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REMARK ON A PROBLEM OF NIVEN AND ZUCKERMAN

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R. T. BUMBY AND E. C. DADE

An integer of an algebraic number field K is called irreducible if it has no proper integer divisors in K. Every integer of K can be written as a product of irreducible integers, usually in many different ways. Various problems have been inspired by this lack of unique factorization. This paper studies the question: When are the irreducible integers of K determined by their norms? Attention is confined to the case in which K is a quadratic field. With this assumption it is possible to give a complete answer in terms of the ideal class group of K and the nature of the units of K.

The fields sought in this problem are those quadratic fields K (with $N: K \rightarrow Q$ denoting the norm) which satisfy

Property N: If α is an irreducible integer of K and β is another integer of K such that $N\alpha = N\beta$, then β is also irreducible.

In many cases Property N can be studied by looking at the class group H of K. However the study is complicated by the existence of quadratic number fields K satisfying:

(1) K is real and $N\varepsilon = +1$, for every unit ε of K.

When K satisfies (1), we are forced to consider an extended class group H' of K defined as follows:

Two nonzero fractional ideals \mathfrak{a} , \mathfrak{b} are said to be strongly equivalent if $\mathfrak{a} \cdot \mathfrak{b}^{-1} = (\gamma)$ is a principal ideal generated by an element γ of positive norm. This is clearly an equivalence relation. The strong equivalence classes form the group H' under the usual multiplication. There are two strong equivalence classes of principal ideals: the class σ consisting of all principal ideals (α) such that one, and hence all, generators of (α) have negative norm; and the identity class 1 of principal ideals (α) all of whose generators have positive norm. Clearly $\sigma^2 = 1$, and the class group H is naturally isomorphic to $H'/\langle \sigma \rangle$.

If K does not satisfy (1), notice that H', as defined above, and the class group H coincide.

In any case, if \mathfrak{p} is any prime ideal of K and \mathfrak{p}' is the conjugate prime ideal, then $\mathfrak{p} \cdot \mathfrak{p}' = (N\mathfrak{p})$. But $N(N\mathfrak{p}) = (N\mathfrak{p})^2 > 0$. So

(2) \mathfrak{p} and \mathfrak{p}' lie in inverse strong equivalence classes.

Our main result is

THEOREM. Let K be a quadratic number field. Then K satisfies property N if and only if:

(a) H has exponent 2

- or (b) H is odd
- or (c) K satisfies (1) and the 2-Sylow subgroup of H' is cyclic

Proof. First we assume that one of (a), (b), and (c) holds. If K does not satisfy property N then there exist an irreducible integer α and a reducible integer β such that $N\alpha = N\beta$. Let $(\alpha) = \mathfrak{p}_1 \cdots \mathfrak{p}_t$, where the \mathfrak{p}_i are prime ideals. Since $N\beta = N\alpha$, the ideal (β) must equal $\mathfrak{q}_1 \cdots \mathfrak{q}_t$, where, for each i, either \mathfrak{q}_i is \mathfrak{p}_i , or \mathfrak{q}_i is \mathfrak{p}'_i . But $\beta = \gamma \cdot \delta$, where γ , δ are nonunit integers. Hence we may assume:

$$(\gamma) = \mathfrak{q}_1 \cdots \mathfrak{q}_s$$
, $(\delta) = \mathfrak{q}_{s+1} \cdots \mathfrak{q}_t$, where $1 \leq s < t$.

Let e_i be +1 if $q_i = p_i$ and -1 if $q_i = p'_i$. By (2) there are numbers ε, ζ in K such that:

(3) $(\varepsilon) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_s^{e_s}, \ (\zeta) = \mathfrak{p}_{s+1}^{e_{s+1}} \cdots \mathfrak{p}_t^{e_t}, \ and \ (\gamma), \ (\delta) \ are \ strongly \ equiva$ $lent to \ (\varepsilon), \ (\zeta), \ respectively.$

In case (a), $\mathfrak{p}_i^{e_i}$ is equivalent to \mathfrak{p}_i . Therefore (3) implies that $\mathfrak{p}_1 \cdots \mathfrak{p}_s = (\eta)$ is a principal ideal. Clearly η is an integer and a proper divisor of α , contradicting its irreducibility.

