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A SUM OF A CERTAIN DIVISOR FUNCTION FOR ARITHMETICAL SEMI-GROUPS

E. M. HORADAM

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Let $\{b_n\}$ denote the set of elements of a free ordered arithmetical semi-group with multiplication and a known counting function. Using the corresponding terminology of arithmetic let $b_n = d\delta$ and let $\tau'(b_n)$ denote the number of divisors d of b_n where both d and δ are square free. Then it is shown here that $T(x)$ defined by

$$T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n) \sim Ax \log x$$

where A is a constant depending on b_u .

A more explicit definition of the semi-group is as follows. Suppose there is an infinite sequence $\{p\}$ of real numbers, which we will call generalised primes, such that

$$1 < p_1 < p_2 < \dots$$

Form the set $\{b\}$ of all p -products, i.e., products $p_1^{v_1} p_2^{v_2} \dots$, where v_1, v_2, \dots are integers ≥ 0 of which all but a finite number are 0. Call these numbers generalised integers and suppose that no two generalised integers are equal if their v 's are different. Then assume $\{b\}$ may be arranged as an increasing sequence:

$$1 = b_1 < b_2 < \dots < b_n < \dots$$

We say $d | b_n$ if $d \in \{b\}$ and there exists $\delta \in \{b\}$ such that $d\delta = b_n$; d and δ are then called complementary divisors of b_n . Let $\tau'(b_n)$ be the number of divisors d of b_n where both d and its complementary divisor are square free. In fact

$$(1.1) \quad \tau'(b_n) = \sum_{d\delta=b_n} 1.$$

d square free
 δ square free

This means that $\tau'(b_n) = 0$ unless b_n is of the form $\prod_{i,j} p_i p_j^2$. Let x be any positive number and b_u any generalised integer. The sum to be evaluated, $T(x)$ is defined by

$$(1.2) \quad T(x) = \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \tau'(b_n)$$

where (b_n, b_u) denotes the greatest common divisor of b_n and b_u . In

order to evaluate this sum a further assumption on the number of generalised integers less than or equal to x is required. Let $[x]$ denote the number of generalised integers $\leq x$.

Assume

$$(1.3) \quad [x] = x + R(x), R(x) = O(x^\alpha) \text{ and } 0 < \alpha < 1.$$

Using (1.3) it will be shown that when b_u is square free

$$(1.4) \quad T(x) = Ax \log x + O\left(x \exp. \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\}\right)$$

where

$$A = \prod_{p|b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4}.$$

This sum is similar to that found by Gordon and Rogers in [2]. Also using the methods of [2] exactly analogous results for arithmetical semi-groups can be found to those shown by Gordon and Rogers. The only extra difficult result required is the prime number theorem for generalised integers. This is proved in [6] and is

$$(1.5) \quad \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

2. Supplementary definitions and results. Define the Möbius function $\mu(b_n)$ for the semi-group as follows: $\mu(b_n) = 0$ if b_n has a square factor $\mu(b_n) = (-1)^k$, where k denotes the number of prime divisors of b_n and b_n has no square factor; $\mu(1) = 1$. Let $\phi(x, b_u)$ denote the number of generalised integers $\leq x$ which are prime to b_u . Then it is proved in [3] that

$$(2.1) \quad \sum_{d|b_n} \mu(d) = \begin{cases} 0 & \text{when } b_n \neq 1 \\ 1 & \text{when } b_n = 1, \end{cases}$$

and in [4] that

$$(2.2) \quad \phi(x, b_u) = \sum_{d|b_u} \mu(d) \left[\frac{x}{d} \right].$$

Hence using assumption (1.3) we have

$$(2.3) \quad \begin{aligned} \phi(x, b_u) &= x \sum_{d|b_u} \frac{\mu(d)}{d} + O\left(x^\alpha \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha}\right) \\ &= xf(b_u) + O(x^\alpha f_\alpha(b_u)) \text{ say.} \end{aligned}$$

