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In a category with a zero object, products and coproducts and in which the map

$$A + B \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A \times B$$

is an epimorphism, we define abelian objects. We show that the product of abelian objects is also a coproduct for the subcategory consisting of all the abelian objects. Moreover, we prove that abelian objects constitute abelian subcategories of certain not-necessarily abelian categories, thus obtaining a generalization of the subcategory of the category of groups consisting of all abelian groups.

2. Definition and properties of Abelian objects.

The direct product is not a coproduct in the category of groups as it is in the category of abelian groups. What is lacking is a canonical map from the product, i.e., the sum map of abelian groups; in particular, we need a map $A \times A \rightarrow A$ which when composed with $(1, 0)$ or $(0, 1)$ is the identity on A. For abelian groups this is the map $(1_A + 1_A)$ (where $(a, b)(f + g) = af + bg$). On the other hand if such a map x exists, then for $a, b \in A$, since $(0, a) + (b, 0) = ((0 + b), (a + 0))$, $a + b = ((0, a) + (b, 0))x = ((0 + b), (a + 0))x = b + a$ since $(1,0)x = (0,1)x = 1_A$, i.e., A is abelian.

This suggests that if we consider only objects where there is always a unique morphism from the product of the object with itself to either component which composes with either $(1, 0)$ or $(0, 1)$ to give the identity, we should get a generalization of abelian groups, provided the original category has certain properties which the category of groups has. Isbell [3] has also considered the existence of this map.

Let $\mathscr{C}$ be a category with a zero object, products and coproducts and in which the map

$$A_1 + A_2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A_1 \times A_2$$

is an epimorphism for each $A_1, A_2 \in \mathscr{C}$. We assume that all categories considered are sufficiently small that the (representatives of) subobjects (and quotient objects) of a given object form a set.

DEFINITION. Let $\mathscr{A}$ be the full subcategory of $\mathscr{C}$ determined by those $A \in \mathscr{C}$ which have a morphism j from $A \times A \rightarrow A$ such that $(1, 0)j = (0, 1)j = 1_A$. We call the objects of $\mathscr{A}$ *abelian objects.*

THEOREM 1. *The product of abelian objects is abelian.*

Proof. Suppose $A_1 \times A_2$ is the product of abelian objects A_i with projection maps p_i, $i = 1, 2$. We form the following products:

$$(A_1 \times A_2)_k \longrightarrow (A_1 \times A_2) \times (A_1 \times A_2) \xrightarrow{p'_i} (A_1 \times A_2)_i$$

$$(A_i)_k \longrightarrow A_i \times A_i \xrightarrow{p^j_i} (A_i)^j$$

$$A_k \times A_k \longrightarrow (A_1 \times A_1) \times (A_2 \times A_2) \xrightarrow{p''_i} A_i \times A_i$$

$i = 1, 2$, $j = 1, 2$, $k = 1, 2$, and we use the symbol $A_k \rightarrow A_1 \times A_2$ to mean the map $(1_{A_1}, 0)$ for $k = 1$, $(0, 1_{A_2})$ for $k = 2$. Then we have

$$z_i = (p'_1 p_i, p'_2 p_i) \colon (A_1 \times A_2) \times (A_1 \times A_2) \longrightarrow A_i \times A_i$$

so that

$$(A_1 \times A_2)_k \longrightarrow (A_1 \times A_2) \times (A_1 \times A_2) \xrightarrow{z_i} A_i \times A_i \xrightarrow{p^j_i} (A_i)^j$$

$$= (A_1 \times A_2)_k \longrightarrow (A_1 \times A_2) \times (A_1 \times A_2) \xrightarrow{p'_j} (A_1 \times A_2)_j \xrightarrow{p_i} A_i$$

(by definition of z_i) and this is equal to

$$(A_1 \times A_2)_k \xrightarrow{p_i} (A_i)^k \longrightarrow A_i \times A_i \xrightarrow{p^j_i} (A_i)^j$$

since both are projections or zero depending upon whether or not $j = k$. Moreover, the p^j_i are right cancellable since the results hold for both $j = 1$, $j = 2$, and $A_i \times A_i$ is a product. Since the A_i are abelian, there is a morphism $x_i \colon A_i \times A_i \rightarrow A_i$ such that $(1_{A_i}, 0)x_i = (0, 1_{A_i})x_i = 1_{A_i}$. So we define $y = (p''_1 x_1, p''_2 x_2)$, $z = (z_1, z_2)$. Then we have