In any case, if $e_1 = \cdots = e_s$, then $\mathfrak{p}_1 \cdots \mathfrak{p}_s$ is principal, and we arrive at a contradiction. Therefore we may assume

Define the integral ideals a, b by:

$$a = (\mathfrak{p}_1 \cdots \mathfrak{p}_r)(\mathfrak{p}_{s+1} \cdots \mathfrak{p}_u)$$

$$\mathfrak{b} = (\mathfrak{p}_{r+1} \cdots \mathfrak{p}_s)(\mathfrak{p}_{u+1} \cdots \mathfrak{p}_t).$$

By (4), both a and b are proper integral ideals. By (3), $a \cdot b^{-1} = (\epsilon \zeta)$ is strongly equivalent to $(\gamma \cdot \delta) = (\beta)$. Since $N\beta = N\alpha$, the ideals (α) , (β) are strongly equivalent. Therefore $a \cdot b^{-1}$ is strongly equivalent to $(\alpha) = a \cdot b$. So:

(5)
$$\mathfrak{b}^2 = (\mathfrak{a} \cdot \mathfrak{b})(\mathfrak{a} \cdot \mathfrak{b}^{-1})^{-1} = (\lambda), \text{ where } N\lambda > 0.$$

In case (b), this implies that b is principal. Hence α has a proper divisor.

In case (c), the only strong equivalence classes of orders dividing 2 are 1 and σ . By (5), b must lie in one of them. So it is principal, and α has a proper divisor.

In each of the three cases, α must have a proper divisor, contradicting its irreducibility. So K must satisfy property N.

Now suppose that K satisfies property N. We first show that H' cannot contain an element π satisfying:

(6) π has even order 2n > 2 and, if K satisfies (1), then $\pi^n \neq \sigma$.

Suppose such a π exists. By Dirichlet's theorem there exists a prime ideal \mathfrak{p} in the class π (or, if K satisfies (1), in the class $\pi\langle\sigma\rangle$). Evidently $\mathfrak{p}^{2n} = (\alpha)$ is generated by an irreducible element α satisfying $N\alpha = p^{2n}$, where $p = N\mathfrak{p}$. But $p^{2n} = N(p^n)$, and, since n > 1, $p^n = p \cdot p^{n-1}$ is reducible. This contradicts property N. So no π satisfying (6) can exist.

Suppose K does not satisfy (1). It follows immediately from (6) that, if H has even order, then it must have exponent 2. So one of (a) or (b) must hold.

Now we assume that K satisfies (1). Then H' cannot contain elements τ , ρ satisfying:

(7)
$$\tau^{2^m} = \sigma$$
, where $m \geq 2$, and $\rho^2 = 1$, $\rho \notin \langle \sigma \rangle$.

Suppose τ , ρ exist. Choose prime ideals $\mathfrak{p}_1, \mathfrak{p}_2$ in the classes $\tau \langle \sigma \rangle$, $\tau^{-1}\rho \langle \sigma \rangle$, respectively. Then $\mathfrak{p}_1^2 \cdot \mathfrak{p}_2^2$ lies in the strong equivalence class 1. So it is a principal ideal (α), where $N\alpha = p_1^2 p_2^2 = N(p_1 p_2)$ and $p_i = N\mathfrak{p}_i$, i = 1, 2. By property N, α must be reducible. One of its proper divisors must generate an ideal from the list: $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1^2, \mathfrak{p}_1 \cdot \mathfrak{p}_2$. But these lie in the classes $\tau \langle \sigma \rangle, \tau^{-1}\rho \langle \sigma \rangle, \tau^2 \langle \sigma \rangle, \rho \langle \sigma \rangle$, respectively. By (7), none of these classes is $\langle \sigma \rangle$. So none of the ideals in our list can be principal. This contradiction shows that τ, ρ cannot exist.

Now we can finish the proof. Assume that the 2-Sylow subgroup S of H' is not cyclic. Choose an element $\tau \in S$ of largest possible order such that $\sigma \in \langle \tau \rangle$. Then $\langle \tau \rangle$ is a direct factor of S. Let S' be a complementary subgroup. Since $S' \cap \langle \sigma \rangle = \{1\}$, no element of S' can have order greater than 2 (by (6)). S' must contain some element $\rho \neq 1$, since S is not cyclic. If H' contains an element $\omega \neq 1$ of odd order, then $\pi = \rho \cdot \omega$ satisfies (6), which is impossible. So H' = S is a 2-group. If $\sigma = \tau^{2^m}$, where $m \geq 2$, then τ, ρ satisfy (7), which is impossible. So $\sigma = \tau^2$ or τ . Therefore

$$H = S/\langle \sigma \rangle \cong S' \times (\langle \tau \rangle / \langle \sigma \rangle)$$
 has exponent 2.

We conclude that, if K satisfies (1) and property N, then (a) or (c) must hold.

A simple modification of the above argument shows that the irreducible integers α of a quadratic number field K are determined by the absolute values $|N\alpha|$ of their norms if and only if the class group H is of type (a) or (b) in the theorem above.

The problem considered in this paper was raised by Niven and Zuckerman in [2]. A more general form of this problem was treated by other methods in [1].

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