Then as is shown in [3], and in any case as the functions are multiplicative

$$(2.4) \quad \begin{aligned} f(b_u) &= \sum_{d|b_u} \frac{\mu(d)}{d} = \prod_{p|b_u} \left(1 - \frac{1}{p}\right), \\ f_\alpha(b_u) &= \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p|b_u} \left(1 + \frac{1}{p^\alpha}\right). \end{aligned}$$

Define $\zeta(s) = \sum_{n=1}^\infty b_n^{-s}$ ($s > 1$). Then it is proved in [1], using an assumption equivalent to (1.3) that

$$\zeta(s) = \prod_{r=1}^\infty (1 - p_r^{-s})^{-1}.$$

Hence

$$\frac{1}{\zeta(s)} = \prod_{r=1}^\infty (1 - p_r^{-s}) = \sum_{n=1}^\infty \mu_n b_n^{-s}.$$

Abel's transformation, in the following form, will be used to give some necessary estimates. Suppose $\{b_n\}$ and $\{a_n\}$ are given with $b_1 \leq b_2 \leq \dots, b_n \rightarrow \infty$. Let $A(x) = \sum_{b_n \leq x} a_n$. Suppose $\psi(x)$ has a continuous derivative $\psi'(x)$ for all x involved. Then

$$\sum_{b_n \leq x} a_n \psi(b_n) = A(x)\psi(x) - \int_{b_1}^x A(u)\psi'(u)du.$$

Using (1.3) and this transformation, we obtain the following results.

$$(2.5) \quad \sum_{b_n \leq x} \frac{1}{b_n^\beta} = \frac{x^{1-\beta}}{1-\beta} + \gamma_\beta + 0(x^{\alpha-\beta}), \quad \begin{cases} \beta \neq 1 \\ \beta \neq \alpha, \end{cases}$$

and γ_β is a constant equal to $\zeta(\beta)$ when $\beta > 1$.

$$(2.6) \quad \sum_{b_n \leq x} \frac{1}{b_n^\alpha} = \frac{x^{1-\alpha}}{1-\alpha} + 0(\log x).$$

$$(2.7) \quad \sum_{b_n > x} \frac{1}{b_n^\beta} = \zeta(\beta) - \sum_{b_n \leq x} \frac{1}{b_n^\beta} = 0(x^{1-\beta}) \quad \text{for } \beta > 1.$$

Again using (1.5) and Abel's transformation we obtain

$$(2.8) \quad \sum_{p \leq x} \frac{1}{p^\alpha} = \frac{x^{1-\alpha}}{(1-\alpha)\log x} + 0\left(\frac{x^{1-\alpha}}{\log^2 x}\right)$$

$$(2.9) \quad \sum_{p \leq x} \log p = x + 0(x/\log x).$$

Define

$$(2.10) \quad \lambda(b_u) = \sum_{\substack{b_n=1 \\ (b_n, b_u)=1}}^\infty \frac{\mu(b_n)}{b_n^2} = \prod_{p|b_u} \left(1 - \frac{1}{p^2}\right).$$

Then from (2.7) we have

$$(2.11) \quad \sum_{\substack{b_n \leq x \\ (b_n, b_u)=1}} \frac{\mu(b_n)}{b_n^2} = \lambda(b_u) + O(x^{-1}) .$$

3. The Q function. Let

$$q_u(b_n) = \begin{cases} 1 & \text{if } b_n \text{ is square free and } (b_n, b_u) = 1 \\ 0 & \text{otherwise .} \end{cases}$$

$$Q_u(x) = \sum_{b_n \leq x} q_u(b_n) .$$

$$e(b_n) = \begin{cases} 1 & \text{if } b_n = 1 \\ 0 & \text{if } b_n \neq 1 . \end{cases}$$

