T_{ix}$, so $S_{ix} = \delta_{ix} C'$. Thus $C' \in \mathscr{D}'$, $C' \otimes I \in \mathscr{D}' \otimes \boldsymbol{C}$.

In order to show $\mathscr{B}$ is of type I it thus suffices to show $\mathscr{B} \cap \mathscr{R}'_{SA}$ is of type I. Let $B \in \mathscr{B} \cap \mathscr{R}'_{SA}$. By Lemma 2.2 $B = S + iT$ with $S, T \in \mathscr{R}$. As $\mathscr{R} \cap i\mathscr{R} = \{0\}$, the argument of Lemma 2.9 (i) shows $S, T \in \mathscr{R} \cap \mathscr{R}'_{SA}$. In particular

$$\mathscr{B} \cap \mathscr{R}'_{SA} = \mathscr{R} \cap \mathscr{R}'_{SA} + i\mathscr{R} \cap \mathscr{R}'_{SA} .$$

Now $(\mathscr{R} \cap \mathscr{R}'_{SA})_{SA}$ is abelian. By Theorem 2.8 $\mathscr{B} \cap \mathscr{R}'_{SA}$ is of type I; the proof is complete.

LEMMA 2.12. *Let $\mathscr{R}$ be a self-adjoint weakly closed real algebra. Let $\mathscr{B} = \mathscr{R} + i\mathscr{R}$. Assume $\mathscr{B}$ has no type I portion. Then there exists a unitary operator U in $\mathscr{R}$ such that $U^* = -U$.*

Proof. $\mathscr{R}_{SA}$ has no type I portion, for if P is a central projection in $\mathscr{R}_{SA}$ such that $\mathscr{R}_{SA}P$ is of type I, then by Lemma 2.3 P is central in $\mathscr{B}$. Since $\mathscr{R}P + i\mathscr{R}P = \mathscr{B}P$, $\mathscr{B}P$ is of type I by Lemma 2.11. Thus $P = 0$. By the "halving lemma" then, [10, Theorem 17] there exist two orthogonal projections E and F in $\mathscr{R}_{SA}$ such that $E + F = I$, and a self-adjoint unitary operator S in $\mathscr{R}_{SA}$ such that $E = SFS$. Let $U = (E - F)S$. Then U, being the product of two unitary operators in $\mathscr{R}$, is a unitary operator in $\mathscr{R}$, and

$$U^* = ((E - F)S)^* = SE - SF = FS - ES = -(E - F)S = -U .$$

3. Anti-automorphisms of order 2. We classify all anti-automorphisms of order 2 of von Neumann algebras leaving the centers elementwise fixed. Our first lemma is of general nature.

LEMMA 3.1. *Let V be a conjugate linear isometry of a Hilbert space $\mathscr{H}$ onto itself. Then V^2 is a unitary operator. If $\mathscr{R}$ denotes the self-adjoint weakly closed (abelian) real operator algebra generated by V^2, then $VA = AV$ for all A in $\mathscr{R}$.*

Proof. Since V is a conjugate linear isometry of $\mathscr{H}$ onto itself V^2 is a (complex) linear isometry of $\mathscr{H}$ onto itself, hence is a unitary operator. Clearly $VV^2 = V^2V$ and $VV^{-2} = V^{-2}V$. Since V^{-2} is unitary and $V^{-2}V^2 = I$, $V^{-2} = (V^2)^*$. Since operators in $\mathscr{R}$ are weak limits of real polynomials in V^2 and $(V^2)^*$, V commutes with every operator in $\mathscr{R}$.

It was noted in [9, Lemma 3.2] that if $\mathfrak{A}$ is a von Neumann algebra, $\mathscr{R}$ a self-adjoint weakly closed real subalgebra of $\mathfrak{A}$ such that $\mathscr{R} + i\mathscr{R} = \mathfrak{A}$, $\mathscr{R} \cap i\mathscr{R} = \{0\}$, then the map $A + iB \to A^* + iB^*$, $A, B \in \mathscr{R}$, is an anti-automorphism of order 2 of $\mathfrak{A}$. The next lemma shows that all anti-automorphisms of order 2 are of this form.

LEMMA 3.2. *Let $\mathfrak{A}$ be a von Neumann algebra, and let ϕ be a $*$-anti-automorphism of order 2 of $\mathfrak{A}$. Let $\mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$. Then $\mathscr{R}$ is a self-adjoint ultra-weakly closed real operator algebra, $\mathscr{R} + i\mathscr{R} = \mathfrak{A}$, $\mathscr{R} \cap i\mathscr{R} = \{0\}$, and $\phi(A + iB) = A^* + iB^*$, $A, B \in \mathscr{R}$.*

Proof. By [1, Théorème 2, p. 56] ϕ is ultra-weakly continuous. Clearly $\mathscr{R}$ is a self-adjoint real algebra, and is ultra-weakly closed. Since every operator A in $\mathfrak{A}$ is of the form

$$A = \frac{1}{2}(A + \phi(A^*)) + i\left[\frac{1}{2i}(A - \phi(A^*))\right]$$

with

$$\frac{1}{2}(A + \phi(A^*)) \in \mathscr{R}$$

and

$$\frac{1}{2i}(A - \phi(A^*)) \in \mathscr{R}, \mathfrak{A} = \mathscr{R} + i\mathscr{R} .$$

The rest of the proof is equally simple.

From now on the anti-automorphisms will leave the center elementwise fixed. This is because of the next lemma.

