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Suppose that Σ is a field of subsets of the set S, and suppose that μ and γ are complex-valued finitely additive set functions defined on Σ. Assume that μ is bounded and γ is finite and absolutely continuous with respect to μ. (A word of warning is in order here. The statement "γ is absolutely continuous with respect to μ" is often interpreted as "$\mu(E) = 0$ implies $\gamma(E) = 0$". This is not the meaning used here. Our definition is "for every $\varepsilon > 0$ there is a $\delta > 0$ such that $|\mu(E)| < \delta$ implies $|\gamma(E)| < \varepsilon$." Unless μ is bounded and countably additive, the two definitions are not equivalent.)

THEOREM 1. **There exists a sequence $\{f_n\}$ of μ-simple functions on S, such that**

1. $$\lim_{n\to\infty} \int_E f_n(s)\mu(ds) = \gamma(E) ,$$

uniformly for $E \in \Sigma$

2. $$\lim_{n,m\to\infty} \int_S |f_n(s) - f_m(s)|\, v(\mu, ds) = 0 ,$$

where $v(\mu)$ is the total variation of μ.

Theorem 1 is established by a pure existence proof, and gives no indication of how to find f_n. A more constructive result is given below.

A partition of S is a finite collection of sets E_i belonging to Σ, such that S is the disjoint union of the E_i, and such that $\mu(E_i) \neq 0, i = 1, \cdots, n$.

The set $\mathscr{P}$ of partitions may be directed by refinement, that is, by the following partial order: $P_1 < P_2$ if for every $E \in P_1$ there exist $F_1, \cdots, F_r \in P_2$ (r may depend on E) such that E and $\bigcup_{i=1}^r F_i$ differ by a μ-null set.

If P is a partition of S, define the μ-simple function f_P^γ to be $\sum_{E\in P} (\gamma(E)/\mu(E))\chi_E$, where χ_E is the characteristic function of E.

THEOREM 2. **If μ is positive, then**

$$\lim_{P\in\mathscr{P}} \int_E f_P^\gamma(s)\mu(ds) = \gamma(E) ,$$

uniformly for $E \in \Sigma$, where $\mathscr{P}$ is directed as explained above.

(The notation throughout is essentially that of [2].) For positive μ, Theorem 1 reduces to Bochner's Radon-Nikodym Theorem. See [1].

Proofs of the theorems. Theorem 2 follows with little difficulty from Theorem 1. Therefore, for the sake of clarity, we shall first establish Theorem 2, assuming Theorem 1 to be valid.

Proof of Theorem 2. The result is proved by translating the situation into an abstract problem on the convergence of operators in a Banach space.

Throughout the proof, we assume γ bounded. This assumption is justified by Lemma 1 in the proof of Theorem 1, which shows that boundedness of γ follows from the hypothesis of Theorem 2.

Let X be the vector space of bounded μ-absolutely continuous set functions on Σ with the norm $|\gamma| = v(\gamma, S)$. Define continuous linear operators T_P on $X, P \in \mathscr{P}$, by the relation

$$(T_P(\gamma))(E) = \int_E f_P^\gamma(s)\mu(ds) .$$

Theorem 2 asserts that for each $\gamma \in X, \lim_{P \in \mathscr{P}} T_P(\gamma) = \gamma$. In order to show this, it suffices to prove

(1) that $\lim_{P \in \mathscr{P}} T_P(\gamma) = \gamma$ for all γ in a dense subset of γ;

(2) that the operators $T_P, P \in \mathscr{P}$ are equicontinuous on the unit ball of X; that is, $|T_P|, P \in \mathscr{P}$ are uniformly bounded.