Then from (2.1)

$$q_u(b_n) = e((b_n, b_u)) \sum_{d^2 \delta = b_n} \mu(d) .$$

This gives

$$\begin{aligned} Q_u(x) &= \sum_{b_n \leq x} e((b_n, b_u)) \sum_{d^2 \delta = b_n} \mu(d) \\ &= \sum_{\substack{d^2 \delta \leq x \\ (d, b_u) = (\delta, b_u) = 1}} \mu(d) \\ &= \sum_{\substack{d \leq \sqrt{x} \\ (d, b_u) = 1}} \mu(d) \phi\left(\frac{x}{d^2}, b_u\right) \\ &= \sum_{\substack{d \leq \sqrt{x} \\ (d, b_u) = 1}} \mu(d) \left\{ \frac{x}{d^2} f(b_u) + O\left(\frac{x^\alpha}{d^{2\alpha}} f(b_u)\right) \right\} \end{aligned}$$

from (2.3)

$$= x f(b_u) \{ \lambda(b_u) + O(x^{-1/2}) \} + O\left(x^\alpha f_\alpha(b_u) \left\{ \frac{x^{1-2\alpha/2}}{1-2\alpha} + \gamma_{2\alpha} \right\} \right)$$

from (2.11) and (2.5). Hence

$$(3.1) \quad Q_u(x) = x f(b_u) \lambda(b_u) + O(x^{1/2} f_\alpha(b_u)) + O(x^\alpha f_\alpha(b_u)) .$$

4. The evaluation of the sum of the divisor function $T(x)$.

Replacing in (1.2) the value for $\tau'(b_n)$ defined in (1.1), we have the result that $T(x)$ is the number of elements in the class satisfying $d\delta = b_n$, $\mu^2(d) = \mu^2(\delta) = 1$, where $b_n \leq x$, $(b_n, b_u) = 1$. This is the same class as that for which $d\delta \leq x$, $(d, b_u) = (\delta, b_u) = 1$ and $\mu^2(d) = \mu^2(\delta) = 1$. Rearranging the order of summation we have that $T(x)$ is the number of elements in the class satisfying $\delta \leq x/d$, $(\delta, b_u) = 1$, δ square free, where $d \leq x$, $(d, b_u) = 1$ and d is square free. Hence

$$\begin{aligned}
 T(x) &= \sum_{d \leq x} q_u(d) \sum_{\delta \leq x/d} q_u(\delta) \\
 &= \sum_{d \leq x} q_u(d) \left\{ \frac{x}{d} f(b_u) \lambda(b_u) + o\left(\frac{x^{1/2}}{d^{1/2}} f_\alpha(b_u)\right) + o\left(\frac{x^\alpha}{d^\alpha} f_\alpha(b_u)\right) \right\}
 \end{aligned}$$

from (3.1)

$$\begin{aligned}
 &= x f(b_u) \lambda(b_u) \sum_{d \leq x} \frac{q_u(d)}{d} + o\left(x^{1/2} f_\alpha(b_u) \sum_{d \leq x} \frac{1}{d^{1/2}}\right) \\
 &\quad + o\left(x^\alpha f_\alpha(b_u) \sum_{d \leq x} \frac{1}{d^\alpha}\right) \\
 &= x f(b_u) \lambda(b_u) \sum_{d \leq x} \frac{q_u(d)}{d} + o(x f_\alpha(b_u))
 \end{aligned}$$

from (2.5) and (2.6).

Now from (3.1) and Abel’s transformation we have

$$\begin{aligned}
 \sum_{d \leq x} \frac{q_u(d)}{d} &= f(b_u) \lambda(b_u) \log x + o(f_\alpha(b_u)) \\
 &\quad + o(x^{-1/2} f_\alpha(b_u)) + o(x^{\alpha-1} f_\alpha(b_u)).
 \end{aligned}$$

Substituting this result in the expression for $T(x)$ we obtain

$$(4.1) \quad T(x) = f^2(b_u) \lambda^2(b_u) x \log x + o(x f_\alpha(b_u)).$$

From the definition in (2.4) we have

$$(4.2) \quad f_\alpha(b_u) = \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} \leq \sum_{d|b_u} \frac{1}{d^\alpha} \leq \sum_{d|b_u} 1 = o(b_u^\delta)$$

where δ is any positive real number. This is proved in [5, Th. 5] and is true for all b_u . However, when b_u is square free we can obtain a better value for $f_\alpha(b_u)$ by using the prime number theorem. Suppose b_u is square free and let $b_u = p_{u1} p_{u2} \cdots p_{uk} \geq p_1 p_2 \cdots p_k$. Then

