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Two teams A and B can be compared by matching each player in A against each player in B. We say that $A > B$ if and only if the players of A collectively win more games against players of B than they lose. If there are n teams $T_1, T_2, \cdots, T_n$, then the outcomes of the matches between the various teams may be represented by an oriented graph H_n on n nodes in which an arc goes from the i-th node to the j-th node if and only if $T_i > T_j$. It is shown that any oriented graph can be generated in this way, and that the minimum number of players necessary to generate any oriented graph H_n is of the order of $n^2/\log n$.

If $A = \{a_1, a_2, \cdots, a_r\}$ and $B = \{b_1, b_2, \cdots, b_s\}$ are two nonempty finite sets of real numbers, we say $A > B$ if and only if the number of solutions of $a_i > b_j$ exceeds the number of solutions of $a_i < b_j$. We think of the sets A and B as teams of players. The numbers in the sets denote both the names and the strengths of the players; we assume the stronger player always wins in any game between two players. A match between the teams A and B consists of rs individual games between the players of A and B. The stronger team is the team whose players win a majority of the games. (We admit the possibility of draws, both between individual players and between teams.)

Let N players $x_1, x_2, \cdots, x_N$ be split into n teams $T_1, T_2, \cdots, T_n$ and suppose that every team plays against every other team. (We assume throughout that $n > 1$.) The results of these matches may be represented by an oriented graph H_n on n nodes $t_1, t_2, \cdots, t_n$ in which an arc goes from t_i to t_j if and only if $T_i > T_j$. For example, the teams $T_1 = \{6, 7, 2\}$, $T_2 = \{1, 5, 9\}$, and $T_3 = \{8, 3, 4\}$ generate the graph H_3 shown in Figure 1.

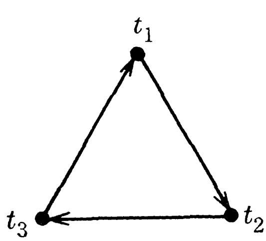

Figure 1.

In § 2 we show that any oriented graph can be generated by means of team comparisons and in § 3 we consider the problem of determining the minimum number of players necessary to generate any oriented graph H_n.

2. Generating arbitrary oriented graphs.

If the teams

$$T_1, T_2, \cdots, T_n$$

generate the oriented graph H_n let $\alpha(i, j)$ denote the net score of T_i against T_j, i.e., the number of games won minus the number of games lost by players of T_i against players of T_j. Let w and s denote the strengths of the weakest and strongest players on the n teams and choose numbers $w_1, w_2,$ s_1 and s_2 such that $w_1 = w_2 < w$ and $s_1 > s_2 > s$. If we add two players of strength s_1 and w_1 to T_i and two players of strength s_2 and w_2 to T_j, it is readily verified that the only affect this has upon the net scores between the different teams is to increase $\alpha(i, j)$ by one.

This process can of course be repeated. It follows that if the net scores between the teams are prescribed in advance, and if their sum is β, then no more than $n + 4\beta$ players are necessary to realize these scores, since we may assume that initially there are n players of equal strength, one on each team. (We remark that although the net scores can be prescribed arbitrarily, the win-loss ratios for the matches between the various teams cannot all be prescribed arbitrarily in general; this follows from results of Steinhaus and Trybula [4] and Usiskin [5].) In particular, therefore, any oriented graph H_n can be generated by $n + 4\binom{n}{2} = 2n^2 - n$, or fewer, players. A simple induction argument, using a refinement of this construction, shows that no more than $n^2 + 3n - 11$ players are necessary to generate any oriented graph H_n if $n \geqq 3$. Our main result gives a sharper bound (for large n) that, in a sense, is best possible.

3. Main result.

THEOREM. *If $\lambda(n)$ denotes the least integer N such that the number of players needed to generate any oriented graph H_n is at most N, then there exist positive constants c_1 and c_2 such that*

$$\frac{c_1 n^2}{\log n} < \lambda(n) < \frac{c_2 n^2}{\log n}$$

Proof. If N players can generate the graph H_n, then the strengths of the N players can be taken from the integers $1, 2, \cdots, N$. (Some of the players may have the same strength.) The number of ways of forming n teams from not more than N players, whose strengths are taken from the integers $1, 2, \cdots, N$, is certainly not more than

$(2n)^N$. There are $3^{\binom{n}{2}}$ oriented graphs H_n. Consequently, if N, or fewer, players suffice to generate every oriented graph H_n, it must be that

$$(2n)^N \geqq 3^{\binom{n}{2}} ,$$

or

$$N \geqq \frac{\log 3}{2} \frac{n(n-1)}{\log (2n)} , \tag{1}$$

since each allocation of players determines at most one graph.

This implies the lower bound of the theorem; the upper bound will follow from three lemmas.

Consider a special oriented graph that consists of two disjoint sets of nodes, A and B, such that an arc goes from each node of A to each node of B; any oriented graph with n nodes that can be expressed as the union of disjoint special graphs will be called a *bilevel graph* B_n. (We admit the possibility that one of the node-sets of one of the special graphs composing B_n is empty.) The structure of a typical bilevel graph, composed of four special graphs, is indicated in Figure 2.