To establish (1), we select as our dense subset the set of all $\gamma \in X$ of the form

$$\gamma(E) = \int_E f(s)\mu(ds) ,$$

where f is a μ-simple function on S. That this subset is dense follows from Theorem 1. We must show that if γ is of the above form, then $\lim_{P \in \mathscr{P}} T_P(\gamma) = \gamma$. But this is immediate from the following argument: Let

$$\gamma(E) = \int_E f(s)\mu(ds) ,$$

$E \in \Sigma$, where (since f is a μ-simple function), we can write

$$f = \sum_{i=1}^N \alpha_i \chi_{E_i} ,$$

with the E_i pairwise disjoint, and $S = \bigcup_{i=1}^N E_i$. Then $\{E_i\}$ is a partition of S. By trivial algebra,

$$T_P(\gamma) = \int_{(\cdot)} f_P^\gamma(s)\mu(ds) = \gamma$$

whenever P is a refinement of $\{E_i\}$, that is, $\{E_i\} > P$. This proves (1).

To establish (2), we shall prove that $|T_P| \leqq 1, P \in \mathscr{P}$, i.e., $|T_P\gamma| \leqq |\gamma|, \gamma \in X, P \in \mathscr{P}$. Any $\gamma \in X$ may be separated into its real and complex parts, and so it suffices to show that $|T_P\gamma| \leqq |\gamma|$ when γ is real. Since any real γ is the difference of its positive and negative parts, $\gamma = (\gamma_+) - (\gamma_-)$, and since $|\gamma_+| + |\gamma_-| = |\gamma|$ (by definition of the norm in X), it suffices to show that $|T_P\gamma| \leqq \gamma$ when γ is positive, and $P \in \mathscr{P}$. Now we must show that if γ is positive, and $P \in \mathscr{P}$, then $|T_P\gamma| \leqq |\gamma| = v(\gamma, S) = \gamma(S)$. By inspection of the definition of T_P, we see that since μ and γ are positive, $T_P\gamma$ is positive. So we must show that $(T_P\gamma)(S) \leqq \gamma(S)$, when γ is positive. By trivial algebra, it follows from the definition of T_P that $(T_P\gamma)(S) = \gamma(S)$. This proves Theorem 2.

We can now proceed to the most important part of the paper.

Proof of Theorem 1. The plan of attack is as follows: First we prove the theorem for the case in which μ is positive, and there is an $N > 0$ such that $|\gamma(E)| \leqq N\mu(E), E \in \Sigma$. Next, we remove the restriction on γ, and show that the theorem holds for all positive μ. Finally, we extend to the general case of a complex μ, using the approach of [2].

Let us establish a simplified notation. If γ is any bounded μ-absolutely continuous set function on Σ, and f is a μ-simple function on S, define the μ-absolutely continuous finitely additive set function $f\gamma$ by the relation

$$(f\gamma)(E) = \int_E f(s)\gamma(ds) .$$

This operation makes X, the space defined earlier, a module over the set of μ-simple functions on S.

This notation allows us to reformulate Theorem 1 in terms of the space X. Conclusion (1) states that $\lim_{n\to\infty} f_n\mu = \gamma$ in X, that is, $\lim_{n\to\infty} v(\gamma - f_n\mu, S) = 0$. Conclusion (2) asserts that the sequence $\{f_n\mu\}$ is Cauchy in the X-norm. Since convergent sequence in X are Cauchy, conclusion (2) follows from conclusion (1).

The fact that $\lim_{n\to\infty} v(\gamma - f_n\mu, S) = 0$ if and only if conclusion (1) of the theorem holds, should be noted carefully, since this is the form in which the theorem will be proved.

We now present the series of lemmas which establishes Theorem 1.

LEMMA 1. *Let μ and γ be finite real-valued finitely additive set functions on Σ. Suppose that μ is bounded and positive, and γ is μ-absolutely continuous. Then γ is bounded.*