LEMMA 3.3. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$, and let ϕ be a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 leaving the center of $\mathfrak{A}$ elementwise fixed. Then*

(i) *If E is a projection in $\mathfrak{A}$ then $E \sim \phi(E)$.*

(ii) *If E' is a projection in $\mathfrak{A}'$ then the map $AE' \to \phi(A)E'$ is a $*$-anti-automorphism of $\mathfrak{A}E'$ of order 2 leaving the center of $\mathfrak{A}E'$ elementwise fixed. It is denoted by $\phi_{E'}$.*

Proof. Let E be a projection in $\mathfrak{A}$. Let $F = \phi(E)$. Then $E = \phi(F)$. By the Comparison Theorem [1, Théorème 1. p. 228] there exist central projections P and Q in $\mathfrak{A}$ such that $P + Q = I$, $PF \precsim PE$, $QF \succsim QE$. There exists a projection $E_1 \leqq E$ in $\mathfrak{A}$ such that $PF \sim PE_1 \leqq PE$. Hence there exists a partial isometry V in $\mathfrak{A}$ such that $V^*V = PF$, $VV^* = PE_1$. As $P = \phi(P)$,

$$\begin{aligned} PE = \phi(PF) = \phi(V^*V) &= \phi(V)\phi(V)^* \sim \phi(V)^*\phi(V) \\ &= \phi(VV^*) = \phi(PE_1) \leqq \phi(PE) = PF . \end{aligned}$$

Thus $PE \precsim PF \precsim PE$, so $PE \sim PF$ [1, Proposition 1, p. 226]. Similarly $QE \sim QF$. $E \sim F$, and (i) is proved.

Let E' be a projection in $\mathfrak{A}'$. Let $A \in \mathfrak{A}$. Following [5] we define C_A to be the intersection of all central projections Q with the property $QA = A$. Clearly $C_A = C_{\varphi(A)}$. By [5, Lemma 3.1.1] $AE' = 0$ if and only if $C_{\varphi(A)}C_{E'} = C_AC_{E'} = 0$ if and only if $\phi(A)E' = 0$. (ii) follows.

LEMMA 3.4. *Let $\mathfrak{A}$ and ϕ be as in Lemma 3.3. Let ω_x be a*

vector state on $\mathfrak{A}$. *Then there exists a unit vector* y *such that* $\omega_x\phi = \omega_y$ *on* $\mathfrak{A}$.

Proof. Let $\omega = \omega_x\phi$. Then ω is a normal state of $\mathfrak{A}$. Let E be the support of ω_x in $\mathfrak{A}$ [1, p. 61]. Let $F = \phi(E)$. By Lemma 3.3 $E \sim F$. Hence there exists a partial isometry V in $\mathfrak{A}$ such that $E = V^*V$, $F = VV^*$. Consider the state ω_{Vx} on $\mathfrak{A}$.

$$\omega_{Vx}(F) = (VV^*Vx, Vx) = (Ex, x) = 1 ,$$

so $Vx \in F$. Moreover, if $\omega_{Vx}(S^*S) = 0$ for $S \in \mathfrak{A}$, then $SVx = 0$. Since E is the support of ω_x in $\mathfrak{A}$, $SVE = 0 = SFV$. Hence $SF = 0$. Thus F is the support of ω_{Vx} in $\mathfrak{A}$, hence Vx is a separating vector for the von Neumann algebra $F\mathfrak{A}F$. Since ω is a normal state of $F\mathfrak{A}F$, there exists by [1, Théorème 4, p. 233] a vector y in F such that $\omega = \omega_y$.

LEMMA 3.5. *Let* $\mathfrak{A}$ *and* ϕ *be as in Lemma* 3.3. *Let* x *be a unit vector in* $\mathscr{H}$. *Assume* $[\mathfrak{A}x] = I$. *Let* y *be the unit vector constructed in Lemma* 3.4. *Then the mapping*

$$(S + iT)x \to (S - iT)y$$

where $S, T \in \mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$, *is isometric, and extends to a conjugate linear isometry* V *of* $\mathscr{H}$ *onto* $[\mathfrak{A}y]$, *such that for* $A \in \mathfrak{A}$,

$$\phi(A) = V^{-1}A^*V .$$

Moreover, if $\mathfrak{A}'$ *is finite then* V *maps* $\mathscr{H}$ *onto* $\mathscr{H}$.

Proof. By Lemma 3.2 $\mathfrak{A} = \mathscr{R} + i\mathscr{R}$. Let $S, T \in \mathscr{R}$. Then $\phi(S + iT) = S^* + iT^*$, hence

$$\begin{aligned} \|(S + iT)x\|^2 &= ((S + iT)^*(S + iT)x, x) \\ &= ((S^*S + T^*T)x, x) + i((S^*T - T^*S)x, x) \\ &= \overline{((S^*S + T^*T)y, y)} + i\overline{((S^*T - T^*S)y, y)} \\ &= ((S^*S + T^*T)y, y) - i((S^*T - T^*S)y, y) \\ &= \|(S - iT)y\|^2 . \end{aligned}$$

Since vectors of the form $(S + iT)x$ are dense in $\mathscr{H}$, the mapping $(S + iT)x \to (S - iT)y$ extends by continuity to an isometry V of $\mathscr{H}$ onto $[\mathfrak{A}y]$. Clearly V is real linear, and

$$V(i(S + iT))x = V(iS - T)x = (-T - iS)y = -iV(S + iT)x ,$$

so V is conjugate linear. If $A \in \mathscr{R}$, $S, T \in \mathscr{R}$, then

$$\begin{aligned} V^{-1}AV(S+iT)x &= V^{-1}A(S-iT)y \\ &= V^{-1}(AS-iAT)y \\ &= (AS+iAT)x \\ &= A(S+iT)x\ . \end{aligned}$$

By continuity and density, $V^{-1}AV = A$ for all $A \in \mathscr{R}$, i.e. $\phi(A) = A^* = V^{-1}A^*V$ for all $A \in \mathscr{R}$. Thus $\phi(A) = V^{-1}A^*V$ for all $A \in \mathfrak{A}$.

Since ϕ is of order 2, $A = V^{-2}AV^2$ for all $A \in \mathfrak{A}$, hence $V^2A = AV^2$; and $V^2 \in \mathfrak{A}'$. Moreover, V^2 is an isometry of $\mathscr{H}$ onto E, the range of V^2. Thus E, being a projection in $\mathfrak{A}'$, is equivalent to I. Clearly $E \leqq V(\mathscr{H}) = [\mathfrak{A}y]$. Since $[\mathfrak{A}y] \in \mathfrak{A}'$, $[\mathfrak{A}y] \sim I$, as projections in $\mathfrak{A}'$. Consequently, if $\mathfrak{A}'$ is finite $[\mathfrak{A}y] = I$. The proof is complete.

