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FIXED POINTS AND FIBRE MAPS

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Let $\mathcal{F} = (E, p, B)$ be a (Hurewicz) fibre space and let λ be a lifting function for \mathcal{F} . For W a subset of B , a map $f: p^{-1}(W) \rightarrow E$ is called a fibre map if $p(e) = p(e')$ implies $p(f(e)) = p(f(e'))$. Define $\tilde{f}: W \rightarrow B$ to be the map such that $\tilde{f}p = pf$. If $[W \cup \tilde{f}(W)] \subseteq V \subseteq B$ where V is pathwise connected, define $f_b^V: p^{-1}(b) \rightarrow p^{-1}(b)$, for $b \in W$, by $f_b^V(e) = \lambda(f(e), \omega)(1)$ where $\omega: I \rightarrow V$ is a path such that $\omega(0) = \tilde{f}(b)$ and $\omega(1) = b$. Let i be a fixed point index defined on the category of compact ANR's and let Q denote the rationals. The main result of this paper is:

THEOREM 1. Let $\mathcal{F} = (E, p, B)$ be a fibre space such that E, B , and all the fibres are compact ANR's. Let $f: E \rightarrow E$ be a fibre map. If U is an open subset of B such that $\tilde{f}(b) \neq b$ for all $b \in bd(U)$ and $cl[U \cup \tilde{f}(U)] \subseteq V \subseteq B$ where V is open and pathwise connected and $\mathcal{F}|_V = (p^{-1}(V), p, V)$ is Q -orientable, then

$$i(f, p^{-1}(V)) = i(\tilde{f}, U) \cdot L(f_b^V)$$

where $L(f_b^V)$ is the Lefschetz number of f_b^V for any $b \in U$.

Independence of $L(f_b^V)$. For $\mathcal{F} = (E, p, B)$ a Hurewicz fibre space with lifting function λ [7] and ω a loop in B based at b , define $\varphi: p^{-1}(b) \rightarrow p^{-1}(b)$ by $\varphi(e) = \lambda(e, \omega)(1)$. The fibre space \mathcal{F} is called Q -orientable if

$$\varphi^*: H^*(p^{-1}(b); Q) \rightarrow H^*(p^{-1}(b); Q)$$

is the identity isomorphism for all pairs (b, ω) where $b \in B$ and ω is a loop in B based at b .

LEMMA. Let $\mathcal{F} = (E, p, B)$ be a Q -orientable fibre space and let $\omega_i: I \rightarrow B, i = 1, 2$, be paths such that $\omega_i(0) = b$ and $\omega_i(1) = b'$. Define $\varphi_i: p^{-1}(b) \rightarrow p^{-1}(b')$ by $\varphi_i(e) = \lambda(e, \omega_i)(1)$, then

$$\varphi_1^* = \varphi_2^*: H^*(p^{-1}(b'); Q) \xrightarrow{\cong} H^*(p^{-1}(b); Q).$$

Proof. By Proposition 2 of [4], each φ_i is a homotopy equivalence with homotopy inverse $\psi_i: p^{-1}(b') \rightarrow p^{-1}(b)$ given by $\psi_i(e') = \lambda(e', \bar{\omega}_i)(1)$ where $\bar{\omega}_i(s) = \omega_i(1 - s)$. Therefore, $\varphi_i^*: H^*(p^{-1}(b'); Q) \rightarrow H^*(p^{-1}(b); Q)$ is an isomorphism and $\psi_i^* = (\varphi_i^*)^{-1}$. Consider $\omega: I \rightarrow B$ defined by

$$\omega(s) = \begin{cases} \omega_1(2s) & 0 \leq s \leq 1/2 \\ \bar{\omega}_2(1 - 2s) & 1/2 \leq s \leq 1 \end{cases}$$

then ω is a loop in B based at b and since \mathcal{F} is Q -orientable, for $\varphi(e) = \lambda(e, \omega)(1)$, φ^* is the identity isomorphism. It follows from [4] that φ is homotopic to $\psi_2\varphi_1$ so $\varphi^* = \varphi_1^*\psi_2^*$ and $\psi_2^* = (\varphi_1^*)^{-1}$. Hence $\psi_2^* = \psi_1^*$ and $\varphi_2^* = \varphi_1^*$.