$$(4.3) \quad \log b_u \geq \sum_{p \leq p_k} \log p = p_k + o(p_k / \log p_k)$$

from (2.9). Hence

$$\begin{aligned}
 f_\alpha(b_u) &= \sum_{d|b_u} \frac{|\mu(d)|}{d^\alpha} = \prod_{p|b_u} \left(1 + \frac{1}{p^\alpha}\right) \leq \prod_{p \leq p_k} \left(1 + \frac{1}{p^\alpha}\right) \\
 &\leq \prod_{p \leq (1+o(1)) \log b_u} \left(1 + \frac{1}{p^\alpha}\right)
 \end{aligned}$$

from (4.3), and so

$$\log f_\alpha(b_u) \leq \sum_{p \leq (1+o(1)) \log b_u} \frac{1}{p^\alpha} (1 + o(1)).$$

Then from (2.8)

$$(4.4) \quad f_\alpha(b_u) = 0 \left(\exp. \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\} \right)$$

for b_u square free. Now from (2.4) and (2.10)

$$\begin{aligned} f^2(b_u)\lambda^2(b_u) &= \prod_{p|b_u} \left(1 - \frac{1}{p}\right)^2 \prod_{\substack{p \\ p|b_u}} \left(1 - \frac{1}{p^2}\right)^2 \\ &= \prod_{p|b_u} \frac{p^2}{(p+1)^2} \prod_p \frac{(p^2-1)^2}{p^4} \\ &= A \text{ (say) .} \end{aligned}$$

Hence from (4.1), (4.2) and (4.4) we have

$$(4.5) \quad T(x) = A x \log x + 0(xb_u^\delta)$$

for all b_u and all positive real numbers δ and

$$(4.6) \quad T(x) = Ax \log x + 0 \left(x \exp \left\{ \frac{(\log b_u)^{1-\alpha}}{\log \log b_u} \right\} \right)$$

for all square free b_u .

This is the result given for $T(x)$ in (1.4). Since

$$\prod_p \left(1 - \frac{1}{p^2}\right)^2 = \frac{1}{\zeta^2(2)} ,$$

the value for A may also be written

$$A = \frac{1}{\zeta^2(2)} \prod_{p|b_u} \frac{p^2}{(p+1)^2} .$$

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Wai-Mee Ching and James Sai-Wing Wong, <i>Multipliers and H^* algebras</i>	387
P. H. Doyle, III and John Gilbert Hocking, <i>A generalization of the Wilder arcs</i>	397
Irving Leonard Glicksberg, <i>A Phragmén-Lindelöf theorem for function algebras</i>	401
E. M. Horadam, <i>A sum of a certain divisor function for arithmetical semi-groups</i>	407
V. Istrăţescu, <i>On some hyponormal operators</i>	413
Harold H. Johnson, <i>The non-invariance of hyperbolicity in partial differential equations</i>	419
Daniel Paul Maki, <i>On constructing distribution functions: A bounded denumerable spectrum with n limit points</i>	431
Ronald John Nunke, <i>On the structure of Tor. II</i>	453
T. V. Panchapagesan, <i>Unitary operators in Banach spaces</i>	465
Gerald H. Ryder, <i>Boundary value problems for a class of nonlinear differential equations</i>	477
Stephen Simons, <i>The iterated limit condition and sequential convergence</i>	505
Larry Eugene Snyder, <i>Stolz angle convergence in metric spaces</i>	515
Sherman K. Stein, <i>Factoring by subsets</i>	523
Ponnaluri Suryanarayana, <i>The higher order differentiability of solutions of abstract evolution equations</i>	543
Leroy J. Warren and Henry Gilbert Bray, <i>On the square-freeness of Fermat and Mersenne numbers</i>	563
Tudor Zamfirescu, <i>On l-simplicial convexity in vector spaces</i>	565
Eduardo H. Zarantonello, <i>The closure of the numerical range contains the spectrum</i>	575