LEMMA 3.6. *Let $\mathfrak{A}$ and ϕ be as in Lemma 3.3. Suppose $\mathfrak{A}$ has no portion of type III. Then there exists a conjugate linear isometry V of $\mathscr{H}$ onto itself such that*

$$\phi(A) = V^{-1}A^*V$$

for all $A \in \mathfrak{A}$.

Proof. Since $\mathfrak{A}$ has no portion of type *III*, neither does $\mathfrak{A}'$ [1, Corollaire 3, p. 102]. Since every projection in $\mathfrak{A}'$ is a sum of finite projections, [1, Corollaire 1, p. 244] and every projection is a sum of cyclic projections, we may choose a family $\{x_\alpha\}_{\alpha \in J}$ of unit vectors in $\mathscr{H}$ such that $\sum_\alpha [\mathfrak{A}x_\alpha] = I$, and $[\mathfrak{A}x_\alpha]\mathfrak{A}'[\mathfrak{A}x_\alpha]$ is finite. Let $\phi[\mathfrak{A}x_\alpha]$ be the anti-automorphism of $[\mathfrak{A}x_\alpha]\mathfrak{A}$ constructed in Lemma 3.3. Since $([\mathfrak{A}x_\alpha]\mathfrak{A})' = [\mathfrak{A}x_\alpha]\mathfrak{A}'[\mathfrak{A}x_\alpha]$, [1, Proposition 1, p. 18] is finite, there exists by Lemma 3.5 a conjugate linear isometry V_α of $[\mathfrak{A}x_\alpha]$ onto itself such that

$$\phi[\mathfrak{A}x_\alpha](A) = V_\alpha^{-1}A^*V_\alpha$$

for each $A \in [\mathfrak{A}x_\alpha]\mathfrak{A}$. Let $V = \sum_\alpha V_\alpha$. Then V is a conjugate linear isometry of $\mathscr{H}$ onto itself, and

$$\begin{aligned} \phi(A) &= \sum_\alpha \phi[\mathfrak{A}x_\alpha](A[\mathfrak{A}x_\alpha]) \\ &= \sum_\alpha V_\alpha^{-1}A^*[\mathfrak{A}x_\alpha]V_\alpha \\ &= \Big(\sum_\alpha V_\alpha^{-1}\Big)A^* \sum_\beta V_\beta \\ &= V^{-1}A^*V\ . \end{aligned}$$

The proof is complete.

THEOREM 3.7. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a*

Hilbert space $\mathscr{H}$. Let ϕ be a $*$-anti-automorphism of order 2 of $\mathfrak{A}$ leaving the center elementwise fixed. Then there exist two orthogonal projections P' and Q' in $\mathfrak{A}'$ with $P' + Q' = I$, a conjugation J of the Hilbert space P', a conjugate linear isometry J' of the Hilbert space Q' such that $J'^2 = -Q'$, such that

$$\phi(A) = JA^*J - J'A^*J' .$$

for all A in $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ is of type III we may assume $Q' = 0$.

Proof. The two cases when $\mathfrak{A}$ is of type *III* and $\mathfrak{A}$ has no type *III* portion, may be treated separately. First assume $\mathfrak{A}$ has no type *III* portion. By Lemma 3.6 there exists a conjugate linear isometry V of $\mathscr{H}$ onto itself such that $\phi(A) = V^{-1}A^*V$ for $A \in \mathfrak{A}$. Since ϕ is of order 2, V^2 is a unitary operator in $\mathfrak{A}'$. Let $\mathscr{R}$ denote the weakly closed self-adjoint real algebra generated by V^2. Let

$$Q' = \{x \in \mathscr{H} : V^2x = -x\} .$$

Then Q' is a spectral projection of V^2, and by routine calculations $VQ' = Q'V$, a fact which also follows from Theorem 2.6 and Lemma 3.1. Let $J' = VQ'$. Then J' is a conjugate linear isometry of Q' onto itself such that $J'^2 = V^2Q' = -Q'$. Let $P' = I - Q'$. Then $P' \in \mathfrak{A}'$. By Corollary 2.7 $V^{-2}P'$ has a square root W in $\mathscr{R}P'$. Put $J = WVP'$.

Then since W, V, and P' all commute, simple calculations give

$$J^2 = P' ,$$
$$V = J'Q' + W^*JP' = J'Q' + JW^*P' ,$$

and

$$V^{-1} = -J'Q' + JWP' .$$

Hence, $V^{-1}A^*V = -J'A^*J' + JA^*J$. This completes the proof when $\mathfrak{A}$ has no portion of type *III*.

Assume $\mathfrak{A}$ is of type *III*, hence $\mathfrak{A}'$ is of type *III* [1, Corollaire 3, p. 102]. Thus for every projection E' in $\mathfrak{A}'$, $E'\mathfrak{A}$ and $E'\mathfrak{A}'E'$ are of type *III*. Let E' be a maximal projection in $\mathfrak{A}'$ such that $\phi_{E'}$ is induced by a conjugation. If $E' \neq I$ there exists a unit vector $x \in I - E'$. By Lemma 3.4 there exists a unit vector y in $[\mathfrak{A}x]$ such that $\omega_x + \omega_y : \mathscr{R} \to \boldsymbol{R}$, $\mathscr{R}$ denoting the real algebra $\{A \in \mathfrak{A} : \phi(A^*) = A\}$. Since $\omega_x + \omega_y$ is normal, and every normal state of $(I - E')\mathfrak{A}$ is a vector state [1, Corollaire 9, p. 322], there exists a vector $z \in [\mathfrak{A}x]$ such that $\omega_x + \omega_y = \omega_z$. Thus $\omega_z : \mathscr{R} \to \boldsymbol{R}$. Define J by $J(S + iT)z = (S - iT)z$. As in Lemma 3.5 J is a conjugation of $[\mathfrak{A}z]$ such that

$$JA^*[\mathfrak{A}z]J = \phi(A)[\mathfrak{A}z] .$$

Since $z \neq 0$, $[\mathfrak{A}z] \neq 0$, and the maximality of E' is contradicted. Thus $E' = I$, the proof is complete.