Proof. Suppose not. Then $\mathscr{M} = \{E \in \Sigma \mid v(\gamma, E) = +\infty\}$ is non-empty. Let $M = \inf\{\mu(E) \mid E \in \mathscr{M}\}$. Pick a $\delta > 0$ so small that $\mu(E) < \delta$ implies $|\gamma(E)| \leqq 1$. Pick an $E_0 \in \mathscr{M}$ such that $\mu(E_0) < M + (\delta/2)$. Since $E_0 \in \mathscr{M}$, $v(\gamma, E) = +\infty$. On the other hand, we shall show that for every $F \in \Sigma$, $F \subseteqq E_0$, $|\gamma(F)| + |\gamma(E_0 - F)| \leqq |\gamma(E_0)| + 2$. This will be a contradiction, since, γ being real,

$$v(\gamma, E_0) = \operatorname*{Sup}_{\substack{F \in \Sigma \\ F \subseteqq E_0}} (|\gamma(F)| + |\gamma(E_0 - F)|) \leqq |\gamma(E_0)| + 2 < +\infty .$$

So to establish Lemma 1, we need only show that if $F \in \Sigma$, $F \subseteqq E_0$, then $|\gamma(F)| + |\gamma(E_0 - F)| \leqq |\gamma(E_0)| + 2$. Pick an $F \in \Sigma$, $F \subseteqq E_0$. If we had $\mu(F) \geqq \delta$, $\mu(E_0 - F) \geqq \delta$, then by finite additivity,

$$\begin{aligned} \mu(F) &= \mu(E_0) - \mu(E_0 - F) \\ &< \left(M + \frac{\delta}{2}\right) - \mu(E_0 - F) \leqq \left(M + \frac{\delta}{2}\right) - \delta < M \\ \mu(E_0 - F) &= \mu(E_0) - \mu(F) \\ &< \left(M + \frac{\delta}{2}\right) - \mu(F) \leqq \left(M + \frac{\delta}{2}\right) - \delta < M \end{aligned}$$

so that, by definition of $M, F \notin \mathscr{M}$ and $E - F \notin \mathscr{M}$. But then, $v(\gamma, F) < +\infty$ and $v(\gamma, E_0 - F) < +\infty$, so that $v(\gamma, E_0) < +\infty$, contradicting $E_0 \in \mathscr{M}$.

Therefore, if $F \in \Sigma$, $F \subseteqq E_0$, then either $\mu(F) < \delta$ or $\mu(E_0 - F) < \delta$. Suppose that $\mu(F) < \delta$. Then by definition of δ, $|\gamma(F)| \leqq 1$. On the other hand,

$$\begin{aligned} |\gamma(E_0 - F)| &= |\gamma(E_0) - \gamma(F)| \\ &\leqq |\gamma(E_0)| + |\gamma(F)| \leqq |\gamma(E_0)| + 1 . \end{aligned}$$

Therefore,

$$|\gamma(F)| + |\gamma(E_0 - F)| \leqq |\gamma(E_0)| + 2 .$$

Similar arguments show that if $\mu(E_0 - F) < \delta$, then

$$|\gamma(F)| + |\gamma(E_0 - F)| \leqq |\gamma(E_0)| + 2 .$$

So whenever $F \in \Sigma$, $F \subseteqq E_0$, $|\gamma(F)| + |\gamma(E_0 - F)| \leqq |\gamma(E_0)| + 2$.

LEMMA 2. *Let μ be a positive bounded finitely additive set function on Σ, and let γ be a real-valued finite, finitely additive set function on Σ. If there is an $N > 0$ such that $|\gamma(E)| \leqq N\mu(E)$ for all $E \in \Sigma$, then there exists a sequence $\{f_n\}$ of μ-simple functions on S, such that*

$$\gamma(E) = \lim_{n\to\infty} \int_E f_n(s)\mu(ds) ,$$

uniformly for $E \in \Sigma$.

Proof. Let $\mathscr{S}$ be the space of μ-simple functions on S, with the norm

$$| f | = \sqrt{\int_S (f(s))^2 \mu(ds)} ,$$

and let $\bar{\mathscr{S}}$ be the completion of $\mathscr{S}$. $\bar{\mathscr{S}}$ is obviously a Hilbert space. For example, if μ is countably additive, then $\bar{\mathscr{S}}$ is just the space L_2.