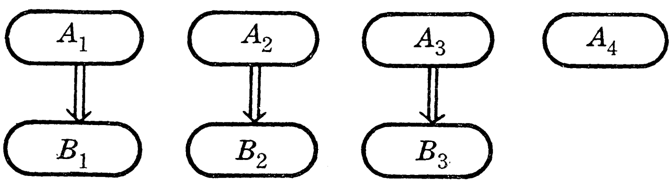

Figure 2.

LEMMA 1. *Any bilevel graph B_n can be generated by $2n$ players, two on a team.*

Proof. We illustrate the proof on the bilevel graph depicted in Figure 2. Associate each node in the various node-sets with the team indicated in the following list.

$A_1 : (1,20)$ $A_2 : (2,18)$ $A_3 : (3,16)$ $A_4 : (4,14)$
$B_1 : (1,19)$ $B_2 : (2,17)$ $B_3 : (3,15)$

One can verify directly that this allocation of players, two on each team, will generate the bilevel graph in Figure 2. An analogous construction will generate any bilevel graph B_n. (We remark that it is easy to modify this construction to show that the lemma remains true even if it is insisted that no two different players have the same strength.)

LEMMA 2. *If the oriented graph H_n can be expressed as the union of l arc-disjoint bilevel graphs $B^{(1)}, B^{(2)}, \cdots, B^{(l)}$, all of which have the same n nodes, then H_n can be generated by $2ln$ players, $2l$ on a team.*

Proof. There exist teams R_{ik} of two players each, according to Lemma 1, such that the teams $R_{ik}, i = 1, 2, \cdots, n$, generate the graphs $B^{(k)}$, for $k = 1, 2, \cdots, l$. We may assume that every player on any team R_{jk} is stronger than every player on any team R_{ih}, for $1 \leqq h < k \leqq l$. (This property can be ensured by adding, if necessary, a suitable constant c_k to the strength of every player on the teams R_{ik}, $k = 1, 2, \cdots, l$.) The teams

$$T_i = \bigcup_{k=1}^{l} R_{ik}, \qquad i = 1, 2, \cdots, n$$

each have $2l$ players and it is not difficult to see that they generate the oriented graph H_n.

The following nontrivial result was proved by Erdös and Moser [1].

LEMMA 3. *There exists a (large) constant c such that any oriented graph H_n can be expressed as the union of l arc-disjoint bilevel graphs, all of which have the same n nodes, where*

$$l < \frac{cn}{\log n} .$$

This suffices to complete the proof of the theorem.

4. Remarks. There are certain curious aspects of this mode of comparison arising from its lack of transitivity. In the example given in § 1, the teams T_1, T_2 and T_3 were such that $T_1 > T_2$ and $T_2 > T_3$. One might expect that $T_1 \cup T_2 > T_2 \cup T_3$, and this is indeed the case. However, since $T_1 < T_3$, one might equally well expect that $T_1 \cup T_2 < T_2 \cup T_3$, and this is false.

The following example is perhaps more striking. If $A = \{2, 3, 10\}$ and $B = \{1, 8, 9\}$ then $A > B$ by 5 wins to 4. If $A_1 = A \cup \{5\}$ and $B_1 = B \cup \{4\}$, then the teams A_1 and B_1 are tied with 8 wins each. If $A_2 = A_1 \cup \{7\}$ and $B_2 = B_1 \cup \{6\}$ then $B_2 > A_2$ by 13 wins to 12. Notice that at each stage we added the stronger player to the team that was the stronger originally, yet the net affect was to reverse the relative strengths of the two teams. This process can be continued. If $A_3 = A_2 \cup \{12\}$ and $B_3 = B_2 \cup \{11\}$, then A_3 and B_3 are tied with 18 wins each. Finally, if $A_4 = A_3 \cup \{14\}$ and $B_4 = B_3 \cup \{13\}$, then $A_4 > B_4$ by 25 wins to 24.

We mention briefly another method of comparing two teams A and B of n players each. The players of A are lined up in some fixed order and paired off against all $n!$ orderings of the players of B. The team that wins a majority of the $n!$ matches will be declared winner. The six matches between T_1 and T_2 of § 1 are as follows:

	6,	7,	2
1.	1,	5,	9
2.	1,	9,	5
3.	5,	1,	9
4.	5,	9,	1
5.	9,	1,	5
6.	9,	5,	1

The team $T_1 = \{6, 7, 2\}$ wins matches 1, 3, 4, 6 and loses matches 2 and 5. Thus $T_1 > T_2$ by 4 wins to 2. Similarly we find that

$$T_2 > T_3 \text{ and } T_3 > T_1$$

by 4 wins to 2 also.

We remark in closing that other related ways of generating oriented graphs have been discussed by McGarvey [3], and Erdös and Moser [1].

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