We are indebted to the referee for the proof of the nontype *III* part of Theorem 3.7. Together with the remarks preceding Corollary 2.7 this proof shows that the theorem can be proved without the use of the structure theory in § 2. In addition to the type *III* algebras a great many finite von Neumann algebras have every anti-automorphism like ϕ in Theorem 3.7 induced by a conjugation.

THEOREM 3.8. *Let $\mathfrak{A}$ be a finite von Neumann algebra acting on a Hilbert space $\mathscr{H}$ and having a separating and cyclic vector x. If ϕ is a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 leaving the center of $\mathfrak{A}$ elementwise fixed, then there exists a conjugation J of $\mathscr{H}$ such that*

$$\phi(A) = JA^*J$$

for all $A \in \mathfrak{A}$.

Proof. As in Lemma 3.4 there exists a vector y in $\mathscr{H}$ such that $\omega_x + \omega_y : \mathscr{R} \to \boldsymbol{R}$, $\mathscr{R}$ denoting the real algebra $\{A \in \mathfrak{A} : \phi(A^*) = A\}$. Since x is separating there exists a vector $z \neq 0$ such that $\omega_x + \omega_y = \omega_z$ on $\mathfrak{A}$ [1, Théorème 4, p. 233]. If $A \in \mathfrak{A}$ and $Az = 0$ then

$$0 = \omega_z(A^*A) \geqslant \omega_x(A^*A) \geqslant 0 ,$$

so $Ax = 0$, hence $A = 0$. Thus z is separating for A. By [1, Corollaire, p. 235] z is cyclic for $\mathfrak{A}$. Define J by $J(S + iT)z = (S - iT)z$, $S, T \in \mathscr{R}$. As in Lemma 3.5 J is a conjugation such that $\phi(A) = JA^*J$ for all A in $\mathfrak{A}$.

We next show that not every $*$-anti-automorphism of order 2 leaving the center elementwise fixed is induced by a conjugation. For this purpose the next lemma is helpful.

LEMMA 3.9. *If J' is a conjugate linear isometry of a Hilbert space $\mathscr{H}$ such that $J'^2 = -I$, then there exists no conjugation J of $\mathscr{H}$ such that $-J'AJ' = JAJ$ for all operators A.*

Proof. Assume J exists. Then $-J'AJ' = JAJ$, hence

$$A = -J'JAJJ' = (iJ'J)A(iJJ') .$$

Note that iJJ' is a unitary operator with inverse $iJ'J$. Thus

$$iJ'J = e^{i\theta}I, \ 0 \leqq \theta < 2\pi ,$$

and

$$J' = e^{i\mu}J,\ 0 \leqq \mu < 2\pi\ .$$

Thus

$$J'^2 = e^{i\mu}Je^{i\mu}J = e^{i\mu}e^{-i\mu}J^2 = I\ ,$$

contrary to assumption.

EXAMPLE 3.10. Let M_2 denote the 2×2 complex matrices considered as all bounded operators on $\boldsymbol{C}^2$. Let

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Then ϕ is a $*$-anti-automorphism of M_2 of order 2 leaving the center fixed. Note that $\mathscr{R} = \{A \in M_2 : \phi(A^*) = A\}$ is the quaternions. Let J' be the conjugate linear isometry of $\boldsymbol{C}^2$ defined by

$$J'\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -\bar{\beta} \\ \bar{\alpha} \end{pmatrix}.$$

Then $J'^2 = -I$, and $\phi(A) = -J'A^*J'$ for all $A \in M_2$. By Lemma 3.9 ϕ is not induced by a conjugation.

We are interested in knowing whether there exists a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$ for a von Neumann algebra $\mathfrak{A}$. An affirmative solution of this problem would reduce the study of $*$-anti-automorphisms of $\mathfrak{A}$ to that of $*$-automorphisms, since then a $*$-anti-automorphism can be written in the form $\phi(A) = \rho(JA^*J)$, where ρ is the $*$-automorphism $\rho(B) = \phi(JB^*J)$. For type I algebras the solution is a simple consequence of the structure theory for such algebras.

LEMMA 3.11. *Let $\mathfrak{A}$ be a von Neumann algebra of type I acting on a Hilbert space $\mathscr{H}$. Then there exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$.*

Proof. We first assume $\mathfrak{A}$ is a maximal abelian von Neumann algebra, i.e. $\mathfrak{A} = \mathfrak{A}'$. If E is a projection in $\mathfrak{A}$ then $(E\mathfrak{A})' = E\mathfrak{A}' = E\mathfrak{A}$ when considered as acting on the Hilbert space E, hence $E\mathfrak{A}$ is maximal abelian. By [1, Proposition 9, p. 98] there exists an orthogonal family E_α of projections in $\mathfrak{A}$ such that $\sum E_\alpha = I$ and $E_\alpha\mathfrak{A}$ is countably decomposable. If we can find a conjugation J_α of E_α such that $J_\alpha E_\alpha \mathfrak{A} J_\alpha = E_\alpha\mathfrak{A}$, and $J_\alpha E_\alpha A^* J_\alpha = E_\alpha A$, then $J = \sum J_\alpha$ has all the required properties. We assume therefore that $\mathfrak{A}$ is countably decomposable. By [1, Corollaire, p. 233] $\mathfrak{A}$ has a separating, and hence cyclic, vector x. The identity map of $\mathfrak{A}$ onto itself is a $*$-anti-automorphism of order 2 leaving the center elementwise fixed. Hence an application of Theorem 3.8 completes the proof when $\mathfrak{A}$ is a maximal abelian von

Neumann algebra.