THEOREM 2. *Let $\mathcal{F} = (E, p, B)$ be a Q -orientable fibre space where B is pathwise connected and $H^*(p^{-1}(b); Q)$ is finitely generated for $b \in B$. For $W \subseteq B$, let $f: p^{-1}(W) \rightarrow E$ be a fibre map, then $L(f_b) = L(f_{b'})$ for all $b, b' \in W$, where f_b means f_b^* .*

Proof. Since $f_b = \varphi_i(f|p^{-1}(b))$, the lemma implies that

$$f_b^*: H^*(p^{-1}(b); Q) \rightarrow H^*(p^{-1}(b); Q)$$

is independent of the choice of the path ω_i from $\bar{f}(b)$ to b . Let $\omega_0, \omega_1: I \rightarrow B$ such that $\omega_0(0) = \bar{f}(b)$, $\omega_0(1) = \omega_1(0) = b$, and $\omega_1(1) = b'$. Define $\omega_2: I \rightarrow B$ by

$$\omega_2(s) = \begin{cases} \bar{f}\omega_1(2s) & 0 \leq s \leq 1/2 \\ \omega_0(2s - 1) & 1/2 \leq s \leq 1. \end{cases}$$

We first show that diagram (1) is homotopy commutative, where $\varphi_i(e) = \lambda(e, \omega_i)(1)$, $i = 0, 1, 2$.

$$(1) \quad \begin{array}{ccccc} p^{-1}(b) & \xrightarrow{(f|p^{-1}(b))} & p^{-1}(\bar{f}(b)) & \xrightarrow{\varphi_0} & p^{-1}(b) \\ \varphi_1 \downarrow & & & & \uparrow \varphi_2 \\ p^{-1}(b') & \xrightarrow{(f|p^{-1}(b'))} & & & p^{-1}(\bar{f}(b')) \end{array}$$

Define the homotopy $H: p^{-1}(b) \times I \rightarrow p^{-1}(b)$ by

$$H(e, t) = \lambda[f(\lambda(e, \omega_i)(1 - t)), \omega^t](1)$$

where

$$\omega^t(s) = \begin{cases} \bar{f}(\bar{\omega}_1(2s + t)) & 0 \leq s \leq \frac{1-t}{2} \\ \omega_0\left(\frac{2s + t - 1}{t + 1}\right) & \frac{1-t}{2} \leq s \leq 1. \end{cases}$$

Then $H(e, 0) = \varphi_2 f \varphi_1(e)$ and $H(e, 1) = \varphi_0 f(e)$ as required. By the lemma and [4], $(f_{b'})^* = (\varphi_1 \varphi_2 (f|p^{-1}(b')))^*$. Furthermore,

$$\begin{aligned} (\psi_1 f_{b'} \varphi_1)^* &= (\psi_1 \varphi_1 \varphi_2 (f|p^{-1}(b')) \varphi_1)^* \\ &= (\varphi_2 (f|p^{-1}(b')) \varphi_1)^* = (\varphi_0 (f|p^{-1}(b)))^* = f_b^*. \end{aligned}$$

Since Q is a field, $H^*(p^{-1}(b); Q)$ and $H^*(p^{-1}(b'); Q)$ are finite dimensional

vector spaces and $\varphi_1^*, f_b^*, \psi_1^*$ are linear transformations. Pick bases for $H^k(p^{-1}(b); Q)$ and $H^k(p^{-1}(b'); Q)$ and let Φ, F' , and Ψ be the matrices with respect to these bases representing $\varphi_1^{*,k}, f_b^{*,k}$, and $\psi_1^{*,k}$ respectively. Since $\psi_1^* = (\varphi_1^*)^{-1}, \Psi\Phi = E_n$, the $n \times n$ identity matrix, where n is the dimension of $H^k(p^{-1}(b); Q)$. Therefore, $\text{trace}(\Phi F' \Psi) = \text{trace}(F')$ which implies that $L(f_b) = L(\psi_1 f_b \varphi_1)$. The theorem now follows because $(\psi_1 f_b \varphi_1)^* = f_b^*$ implies $L(\psi_1 f_b \varphi_1) = L(f_b)$.

2. **Extension of a theorem of Leray.** Let B and F be topological spaces and let $(B \times F, \pi^1, B)$ be the trivial fibre space. Suppose W is a subset of B and $f: W \times F \rightarrow B \times F$ is a fibre map. Define $f_b: F \rightarrow F$ by $f_b = \pi^2 f j_b$ where $j_b: F \rightarrow W \times F$ is given by $j_b(x) = (b, x)$ and $\pi^2: B \times F \rightarrow F$ is projection. Theorem 3 is a restatement of Theorem 27 of [9] in the somewhat specialized form in which we shall use it.

