

# Pacific Journal of Mathematics

**REMARK ON A PROBLEM OF NIVEN AND ZUCKERMAN**

RICHARD THOMAS BUMBY AND EVERETT C. DADE

## REMARK ON A PROBLEM OF NIVEN AND ZUCKERMAN

R. T. BUMBY AND E. C. DADE

**An integer of an algebraic number field  $K$  is called irreducible if it has no proper integer divisors in  $K$ . Every integer of  $K$  can be written as a product of irreducible integers, usually in many different ways. Various problems have been inspired by this lack of unique factorization. This paper studies the question: When are the irreducible integers of  $K$  determined by their norms? Attention is confined to the case in which  $K$  is a quadratic field. With this assumption it is possible to give a complete answer in terms of the ideal class group of  $K$  and the nature of the units of  $K$ .**

The fields sought in this problem are those quadratic fields  $K$  (with  $N: K \rightarrow \mathbb{Q}$  denoting the norm) which satisfy

*Property N: If  $\alpha$  is an irreducible integer of  $K$  and  $\beta$  is another integer of  $K$  such that  $N\alpha = N\beta$ , then  $\beta$  is also irreducible.*

In many cases Property  $N$  can be studied by looking at the class group  $H$  of  $K$ . However the study is complicated by the existence of quadratic number fields  $K$  satisfying:

(1)  $K$  is real and  $N\varepsilon = +1$ , for every unit  $\varepsilon$  of  $K$ .

When  $K$  satisfies (1), we are forced to consider an extended class group  $H'$  of  $K$  defined as follows:

Two nonzero fractional ideals  $\alpha, \beta$  are said to be *strongly equivalent* if  $\alpha \cdot \beta^{-1} = (\gamma)$  is a principal ideal generated by an element  $\gamma$  of positive norm. This is clearly an equivalence relation. The strong equivalence classes form the group  $H'$  under the usual multiplication. There are two strong equivalence classes of principal ideals: the class  $\sigma$  consisting of all principal ideals  $(\alpha)$  such that one, and hence all, generators of  $(\alpha)$  have negative norm; and the identity class  $1$  of principal ideals  $(\alpha)$  all of whose generators have positive norm. Clearly  $\sigma^2 = 1$ , and the class group  $H$  is naturally isomorphic to  $H'/\langle\sigma\rangle$ .

If  $K$  does not satisfy (1), notice that  $H'$ , as defined above, and the class group  $H$  coincide.

In any case, if  $\mathfrak{p}$  is any prime ideal of  $K$  and  $\mathfrak{p}'$  is the conjugate prime ideal, then  $\mathfrak{p} \cdot \mathfrak{p}' = (N\mathfrak{p})$ . But  $N(N\mathfrak{p}) = (N\mathfrak{p})^2 > 0$ . So

(2)  $\mathfrak{p}$  and  $\mathfrak{p}'$  lie in inverse strong equivalence classes.

Our main result is





the absolute values  $|N\alpha|$  of their norms if and only if the class group  $H$  is of type (a) or (b) in the theorem above.

The problem considered in this paper was raised by Niven and Zuckerman in [2]. A more general form of this problem was treated by other methods in [1].

#### BIBLIOGRAPHY

1. R. T. Bumby, *Irreducible integers in Galois Extensions*, Pacific J. Math. (to appear).
2. I. Niven and H. S. Zuckerman, *On the lack of unique factorization in quadratic fields*, Oral presentation at American Mathematical Society Symposium on Number Theory, Pasadena, November 1963.

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