We next assume $\mathfrak{A}$ is an abelian von Neumann algebra. Then $\mathfrak{A}'$ is of type I. Hence by [1, Proposition 2, p. 252] there exist central orthogonal projections P_n in $\mathfrak{A}'$ for each cardinal n, so $P_n \in \mathfrak{A}$, such that $P_n\mathfrak{A}'$ is homogeneous of type I_n or $P_n = 0$, and $\sum_{n\geq 1} P_n = I$. As remarked above we can restrict our attention to the case when $\mathfrak{A}'$ is homogeneous. We assume therefore $\mathfrak{A}' = \mathscr{C} \otimes \mathscr{B}(\mathscr{H}_2)$, where $\mathscr{C}$ is an abelian von Neumann algebra acting on a Hilbert space $\mathscr{H}_1$, $\mathscr{B}(\mathscr{H}_2)$ denoting all bounded operators on the Hilbert space $\mathscr{H}_2$. Since $\mathfrak{A} = \mathfrak{A}'' = \mathscr{C}' \otimes \boldsymbol{C}$ is abelian, $\mathfrak{A} \subset \mathfrak{A}'$, hence $\mathscr{C}' \subset \mathscr{C}$. Thus $\mathscr{C}$ is maximal abelian, and $\mathfrak{A} = \mathscr{C} \otimes \boldsymbol{C}$. By the above paragraph there exists a conjugation J_1 of $\mathscr{H}_1$ such that $A = J_1A^*J_1$ for all $A \in \mathscr{C}$. Let J_2 be any conjugation of $\mathscr{H}_2$. Then $J = J_1 \otimes J_2$ is a conjugation of $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ such that $JB^*J = B$ for all B in $\mathfrak{A}$.

In the general case we may by the same argument as above assume $\mathfrak{A}$ is homogeneous, so of the form $\mathfrak{A} = \mathfrak{F} \otimes \mathscr{B}(\mathscr{K}_2)$ with $\mathfrak{F}$ an abelian von Neumann algebra acting on the Hilbert space $\mathscr{K}_1$. By the above paragraph there exists a conjugation J_1 of $\mathscr{K}_1$ such that $J_1A^*J_1 = A$ for all $A \in \mathfrak{F}$. Let J_2 be any conjugation of $\mathscr{K}_2$. Since the center of $\mathfrak{A}$ equals $\mathfrak{F} \otimes \boldsymbol{C}$ the conjugation $J = J_1 \otimes J_2$ has all the required properties. The proof is complete.

The truth of the above lemma without the type I assumption is a deep open problem. We can show that the existence of an anti-automorphism as in Theorem 3.7 implies an affirmative solution.

THEOREM 3.12. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose there exists a $*$-anti-automorphism ϕ of $\mathfrak{A}$ of order 2 leaving the center elementwise fixed. Then there exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$. Moreover, if $\mathfrak{A}$ has no type I portion, and $\mathscr{R} = \{A \in \mathfrak{A} : \phi(A^*) = A\}$ then $J\mathscr{R}J = \mathscr{R}$.*

Proof. By Theorem 3.7 we may assume there exists a conjugate linear isometry J' of $\mathscr{H}$ such that $\phi(A) = -J'A^*J'$, and $J'^2 = -I$. By Lemma 3.11 we may assume $\mathfrak{A}$ has no portion of type I. By Lemma 2.12 there exists a unitary operator U in $\mathscr{R}$ such that $U^* = -U$. Let $J = UJ'$. Then J is a conjugate linear isometry of $\mathscr{H}$ onto itself, and since

$$J'U = J'\phi(U^*) = -J'\phi(U) = -J'(-J'U^*J') = UJ',\ J^2 = I\,,$$

hence J is a conjugation. If $A \in \mathscr{R}$ then

$$JAJ = UJ'AJ'U = U^*\phi(A^*)U = U^*AU \in \mathscr{R}\,,$$

so J leaves $\mathscr{R}$ invariant, hence $\mathfrak{A}$ invariant. Finally, if A belongs to the center of $\mathfrak{A}$, then $JA^*J = U^*AU = A$.

4. Inner anti-automorphisms. In the last section anti-automorphisms of order 2 leaving the center elementwise fixed were analysed. One obviously wants to delete the assumption that anti-automorphisms be of order 2. In the present section we shall do this for the anti-automorphisms which are the analogue of inner automorphisms, and show these anti-automorphisms are compositions of inner automorphisms and anti-automorphisms induced by conjugations.

LEMMA 4.1. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Suppose V is a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^{-1}\mathfrak{A}V = \mathfrak{A}$. Let $U = V^2$, and assume $X^{-1}\mathfrak{A}X = \mathfrak{A}$ for all square roots X of U in the von Neumann algebra $\mathscr{B}$ generated by U. Then there exists a square root $U^{1/2}$ of U in $\mathscr{B}$ such that if $W = VU^{-1/2}$ then $W^4 = I$ and $W^{-1}\mathfrak{A}W = \mathfrak{A}$.*

Proof. Let $\mathscr{R}$ denote the self-adjoint weakly closed real algebra generated by U. By Lemma 3.1 $AV = VA$ for all A in $\mathscr{R}$. By Theorem 2.6 there exist three orthogonal projections E, F, and G in $\mathscr{R}$ such that $E\mathscr{R} = E\mathscr{B}_{SA}$, $F\mathscr{R} = F\mathscr{B}$, note $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, and $G\mathscr{R} = \{AR + \rho(A)Q : A \in \mathscr{B}, \rho$ being a real *-isomorphism of $\mathscr{B}R$ onto $\mathscr{B}Q$, R and Q are orthogonal projections in $\mathscr{B}$ such that $R + Q = G\}$. Now $iF \in F\mathscr{R}$, hence

$$(iF)V = V(iF) = -iVF = -iFV\,,$$

so $F = 0$. By Corollary 2.7 we can choose a square root $U^{1/2}$ of U in $\mathscr{B}$ such that $GU^{1/2} \in G\mathscr{R}$, so commutes with V. EU is self-adjoint so equal to $P_1 - Q_1$, where P_1 and Q_1 are orthogonal projections in $\mathscr{R}$ with sum E. Since we may assume

$$EU^{1/2} = E(P_1 + iQ_1),\ EVU^{1/2} = E(P_1 - iQ_1)V = EU^{-1/2}V\,.$$

Let $W = VU^{-1/2}$. Then by hypothesis $W^{-1}\mathfrak{A}W = \mathfrak{A}$, and

$$\begin{aligned} W^2 &= VU^{-1/2}VU^{-1/2} \\ &= V(EU^{-1/2}V + GU^{-1/2}V)U^{-1/2} \\ &= V(VEU^{1/2} + VGU^{-1/2})U^{-1/2} \\ &= V^2(E + GU^{-1}) \\ &= UE + G\,. \end{aligned}$$

Therefore, $W^4 = (UE + G)^2 = (P_1 - Q_1)^2 + G = P_1 + Q_1 + G = I$. The proof is complete.