THEOREM 3 (Leray). *Let $(B \times F, \pi^1, B)$ be the trivial fibre space where B and F are finite polyhedra. For U an open connected subset of B , let $f: \text{cl}(U) \times F \rightarrow B \times F$ be a fibre map.¹ If $\bar{f}(b) \neq b$ for all $b \in \text{bd}(U)$, then*

$$\bar{i}(f, U \times F) = \bar{i}(\bar{f}, U) \cdot L(f_b)$$

for all $b \in U$, where \bar{i} denotes the Leray fixed point index.

By Theorem 22 and Corollary 26-27 of [9], the Leray index [9, p. 208] satisfies the O'Neill axioms [10, p. 500]. (We will use the formulation of the axioms and the terminology of [1]). Therefore, an index i for the category of compact ANR's, satisfying the O'Neill axioms, may be obtained from the index \bar{i} in the following manner [2, p. 20]. Let X be a compact ANR and let α be a finite open cover of X , then there exists a finite polyhedron K and maps $\varphi: X \rightarrow K, \psi: K \rightarrow X$ such that $\psi\varphi$ is α -homotopic to the identity map on X , i.e. there exists a map $H: X \times I \rightarrow X$ such that $H(x, 0) = x, H(x, 1) = \psi\varphi(x)$, and for each $x \in X$, the set $\{H(x, t) \mid t \in I\}$ lies in a single element of α [5, Theorem 6.1]. Write $\psi\varphi \sim_\alpha 1_x$. For U an open subset of X and $f: X \rightarrow X$ a map such that $f(x) \neq x$ for all $x \in \text{bd}(U)$, let

$$i_\alpha(f, U) = \bar{i}(\varphi f \psi, \psi^{-1}(U)).$$

Browder [2, Theorem 2, p. 20] showed that there exists a finite open cover $\kappa_f(U)$ of X such that if α is a refinement of $\kappa_f(U)$, then $i_\alpha(f, U)$ is well-defined and independent of α, φ , and ψ . Write $i_\alpha = i$ for all

¹ The notation $\text{cl}(U)$ denotes the closure of U . We use $\text{bd}(U)$ for the boundary of U .

such α .

THEOREM 4. *Let $(B \times F, \pi^1, B)$ be the trivial fibre space where B is a finite polyhedron and F is a compact ANR. For U a connected open subset of B , let $f: \text{cl}(U) \times F \rightarrow B \times F$ be a fibre map. If $\bar{f}(b) \neq b$ for all $b \in \text{bd}(U)$, then*

$$i(f, U \times F) = \bar{i}(\bar{f}, U) \cdot L(f_b)$$

for all $b \in U$.

Proof. Let F be dominated by a finite polyhedron K by means of maps $\varphi: F \rightarrow K$ and $\psi: K \rightarrow F$. Define $f^*: B \times K \rightarrow B \times K$ by $f^*(b, k) = (\bar{f}(b), \varphi f_b \psi(k))$ then f^* is a fibre map with respect to $(B \times K, \pi^1, B)$ and $\bar{f}^* = \bar{f}$. Since $\psi\varphi$ is homotopic to the identity map on F , $L(f_b^*) = L(f_b)$ (see the proof that $L(f_b) = L(\psi_1 f_b \varphi_1)$ in Theorem 2). Let α be a finite open cover of B which refines $\kappa_{\bar{f}}(U)$, then $\tau = \{(\pi^1)^{-1}(A) \mid A \in \alpha\}$ refines $\kappa_f(p^{-1}(U))$. Since $f^* = (1_B \times \varphi)f(1_B \times \psi)$ and, trivially,