Define a linear functional $\mathscr{O}$ on $\mathscr{S}$ as follows:

$$\mathscr{O}f = \int_S f(s)\gamma(ds) .$$

To check that $\mathscr{O}$ is continuous, observe that

$$| \mathscr{O}f | \leqq \int_S | f(s) | v(\gamma, ds) \leqq N \int_S | f(s) | \mu(ds) \leqq N | f | \cdot | 1 |$$

where 1 is the μ-simple function which maps all of S to 1. (The last inequality is just Hölder's inequality with $g = 1$.)

Since $\mathscr{O}$ is continuous on $\mathscr{S}$, it has a continuous extension $\bar{\mathscr{O}}$ to all of $\bar{\mathscr{S}}$. Since $\bar{\mathscr{O}}$ is a continuous linear functional on a Hilbert space, there is some $x \in \bar{\mathscr{S}}$ such that for all $y \in \bar{\mathscr{S}}$, $\bar{\mathscr{O}}y = x \cdot y$. Since $\mathscr{S}$ is dense in $\bar{\mathscr{S}}$, there exists a sequence $\{f_n\}$ of elements of $\mathscr{S}$ (i.e., a sequence of μ-simple functions on S) such that $\lim_{n\to\infty} f_n = x$ in $\bar{\mathscr{S}}$. Therefore, for each positive M, $\lim_{n\to\infty} f_n \cdot y = \mathscr{O}f$ uniformly for y in the M-ball of $\bar{\mathscr{S}}$.

For $E \in \Sigma$, $| \chi_E |$ in $\mathscr{S}$ is just

$$\begin{aligned} | \chi_E | &= \sqrt{\int_S (\chi_E(s))^2 \mu(ds)} \\ &= \sqrt{\int_E 1 \mu(ds)} = \sqrt{\mu(E)} \leqq \sqrt{\mu(S)} . \end{aligned}$$

Therefore, $\{\chi_E \mid E \in \Sigma\}$ consists entirely of elements of the M-ball of $\bar{\mathscr{S}}$ where $M = \sqrt{\mu(S)}$. By the result of the last paragraph,

$$\lim_{n\to\infty} f_n \cdot \chi_E = \mathscr{O}\chi_E ,$$

uniformly for $E \in \Sigma$. Recalling the definitions of $\mathscr{O}$ and of the dot product in $\mathscr{S}$, this equation becomes

$$\lim_{n\to\infty} \int_S f_n(s)\chi_E(s)\mu(ds) = \int_S \chi_E(s)\gamma(ds) ,$$

uniformly for $E \in \Sigma$, i.e.

$$\lim_{n\to\infty} \int_E f_n(s)\,\mu(ds) = \gamma(E)$$

uniformly for $E \in \Sigma$.

The next step in the proof of Theorem 1 is to extend Lemma 2 to the case in which we no longer have $|\gamma(E)| \leqq N\mu(E)$. It becomes easier to see how this extension may be accomplished, if we rephrase the conclusion that we are trying to prove. We are trying to show that if μ is as in Lemma 2, then $\{f\mu \mid f \text{ a } \mu\text{-simple function on } S\}$ is dense in X. By Lemma 2, $\{f\mu \mid f \text{ a } \mu\text{-simple function on } S\}$ is dense in $X_0 = \{\gamma \in X \mid \exists N > 0 \ni \forall E \in \Sigma, |\gamma(E)| \leqq N\mu(E)\}$. So we want to show that X_0 is dense in X, i.e., for every μ-absolutely continuous finite γ, there is a sequence γ_n of set functions of X, such that $|\gamma_n(E)| \leqq n\mu(E)$, $E \in \Sigma$, and $\lim_{n\to\infty} v(\gamma - \gamma_n, S) = 0$. We find $\{\gamma_n\}$ as follows:

DEFINITION. Let γ_1 and γ_2 be finite positive finitely additive set functions on Σ. Define the set function $\gamma_1 \wedge \gamma_2$ on Σ by the relation

$$(\gamma_1 \wedge \gamma_2)(E) = \inf_{\substack{F \in \Sigma \\ F \subseteq E}} [\gamma_1(F) + \gamma_2(E - F)] .$$

Observations.