LEMMA 4.2. *Let $\mathfrak{A}$ be a von Neumann algebra with no type I portion acting on a Hilbert space $\mathscr{H}$. Let V be a conjugate linear isometry of $\mathscr{H}$ onto itself such that $V^{-1}\mathfrak{A}V = \mathfrak{A}$ and $V^2 \in \mathfrak{A}$. Then there exists a unitary operator U in $\mathfrak{A}$ and a conjugation J of $\mathscr{H}$ such that $V = JU$ and such that $J\mathfrak{A}J = \mathfrak{A}$.*

Proof. V satisfies the conditions of Lemma 4.1, hence $V = WU_1^{1/2}$ where $U_1 = V^2 \in \mathfrak{A}$, $W^4 = I$, and $W^{-1}\mathfrak{A}W = \mathfrak{A}$. Let S denote the self-adjoint unitary operator W^2. From the proof of Lemma 4.1 $S \in \mathfrak{A}$. Let E and F be projections in $\mathfrak{A}$ such that $E + F = I$, $E - F = S$. Let $\mathscr{B} = \{A \in \mathfrak{A} : SAS = A\}$. Then $\mathscr{B} = E\mathfrak{A}E + F\mathfrak{A}F$. Moreover, the anti-automorphism ϕ defined by $\phi(A) = W^{-1}A^*W$ leaves $\mathscr{B}$ invariant. In fact, if $A \in \mathscr{B}$ then $S(W^{-1}AW)S = W^{-1}(W^{-2}AW^2)W = W^{-1}AW$, hence $W^{-1}AW \in \mathscr{B}$. Since $W^{-2}AW^2 = SAS = A$ for $A \in \mathscr{B}$, ϕ induces an anti-automorphism of order 2 of $\mathscr{B}$. By Lemma 3.2 $\mathscr{B} = \mathscr{R} + i\mathscr{R}$, where $\mathscr{R} = \{A \in \mathscr{B} : W^{-1}AW = A\} = \{A \in \mathscr{B} : AW = WA\}$ is a self-adjoint weakly closed real algebra satisfying $\mathscr{R} \cap i\mathscr{R} = \{0\}$. Since $\mathscr{B} = E\mathfrak{A}E + F\mathfrak{A}F$ with E and F in $\mathfrak{A}$, $\mathscr{B}$ has no type I portion. Hence by Lemma 2.12 there exists a unitary operator U_2 in $\mathscr{R}$ such that $U_2^* = -U_2$. Then $U_2^{1/2} = 2^{-1/2}(I + U_2) \in \mathscr{R}$, and both U_2 and $U_2^{1/2}$ commute with W. Let $W_1 = WU_2^{1/2}$. Then $W_1^2 = WU_2^{1/2}WU_2^{1/2} = SU_2 \in \mathfrak{A}$, and $W_1^{-2} = SU_2^* = -SU_2 = -W_1^2$. As for U_2, $(W_1^2)^{1/2}$ belongs to the self-adjoint real algebra generated by W_1^2. Moreover, $\mathfrak{A} = W_1^{-1}\mathfrak{A}W_1$. Let $J = W_1(W_1^2)^{-1/2}$. Then $\mathfrak{A} = J^{-1}\mathfrak{A}J$, and

$$J^2 = (W_1(W_1^2)^{-1/2})^2 = W_1^2(W_1^2)^{-1} = I\,,$$

since W_1 commutes with $(W_1^2)^{-1/2}$. Thus J is a conjugation, $J = J^{-1}$, and $J\mathfrak{A}J = \mathfrak{A}$.

Finally, if $U_3 = JW$ then a straightforward computation shows $U_3 = (I + U_2)(S - U_2) \in \mathfrak{A}$. Let $U = U_3U_1^{1/2}$. Then $U \in \mathfrak{A}$, and $V = WU_1^{1/2} = JU_3U_1^{1/2} = JU$. The proof is complete.

Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Then an inner $*$-automorphism of $\mathfrak{A}$ is one of the form $A \to U^{-1}AU$, where U is a unitary operator in $\mathfrak{A}$. Clearly such an automorphism leaves the center elementwise fixed. If ϕ is a $*$-anti-automorphism of $\mathfrak{A}$ we say ϕ is *inner* if ϕ leaves the center of $\mathfrak{A}$ elementwise fixed and if there exists a conjugate linear isometry V of $\mathfrak{H}$ onto itself such that $V^2 \in \mathfrak{A}$ and $\phi(A) = V^{-1}A^*V$ for all $A \in \mathfrak{A}$. If U is a unitary operator in $\mathfrak{A}$, and J is a conjugation of $\mathscr{H}$ such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$ and $J\mathfrak{A}J = \mathfrak{A}$, then clearly the $*$-anti-automorphism $A \to U^{-1}JA^*JU$ of $\mathfrak{A}$ is inner. We shall see that every inner $*$-anti-automorphism is of this form. In the type I case every

$*$-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed is inner. The analogous result holds for *-anti-automorphisms.

LEMMA 4.3. *Let $\mathfrak{A}$ be a von Neumann algebra of type I acting on a Hilbert space $\mathscr{H}$. Let ϕ be a $*$-anti-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed. Then there exist a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$, and a unitary operator U in $\mathfrak{A}$, such that*

$$\phi(A) = U^{-1}JA^*JU$$

for all A in $\mathfrak{A}$. In particular, ϕ is inner.

Proof. By Lemma 3.11 there exists a conjugation J of $\mathscr{H}$ with the stated properties. The map $A \to \phi(JA^*J)$ is a $*$-automorphism of $\mathfrak{A}$ leaving the center elementwise fixed, hence is inner [1, Corollaire, p. 256]. Let U be a unitary operator in $\mathfrak{A}$ such that $\phi(JA^*J) = U^{-1}AU$ for $A \in \mathfrak{A}$. Then $\phi(A) = \phi(J(JAJ)J) = U^{-1}(JAJ)^*U = U^{-1}JA^*JU$.