$$(1_B \times \psi)(1_B \times \varphi) \sim_{\tau} 1_B \times 1_F,$$

then $i(f, U \times F) = \bar{i}(f^*, U \times K)$. Therefore, by Theorem 3,

$$i(f, U \times F) = \bar{i}(\bar{f}, U) \cdot L(f_b).$$

3. Proof of Theorem 1. We first assume that B is a finite polyhedron. By a theorem of Hopf [6, Theorem 5], given $\varepsilon > 0$, there exists a map $\bar{g}: B \rightarrow B$ homotopic to \bar{f} by a homotopy $h: B \times I \rightarrow B$ such that $h(b, 0) = \bar{f}(b)$, $h(b, 1) = \bar{g}(b)$ and $\rho[h(b, t), h(b, t')] < \varepsilon$ for $b \in B$, $t, t' \in I$, where ρ is the metric of B . The map \bar{g} has a finite number of fixed points b_1, \dots, b_s where, with respect to some barycentric subdivision of B , each b_j lies in the interior of a different simplex σ_j of B , where σ_j is not a face of any other simplex of B . Since \bar{f} has no fixed points on $\text{bd}(U)$, $\inf \{\rho(b, f(b)) \mid b \in \text{bd}(U)\} = \varepsilon_1 > 0$. Let $\varepsilon_2 > 0$ be the distance from $\text{cl}[U \cup \bar{f}(U)]$ to $B - V$ (if $V = B$, take $\varepsilon_2 = \infty$). Let $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then $h(b, t) \neq b$ for all $b \in \text{bd}(U)$. Hence $i(\bar{f}, U) = i(\bar{g}, U)$ by the homotopy axiom. Furthermore, $\text{cl}[U \cup \bar{g}(U)] \subseteq V$. The homotopy h induces $h': B \rightarrow B^I$. Let λ be regular lifting function for \mathcal{F} and define $H': E \rightarrow E^I$ by

$$H'(e)(t) = \lambda(f(e), h'(p(e)))(t).$$

Define $g: E \rightarrow E$ by $g(e) = H'(e)(1)$ then g is a fibre map homotopic to f by a homotopy without fixed points on $\text{bd}(p^{-1}(U))$ so $i(f, p^{-1}(U)) = i(g, p^{-1}(U))$. Furthermore, $pg = \bar{g}p$. Since f_{b_j} is precisely g_{b_j} if we use the path $h'(b_j)$ to define f_{b_j} and the constant path to define g_{b_j} ,

then $L(f_{b_j}^V) = L(g_{b_j}^V)$. We have shown that when B is a finite polyhedron, it is sufficient to verify the conclusion for the map g .

Let U_j be a δ -neighborhood of b_j where δ is chosen small enough so that $[\text{cl}(U_j) \cup \bar{g}(\text{cl}(U_j))] \subseteq \sigma_j$. We may contract σ_j to b_j so that b_j stays fixed throughout the contraction and such that the restriction to $\text{cl}(U_j)$ contracts $\text{cl}(U_j)$ through itself to b_j . The contraction induces fibre homotopy equivalences

$$\begin{aligned} \alpha: p^{-1}(\sigma_j) &\xrightarrow{\cong} \sigma_j \times F: \beta \\ \alpha': p^{-1}(\text{cl}(U_j)) &\xrightarrow{\cong} \text{cl}(U_j) \times F: \beta' \end{aligned}$$

where the primes denote restriction and $F = p^{-1}(b_j)$ [4, Proposition 4]. Consider the diagram

$$(2) \quad \begin{array}{ccccc} & & g' & & \\ & \overbrace{\hspace{10em}} & & \overbrace{\hspace{10em}} & \\ \text{cl}(U_j) \times F & \xleftarrow[\beta']{\alpha'} & p^{-1}(\text{cl}(U_j)) & \xrightarrow{g} & p^{-1}(\sigma_j) \xleftarrow[\beta]{\alpha} \sigma_j \times F \\ & \searrow \text{proj.} & \downarrow p & & \downarrow p \\ & & \text{cl}(U_j) & \xrightarrow{\bar{g}} & \sigma_j \\ & & & & \swarrow \text{proj.} \end{array}$$

where $g' = \alpha g \beta'$. By Theorem 4,

$$i(g', U_j \times F) = \bar{i}(\bar{g}, U) \cdot L(g_{b_j}^V).$$

If we use the constant path to define g_{b_j} , then $g_{b_j} = g_{b_j}'$, so $L(g_{b_j}^V) = L(g_{b_j}'^V)$. Let $\mu = g\beta': p^{-1}(\text{cl}(U_j)) \rightarrow \sigma_j \times F$, then by the commutativity axiom

$$i(\alpha\mu, U_j \times F) = i(\mu\alpha', p^{-1}(U_j)).$$

Now $i(\alpha\mu, U_j \times F) = i(g', U_j \times F)$ by definition. On the other hand, $\mu\alpha' = g\beta'\alpha'$ is homotopic to g by a homotopy which has no fixed points on $bd(p^{-1}(U_j))$ since \bar{g} has no fixed points on $bd(U_j)$ and the homotopy between $\beta'\alpha'$ and the identity is fibre-preserving, so by the homotopy axiom $i(\mu\alpha', p^{-1}(U_j)) = i(g, p^{-1}(U_j))$. Therefore