1. $\gamma_1 \wedge \gamma_2$ is finitely additive. This is easy, and the proof is contained in the proof of Theorem III 7.3 in [2]. Therefore the proof will not be included here.
2. $\gamma_1 \wedge \gamma_2 \leqq \gamma_1$ and $\gamma_1 \wedge \gamma_2 \leqq \gamma_2$.
3. If $\gamma_2 \leqq \gamma_3$ then $\gamma_1 \wedge \gamma_2 \leqq \gamma_1 \wedge \gamma_3$.

LEMMA 3. *Let μ be a positive bounded finitely additive set function on Σ, and let γ be a positive bounded finitely additive μ-absolutely continuous set function on Σ. Let $\gamma_n = (n\mu) \wedge \gamma$. Then*

$$\lim_{n\to\infty} v(\gamma - \gamma_n, S) = 0 .$$

Proof. $0 \leqq \gamma_n \leqq \gamma_{n+1} \leqq \gamma$, so that if $\lim_{n\to\infty} \gamma_n(S) = \gamma(S)$, the lemma is proved. Let $\varepsilon > 0$ be given. By hypothesis, there is a $\delta > 0$ such that if $\mu(E) < \delta$, $E \in \Sigma$, then $\gamma(E) \leqq \varepsilon$. Let $N \geqq \gamma(S)/\delta$. Consider any $F \in \Sigma$.

Case 1. $\mu(F) > \delta$. Then $N\mu(F) \geqq \gamma(S)$, so that

$$N\mu(F) + \gamma(S - F) \geqq \gamma(S) .$$

Case 2. $\mu(F) < \delta$. Then $\gamma(F) \leqq \varepsilon$, so that $\gamma(S - F) \geqq \gamma(S) - \varepsilon$, and so

$$N\mu(F) + \gamma(S - F) \geqq \gamma(S) - \varepsilon .$$

In either case, $N\mu(F) + \gamma(S - F) \geqq \gamma(S) - \varepsilon$. So

$$\gamma_N(S) = \inf_{F \in \Sigma} (N\mu(F) + \gamma(S - F)) \geqq \gamma(S) - \varepsilon .$$

Therefore, $\lim_{n\to\infty} \gamma_n(S) = \gamma(S)$.

LEMMA 4. *Let μ be a positive bounded finitely additive set function on Σ, and let γ be a complex-valued finite μ-absolutely continuous finitely additive set function on Σ. Then there exists a sequence $\{f_n\}$ of μ-simple functions on S, such that*

$$\lim_{n\to\infty} v(\gamma - f_n\mu, S) = 0 .$$

Proof. γ can be separated into its real and imaginary parts, so that it suffices to consider γ real. Any real γ may be separated into its positive and negative parts, $\gamma = \gamma_+ - \gamma_-$. Lemma 1 assures us that γ_+ and γ_- are bounded. So it suffices to consider γ positive. For $n > 0$, pick N_n so large that

$$v(\gamma - (N_n\mu \wedge \gamma), S) < \frac{1}{2n} .$$

This is possible by Lemma 3. By Lemma 2, and by the fact that $|(N_n\mu \wedge \gamma)(E)| \leqq N_n\mu(E), E \in \Sigma$, there is an f_n, a μ-simple function on S, such that

$$v((N_n\mu \wedge \gamma) - f_n\mu, S) < \frac{1}{2n} .$$

Then

$$\begin{aligned} v(\gamma - f_n\mu, S) &\leqq v((\gamma - N_n\mu \wedge \gamma), S) + v((N_n\mu \wedge \gamma) - f_n\mu, S) \\ &< \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} , \end{aligned}$$

so that $\lim_{n\to\infty} v(\gamma - f_n\mu, S) = 0$.