THEOREM 4.4. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let ϕ be an inner $*$-anti-automorphism of $\mathfrak{A}$. Then there exist a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$ and such that $JA^*J = A$ for all A in the center of $\mathfrak{A}$, and a unitary operator U in $\mathfrak{A}$, such that*

$$\phi(A) = U^{-1}JA^*JU$$

for all A in $\mathfrak{A}$.

Proof. The type I portion is taken care of by Lemma 4.3. We may thus assume $\mathfrak{A}$ has no type I portion. By assumption $\phi(A) = V^{-1}A^*V$ for all A in $\mathfrak{A}$, where V is a conjugate linear isometry of $\mathscr{H}$ such that $V^2 \in \mathfrak{A}$. By Lemma 4.2 there exists a unitary operator U in $\mathfrak{A}$ and a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$, and $V = JU$. Thus $\phi(A) = U^{-1}JA^*JU$. If A is in the center of $\mathfrak{A}$ then $A = UAU^{-1} = U\phi(A)U^{-1} = JA^*J$. The proof is complete.

An examination of the proof of Theorem 4.4 shows that in order to find a conjugation J such that $J\mathfrak{A}J = \mathfrak{A}$, we used the innerness of ϕ mainly because we cannot in general conclude that if U is a unitary operator such that $U^{-1}\mathfrak{A}U = \mathfrak{A}$, then $U^{-1/2}\mathfrak{A}U^{1/2} = \mathfrak{A}$ for some square root of U in the von Neumann algebra generated by U. This is a bit surprising, for if T is a positive invertible operator such that $T^{-1}\mathfrak{A}T = \mathfrak{A}$, then by a theorem of Gardner [3, Theorem 3.5] $T^{-1/2}\mathfrak{A}T^{1/2} = \mathfrak{A}$. In fact, let M_2 be the complex 2×2 matrices acting

on C^2, and let C_2 be the scalar operators in M_2. Let $\mathfrak{A} = M_2 \otimes C_2$. Let E_1, E_2, F_1, and F_2 be 1-dimensional projections in M_2 such that $E_1 + E_2 = F_1 + F_2 = I$. Let $S_1 = E_1 - E_2$, $S_2 = F_1 - F_2$ be self-adjoint unitary operators in M_2. Let $S = S_1 \otimes S_2$. Then S is a self-adjoint unitary operator in $M_2 \otimes M_2$, and the map

$$A \otimes I \to S(A \otimes I)S = S_1 A S_1 \otimes I$$

is an automorphism of order 2 of $\mathfrak{A}$. We show $S^{-1/2}\mathfrak{A}S^{1/2} \neq \mathfrak{A}$. Indeed $S = E - F$, where $E = E_1 \otimes F_1 + E_2 \otimes F_2$, $F = E_1 \otimes F_2 + E_2 \otimes F_1$. S has two square roots, namely $E \pm iF$. A straightforward computation yields $S^{-1/2}(A \otimes I)S^{1/2} = (E_1 A E_1 + E_2 A E_2) \otimes I \pm i(E_1 A E_2 - E_2 A E_1) \otimes S_2$. Since the second term need not be in $\mathfrak{A}$, $S^{-1/2}\mathfrak{A}S^{1/2} \neq \mathfrak{A}$.

We conclude this section with a result which combines the results in § 3 with Theorem 4.4. For simplicity we state the theorem for factors.

THEOREM 4.5. *Let $\mathfrak{A}$ be a factor acting on a Hilbert space $\mathscr{H}$. Then the following four conditions are equivalent.*

(i) *There exists an inner $*$-anti-automorphism of $\mathfrak{A}$.*

(ii) *There exists a conjugation J of $\mathscr{H}$ such that $J\mathfrak{A}J = \mathfrak{A}$.*

(iii) *There exists a self-adjoint weakly closed real algebra $\mathscr{R}$ such that $\mathscr{R} \cap i\mathscr{R} = \{0\}$, and $\mathfrak{A} = \mathscr{R} + i\mathscr{R}$.*

(iv) *There exists a $*$-anti-automorphism of order 2 of $\mathfrak{A}$.*

Proof. By Theorem 4.4 (i) and (ii) are equivalent. By Lemma 3.2 (ii) implies (iii). Assume (iii). Then the mapping $A + iB \to A^* + iB^*$ with $A, B \in \mathfrak{A}$ is a $*$-anti-automorphism of $\mathfrak{A}$ of order 2 [9, Lemma 3.2], By Theorem 3.12 (iv) implies (ii).

5. **Automorphisms of order 2.** One of the key points of the proof of Theorem 4.4 was that $\mathscr{B}$ had no type I portion if $\mathfrak{A}$ had none. In the proof we used that the self-adjoint unitary operator S, for which $\mathscr{B}$ was the fixed point set, belonged to $\mathfrak{A}$. In general it is unnecessary to assume $S \in \mathfrak{A}$. As this result is closely related to Lemma 2.11 we include a proof.

LEMMA 5.1. *Let $\mathfrak{A}$ be a C^*-algebra. Let ψ be a $*$-automorphism of order two of $\mathfrak{A}$. Let $\mathscr{B} = \{A \in \mathfrak{A} : \psi(A) = A\}$. Then $\mathscr{B}$ is a C^*-algebra. If $\mathscr{B}$ is abelian then every irreducible representation of $\mathfrak{A}$ is on a Hilbert space of dimension at most 2.*