$$i(g, p^{-1}(U_j)) = \bar{i}(\bar{g}, U_j) \cdot L(g_{b_j}^V).$$

Renumber the fixed points of \bar{g} so that b_1, \dots, b_q are the fixed points which lie in U . Since $g(e) = e$ implies $p(e) = b_j$ for some $j = 1, \dots, s$, g has no fixed points on $[p^{-1}(\text{cl}(U)) - \bigcup_{j=1}^q p^{-1}(U_j)]$. Hence by the additivity axiom,

$$\begin{aligned}
 i(g, p^{-1}(U)) &= \sum_{j=1}^q i(g, p^{-1}(U_j)) \\
 &= \sum_{j=1}^q \bar{i}(\bar{g}, U_j)L(g_j^v) = \bar{i}(\bar{g}, U) \cdot L(g^v) .
 \end{aligned}$$

Now suppose that B is a compact ANR, let K be a finite polyhedron and let $\varphi: B \rightarrow K, \psi: K \rightarrow B$ be maps such that $\psi\varphi \sim_\alpha 1_B$ where α refines $\kappa_{\bar{\gamma}}(U)$ and $\alpha(\bar{f}(U))$, the union of all $A \in \alpha$ such that $A \cap \bar{f}(U) \neq \emptyset$, is contained in V . Let $\psi^*(\mathcal{S}) = (\psi^*(E), p^*, K)$ where

$$\psi^*(E) = \{(k, e) \in K \times E \mid \psi(k) = p(e)\}$$

and $p^*(k, e) = k$, then $\psi^*(\mathcal{S})$ is a fibre space with lifting function $\lambda^{\#}$ given by

$$\lambda^{\#}((k, e), \omega)(t) = (\omega(t), \lambda(e, \psi\omega)(t))$$

where λ is the lifting function of \mathcal{S} . Let $h: B \times I \rightarrow B$ be the α -homotopy such that $h(b, 0) = b, h(b, 1) = \psi\varphi(b)$, then h induces $h': B \rightarrow B'$. Define $\varphi': E \rightarrow \psi^*(E)$ by

$$\varphi'(e) = (\varphi p(e), \lambda(e, h'(p(e))))$$

Consider

$$\begin{array}{ccc}
 \psi^*(E) & \xleftarrow{\varphi'} & E \\
 \downarrow p^{\#} & \swarrow f^{\#} & \searrow f \\
 \psi^*(E) & \xrightarrow{\psi'} & E \\
 \downarrow p^{\#} & & \downarrow p \\
 K & \xrightarrow{\psi} & B \\
 \downarrow p^{\#} & \swarrow \bar{f}^{\#} & \searrow \bar{f} \\
 K & \xleftarrow{\varphi} & B
 \end{array}$$

(3)

where $\psi'(k, e) = e$ and $f^{\#} = \varphi' f \psi'$. Since $\bar{f}^{\#} = \varphi \bar{f} \psi$ and $\psi\varphi \sim_\alpha 1_B$, then $i(\bar{f}, U) = \bar{i}(\bar{f}^{\#}, \psi^{-1}(U))$. We let $\nu = \varphi' f: E \rightarrow \psi^*(E)$, then by the commutativity axiom,

$$i(\psi'\nu, p^{-1}(U)) = i(\nu\psi', \psi'^{-1}p^{-1}(U)) .$$

Define $H: E \times I \rightarrow E$ by $H(e, t) = \lambda(e, h'(p(e)))(t)$. If $H(f(e), t) = e$ for any $e \in bd(p^{-1}(U)), t \in I$, then $h(\bar{f}(p(e)), t) = p(e)$ which is impossible since α refines $\kappa_{\bar{\gamma}}(U)$ [2, p. 20], so $\psi'\nu = \psi'\varphi'f$ is homotopic to f by a homotopy without fixed points on $bd(p^{-1}(U))$ and by the homotopy axiom

$$i(\psi'\nu, p^{-1}(U)) = i(f, p^{-1}(U)) .$$

On the other hand, $i(\nu\psi', \psi'^{-1}p^{-1}(U)) = i(f^\#, p^{\#-1}(\psi^{-1}(U)))$. If $k \in \psi^{-1}(U)$, then $\bar{f}^\#(k) \in \psi^{-1}(V) = W$ since $\alpha(\bar{f}(U)) \subseteq V$. Let $\omega: I \rightarrow W$ be a path such that $\omega(0) = \bar{f}^\#(k)$ and $\omega(1) = k$. Define $\omega': I \rightarrow V$ by