We have now completed the second step in the proof of Theorem 1, namely, we have proved the theorem for the case $\mu \geqq 0$. The final step is to establish the theorem in the complex case. To accomplish this, we carry over the procedure of [2], which is valid for the countably additive case. In [2], the argument (to prove the usual Radon-Nikodym Theorem in the complex case, given its validity in the case $\mu \geqq 0$) is roughly the following: Let μ and γ be bounded measures on the sigma-field Σ, and suppose that γ is μ-absolutely continuous. Then γ and μ are both $v(\mu, \cdot)$-absolutely continuous. So there are integrable functions g and h on S such that

$$\mu(E) = \int_E h(s) v(\mu, ds)$$

and

$$\gamma(E) = \int_E g(s) v(\mu, ds) ,$$

$E \in \Sigma$. It is then proved that

$$v(\mu, E) = \int_E \frac{1}{h(s)} \mu(ds) ,$$

$E \in \Sigma$, and from this it is shown that

$$\gamma(E) = \int_E \frac{g(s)}{h(s)} \mu(ds) .$$

In order to apply this type of argument to the present situation, we need two estimates:

LEMMA 5. *Let μ be any bounded complex-valued finitely additive set function on Σ, and let u denote the set function $v(\mu, \cdot)$. Then there exists a sequence $\{f_n\}$ of μ-simple functions on S, such that $|f_n(s)| \geqq 1/2$ for $n > 0$ and $s \in S$, and such that*

$$\lim_{n \to \infty} v(\mu - f_n u, S) = 0 .$$

Proof. By Lemma 2, there is a sequence $\{g_n\}$ of μ-simple functions on S, such that $\lim_{n \to \infty} v(\mu - g_n u, S) = 0$. Then

$$\lim_{n \to \infty} | v(\mu, E) - v(g_n u, E) | = 0$$

uniformly in $E \in \Sigma$. Now

$$v(g_n u, E) = \int_E | g_n(s) | v(\mu, ds) ,$$

so that

$$
\begin{aligned}
|v(\mu, E) - v(g_n u, E)| &= \left| v(\mu, E) - \int_E |g_n(s)| \, v(\mu, ds) \right| \\
&= \left| \int_E (1 - |g_n(s)|) v(\mu, ds) \right| .
\end{aligned}
$$

Therefore,

$$\lim_{n\to\infty} \int_E (1 - |g_n(s)|) v(\mu, ds) = 0$$

uniformly for $E \in \Sigma$. Hence,

$$\lim_{n\to\infty} \int_S |1 - |g_n(s)|| \, v(\mu, ds) = 0 .$$

If $|g_n(s)| < 1/2$, then $|1 - |g_n(s)|| > 1/2$. Therefore, setting

$$S_n = \left\{ s \in S \mid |g_n(s)| < \frac{1}{2} \right\} ,$$

$$
\begin{aligned}
\frac{1}{2} v\left(\mu, \left\{ s \in S \mid |g_n(s)| < \frac{1}{2} \right\}\right) &\leqq \int_{S_n} |1 - |g_n(s)|| \, v(\mu, ds) \\
&\leqq \int_S |1 - |g_n(s)|| \, v(\mu, ds) .
\end{aligned}
$$

So $\lim_{n\to\infty} v(\mu, S_n) = 0$. Let χ_n be the characteristic function of S_n. Then

$$\lim_{n\to\infty} \int_S |\chi_n(s)| \, v(\mu, ds) = 0 .$$

Define $f_n = g_n + \chi_n$. By checking the two cases $|g_n(s)| \geqq 1/2$ and $|g_n(s)| < 1/2$, we see that $|f_n(s)| \geqq 1/2, n > 0, s \in S$. Now

$$
\begin{aligned}
v(f_n u - g_n u, S) &= v((f_n - g_n)u, S) \\
&= v(\chi_n u, S) = \int_S |\chi_n(s)| \, v(\mu, ds) .
\end{aligned}
$$

Since $\lim_{n\to\infty} v(\mu - g_n u, S) = 0$, we have $\lim_{n\to\infty} v(\mu - f_n u, S) = 0$.