Proof. Clearly $\mathscr{B}$ is a C^*-algebra. Let $\mathscr{C} = \{A \in \mathfrak{A} : -A = \psi(A)\}$. Then $\mathscr{B} \cap \mathscr{C} = \{0\}$, and $\mathfrak{A} = \mathscr{B} + \mathscr{C}$. In fact, the latter equality

follows since if $A \in \mathfrak{A}$ then

$$A = \frac{1}{2}(A + \psi(A)) + \frac{1}{2}(A - \psi(A)) ,$$

where the first term is in $\mathscr{B}$ and the second in $\mathscr{C}$. Note that if $B, C \in \mathscr{C}$ then $BC \in \mathscr{B}$ since $\psi(BC) = \psi(B)\psi(C) = (-B)(-C) = BC$. By hypothesis $\mathscr{B}$ is abelian. Let φ be an irreducible representation of $\mathfrak{A}$. Then $\varphi(\mathscr{B})$ is an abelian C^*-algebra, hence isomorphic to some $C(X)$. Assume X contains more than two points. Then there exist three positive operators F_1, F_2, and F_3 in $\varphi(\mathscr{B})$ and orthogonal unit vectors x_1, x_2, and x_3 in $\mathscr{K}$- the Hilbert space on which φ represents $\mathfrak{A}$- such that $F_j x_k = \delta_{jk} x_k$. By [4, Theorem 1 and Lemma 5] there exists a unitary operator U in $\mathfrak{A}$ such that $\varphi(U)x_1 = x_2$, $\varphi(U)x_2 = x_3$. By the above $U = A + B$ with $A \in \mathscr{B}$, $B \in \mathscr{C}$. As

$$I = U^*U = (A^*A + B^*B) + (A^*B + B^*A) ,$$

and the first term is in $\mathscr{B}$ and the second in $\mathscr{C}$, $I = A^*A + B^*B$. In particular, $\| B \| \leqq 1$, hence $\| \varphi(B)x_1 \| \leqq 1$. Now

$$\begin{aligned} (\varphi(B)x_1, x_2) &= (\varphi(U)x_1, x_2) - (\varphi(A)x_1, x_2) \\ &= (x_2, x_2) - (\varphi(A)F_1 x_1, x_2) \\ &= 1 - (F_1\varphi(A)x_1, x_2) \\ &= 1 . \end{aligned}$$

Thus $1 = (\varphi(B)x_1, x_2) \leqq \| \varphi(B)x_1 \| \, \| x_2 \| \leqq 1$, so that $\varphi(B)x_1 = x_2$. Similarly $\varphi(B)x_2 = x_3$. Thus

$$\varphi(B^2)x_1 = \varphi(B)\varphi(B)x_1 = \varphi(B)x_2 = x_3 .$$

But $B^2 \in \mathscr{B}$, hence

$$\varphi(B^2)x_1 = \varphi(B^2)F_1 x_1 = F_1\varphi(B^2)x_1 = F_1 x_3 = 0 ,$$

a contradiction. Thus X contains at most two points. Assume $\dim \mathscr{K} \geqq 3$. Let x_1, x_2, x_3 be three orthogonal unit vectors in $\mathscr{K}$. If $\varphi(\mathscr{B}) = \boldsymbol{C}I$, we can find as above B in $\mathscr{C}$ such that $\varphi(B)x_1 = x_2$, $\varphi(B)x_2 = x_3$, hence $\varphi(B^2)x_1 = x_3$. But $B^2 = aI$ with $a \in \boldsymbol{C}$, hence $\varphi(B^2)x_1 = ax_1$, a contradiction. If X is a two point space $\varphi(\mathscr{B}) = \{aE + bF : a, b \in \boldsymbol{C}$, E and F orthogonal projections in $\varphi(\mathscr{B})$ with $E + F = I\}$. We may assume $\dim F \geqq 2$, $x_1 \in E$, $x_2 x_3 \in F$. Then B can be chosen as above, hence $x_3 = \varphi(B^2)x_1 = \varphi(B^2)Ex_1 = E\varphi(B^2)x_1 = Ex_3 = 0$, a contradiction. Thus $\dim \mathscr{K} \leqq 2$.

THEOREM 5.2. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let ψ be a $*$-automorphism of order two of $\mathfrak{A}$*

Let $\mathscr{B} = \{A \in \mathfrak{A} : \psi(A) = A\}$. *If* $\mathscr{B}$ *is a von Neumann algebra of type I then so is* $\mathfrak{A}$.

Proof. Clearly $\mathscr{B}$ is a von Neumann algebra. Let P be the central projection on the type I portion of $\mathfrak{A}$. Then P is invariant under ψ, hence $P \in \mathscr{B}$. Assume $P \neq I$. Then $\mathfrak{A}(I - P)$ has no type I portion while $\mathscr{B}(I - P)$ is of type I. Let E be a nonzero abelian projection in $\mathscr{B}(I - P)$. Then $A \to E\psi(A)E$ is an automorphism of $E\mathfrak{A}E$ leaving operators in $E\mathscr{B}E$ elementwise fixed. Moreover $E\mathscr{B}E$ is abelian. By Lemma 5.1 every irreducible representation of $E\mathfrak{A}E$ is on a Hilbert space of dimension at most 2. Thus $E\mathfrak{A}E$ is of type I (cf. argument in proof of Theorem 2.8), contradicting the fact that $\mathfrak{A}(I - P)$ has no type I portion. Thus $P = I$, $\mathfrak{A}$ is of type I.

The author wants to thank E. Effros and R. Kadison for many stimulating conversations on the subject, and to thank the referee for many valuable suggestions.

References

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
2. E. Effros, *Global structure in von Neumann algebras*, Trans. Amer. Math. Soc. **121** (1966), 434-454.
3. L. T. Gardner, *On isomorphisms of C*-algebras*, Amer. J. Math. **87** (1965), 384-396.
4. J. Glimm and R. V. Kadison, *Unitary operators in C*-algebras*, Pacific J. Math. **10** (1960), 547-556.
5. R. V. Kadison, *Unitary invariants for representations of operator algebras*, Ann. of Math. **66** (1957), 304-379.
6. J. Schwartz, *Two finite, non hyperfinite, non-isomorphic factors*, Comm. Pure Appl. Math. **16** (1963), 19-26.
7. ———, *Non-isomorphism of a pair of factors of type III*, Comm. Pure Appl. Math. **16** (1963), 111-120.
8. E. Størmer, *Jordan algebras of type I*, Acta math. **115** (1966), 165-184.
9. ———, *Irreducible Jordan algebras of self-adjoint operators* (to appear).
10. D. M. Topping, *Jordan algebras of self-adjoint operators*, Mem. Amer. Math. Soc. **53** (1965).

Received March 4, 1966.

University of Oslo and
University of Aarhus

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), No. 6, 2-chome, Fujimi-cho, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics
Vol. 21, No. 2 December, 1967