$$\omega'(s) = \begin{cases} h'(\bar{f}\psi(k))(2s) & 0 \leq s \leq 1/2 \\ \psi\omega(2s - 1) & 1/2 \leq s \leq 1 \end{cases}$$

and let $f_{\psi(k)}$ be given by $f_{\psi(k)}(e) = \lambda(f(e), \omega')(1)$. Define $f'_{\psi(k)}: p^{-1}(\psi(k)) \rightarrow p^{-1}(\psi(k))$ by

$$f'_{\psi(k)}(e) = \lambda[\lambda(f(e), h'(\bar{f}\psi(k)))(1), \psi\omega](1),$$

then by [4], $f'_{\psi(k)}$ is homotopic to $f_{\psi(k)}$. But $f_k^\#(k, e) = \lambda^\#((k, e), \omega)(1) = (k, f'_{\psi(k)}(e))$. Therefore $L(f_k^{\#W})$ is equal to $L(f_k^V)$ and is independent of k and ω . Applying the first part of the proof to the fibre space $\psi^\#(\mathcal{S})$, the map $f^\#$, and the open set $\psi^{-1}(U) \subseteq K$, we get

$$i(f^\#, p^{\#-1}(\psi^{-1}(U))) = \bar{i}(\bar{f}^\#, \psi^{-1}(U)) \cdot L(f_k^{\#W}).$$

Therefore,

$$i(f, p^{-1}(U)) = i(\bar{f}, U) \cdot L(f_k^V)$$

which completes the proof of Theorem 1.

4. The index of a fixed point class. Let X be a compact ANR and let $f: X \rightarrow X$ be a map. Denote the fixed point classes of f by F_1, \dots, F_r . Let $(\tilde{X}, \tilde{p}, X)$ be the universal covering space of X , then by [2, pp. 43-44] there is a map $\tilde{f}^j: \tilde{X} \rightarrow \tilde{X}$ such that $\tilde{p}\tilde{f}^j = f\tilde{p}$ which has the following properties: (1) if $\tilde{f}^j(e) = e$, then $p(e) \in F_j$, (2) for each $b \in F_j$ there exists $e \in \tilde{p}^{-1}(b)$ such that $\tilde{f}^j(e) = e$. We say that \tilde{f}^j covers F_j . There is an open set U_j in X containing F_j such that $\text{cl}(U_j) \cap F_k = \emptyset$ for $k \neq j$. The index of F_j is defined by $i(F_j) = i(f, U_j)$ and is independent of the choice of U_j .

THEOREM 5. *Let X be a compact ANR with finite fundamental group. Let $f: X \rightarrow X$ be a map, let F be a fixed point class of f , and let $\tilde{f}: \tilde{X} \rightarrow \tilde{X}$ cover F . If there exists an open subset U of X such that for $x \in U$, $f(x) = x$ if, and only if, $x \in F$, $f(x) \neq x$ for $x \in \text{bd}(U)$, and $\text{cl}[U \cup f(U)] \subseteq V$, where V is an open connected simply-connected subset of X , then*

$$i(F) = L(\tilde{f})/L(\tilde{f}_x^V)$$

for $x \in U$.

Proof. We first observe that $L(\tilde{f}_x^V) \neq 0$. Take $x \in F$, then since the fibre is discrete $L(\tilde{f}_x^V)$ is just the number of fixed points of \tilde{f}

restricted to $\tilde{p}^{-1}(x)$ which, since \tilde{f} covers F , must be greater than zero. Since $\pi_1(X)$ is finite, \tilde{X} is compact and we can apply Theorem 1 to obtain

$$i(f, U) = i(\tilde{f}, \tilde{p}^{-1}(U))/L(\tilde{f}_x^v).$$

Since \tilde{f} has no fixed points outside of $\tilde{p}^{-1}(U)$, $i(\tilde{f}, \tilde{p}^{-1}(U)) = L(\tilde{f})$.

The existence of the simply-connected set V in the hypotheses of Theorem 5 is not as severe a restriction as it may appear. For example, if X is a finite polyhedron, (or a compact topological manifold, with or without boundary) f is homotopic to a map g which has only isolated fixed points [6, Theorem 5] [3, Theorem 2] and the homotopy carries F to a fixed point class F' of g of the same index [2, Theorem 3, p. 36]. Hence we can apply Theorem 5 to g and F' to compute $i(F)$ (compare Theorem 5.2 of [8]).

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