LEMMA 6. *Let μ and u be as in the previous lemma. Then there exists a sequence $\{g_n\}$ of μ-simple functions on S, such that*

$$\lim_{n\to\infty} v(u - g_n \mu, S) = 0 .$$

Proof. Let $\{f_n\}$ be a sequence of μ-simple functions on S, as in the conclusion of the previous lemma, and define $g_n(s) = 1/f_n(s) \cdot g$ is well-defined, since $f_n(s) \neq 0$.

Now $g_n \mu = g_n f_n u + g_n[\mu - f_n u] = u + g_n[\mu - f_n u]$. Since

$$|f_n(s)| \geqq \frac{1}{2}, \qquad |g_n(s)| = \frac{1}{|f_n(s)|} \leqq 2,$$

so that $v(g_n[\mu - f_n u], S) \leqq 2v(\mu - f_n u, S)$. But

$$\lim_{n\to\infty} v(\mu - f_n u, S) = 0 .$$

So

$$\lim_{n\to\infty} v(g_n\mu - u, S) = \lim_{n\to\infty} v(g_n[\mu - f_n u], S) = 0 .$$

Proof of Theorem 1. By Lemma 4 there is a sequence $\{h_n\}$ of μ-simple functions on S, such that $\lim_{n\to\infty} v(\gamma - h_n u, S) = 0$, where u, as usual, stands for $v(\mu, \cdot)$. By Lemma 6, there is a sequence $\{g_n\}$ of μ-simple functions on S such that $\lim_{n\to\infty} v(u - g_n\mu, S) = 0$. Now

$$\gamma = h_m u + (\gamma - h_m u) = h_m g_N \mu + h_m(u - g_N\mu) + (\gamma - h_m u) .$$

Given $n > 0$, pick m_n so large that $v(\gamma - h_{m_n}u, S) < 1/2n$. Since h_{m_n} is μ-simple, we can pick $M_n > 0$ such that $|h_{m_n}(s)| \leqq M_n$, $s \in S$. Pick N_n so large that $v(u - g_{N_n}\mu, S) < 1/2nM_n$. Then it follows that

$$\begin{aligned} v(\gamma - h_{m_n}g_{N_n}\mu, S) &\leqq v(h_{m_n}(u - g_{N_n}\mu), S) + v(\gamma - h_{m_n}u, S) \\ &\leqq M_n v(u - g_{N_n}\mu, S) + v(\gamma - h_{m_n}u, S) \\ &< M_n \cdot \frac{1}{2nM_n} + \frac{1}{2n} = \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n} . \end{aligned}$$

So, if we define $f_n = h_{m_n}g_{N_n}$, then we have $v(\gamma - f_n\mu, S) < 1/n$. Therefore,

$$\lim_{n\to\infty} v(\gamma - f_n\mu, S) = 0 .$$

Some examples. This section exhibits counterexamples to show that the principal hypothesis, boundedness of μ, is needed to prove Theorem 1.

EXAMPLE 1. Let S be the set of natural numbers, and let Σ be the set of all subsets of S. Define a countably additive set function μ by $\mu(\{n\}) = 1$, and let γ be any finitely additive set function on Σ that is not countably additive. If $v(\mu, E) < 1/2$, then E must be empty, so that $\gamma(E) = 0$. Therefore γ is μ-absolutely continuous. But if γ were representable as in Theorem 1, then we could show that γ is countably additive.

EXAMPLE 2. Let S and Σ be as in Example 1. Define finite set functions μ and γ finitely additive on Σ by setting $\mu(\{n\}) = 1$, $\gamma(\{n\}) = n$, and extending by a Zorn's lemma argument. μ and γ are finite, but not bounded. By the reasoning of Example 1, γ is μ-absolutely continuous. If γ were representable in the form of Theorem 1, then we could write $\gamma = \gamma_1 + \gamma_2$ where $v(\gamma_1, S) < 1$ and $|\gamma_2(E)| \leqq M|\mu(E)|$, $E \in \Sigma$. In fact, the above decomposition of γ is impossible.

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