$$\begin{tikzcd}
 & A_2 \times A_2 \arrow[r, "x_2"] & A_2 \\
(A_1 \times A_2) \times (A_1 \times A_2) \arrow[ur, "z_2"] \arrow[r, dash, "z"] \arrow[dr, "z_1"'] & (A_1 \times A_1) \times (A_2 \times A_2) \arrow[u, "p''_2"'] \arrow[d, "p''_1"] \arrow[r, "y"] & A_1 \times A_2 \arrow[u, "p_2"'] \arrow[d, "p_1"] \\
 & A_1 \times A_1 \arrow[r, "x_1"] & A_1
\end{tikzcd}$$

commutative from the definitions of z_i, y and z. But by the above

$$\begin{aligned}
&(A_1 \times A_2)_k \longrightarrow (A_1 \times A_2) \times (A_1 \times A_2) \xrightarrow{z} (A_1 \times A_1) \times (A_2 \times A_2) \\
&\qquad \xrightarrow{y} (A_1 \times A_2) \xrightarrow{p_i} A_i \\
&= (A_1 \times A_2)_k \longrightarrow (A_1 \times A_2) \times (A_1 \times A_2) \xrightarrow{z_i} A_i \times A_i \xrightarrow{x_i} A_i \\
&= (A_1 \times A_2)_k \longrightarrow (A_i)^k \longrightarrow A_i \times A_i \xrightarrow{x_i} A_i = A_1 \times A_2 \xrightarrow{p_i} A_i \\
&= A_1 \times A_2 \xrightarrow{1} A_1 \times A_2 \xrightarrow{p_i} A_i \,,
\end{aligned}$$

$i = 1, 2,\ k = 1, 2$. Now the p_i are right cancellable since the equations hold for $i = 1, 2$. Hence $(1_{A_1 \times A_2}, 0)zy = 1_{A_1 \times A_2}$ and $(0, 1_{A_1 \times A_2})zy = 1_{A_1 \times A_2}$, i.e., zy is the desired map.

PROPOSITION. X is abelian if and only if every morphism $\begin{pmatrix} f \\ g \end{pmatrix}: A_1 + A_2 \rightarrow X$ can be factored through $A_1 \times A_2$. (A_1, A_2 not necessarily abelian)

Proof. If X is abelian we have $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(f, g)x$, where $X \times X \xrightarrow{x} X$ is the abelianess map. If X has the given property, it is abelian by virtue of factorization of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

THEOREM 2. *The product of abelian objects in $\mathscr{C}$ is also their coproduct in the subcategory of abelian objects.*

Proof. If A_1 and A_2 are abelian, so is their product and since $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an epimorphism the factorization of the proposition above is unique.

3. Abelian subcategories. We now define a type of category in which it will be shown that the abelian objects form an abelian subcategory.

DEFINITION. The *image* of a map $A \rightarrow B$ is the smallest subobject of B such that $A \rightarrow B$ factors through the representative monomorphisms.

We define *coimage* dually.

DEFINITION. Let $\mathscr{S}$ be a category with a zero object, products and coproducts, satisfying the following conditions:

(1) If $K \rightarrow A$ is a kernel and $A \rightarrow B$ is an epimorphism, then

image $(K \to B)$ is a kernel.

(2) Any morphism of $\mathscr{S}$ may be factored into (representatives of) its coimage followed by its image.

(3) Every epimorphism is a cokernel.

Then $\mathscr{S}$ is called a *nearly abelian* category.

Clearly the category of groups and group homomorphisms satisfies these conditions.

THEOREM 3. *Let $\mathscr{S}$ be a nearly abelian category. The subcategory $\mathscr{A}$ of abelian objects of $\mathscr{S}$ is an abelian category.*

Proof. A zero object is clearly abelian.

Products and coproducts are abelian by Theorems 1 and 2 and the following lemma:

LEMMA 0. *In a category $\mathscr{C}$ with zero object, products, coproducts, and satisfying conditions* (2) *and* (3).

$$A_1 + A_2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A_1 \times A_2$$

is an epimorphism, for each $A_1, A_2 \in \mathscr{C}$.

We first prove

LEMMA 1. *If $f\colon A \to B$ and $g\colon B \to C$ are such that g and fg have images, then the image of fg is contained in the image of g.*

Proof. Let $I \to C$ be the image of g. Then $A \to B \to I \to C = A \to B \to C$ so that $I \to C$ contains the image of fg.

LEMMA 2. *In a category $\mathscr{C}$ with coproducts and images the subobjects of a given object form a complete lattice.*

Proof. Let $\{s_j\colon A_j \to A \mid j \in J\}$ represent an arbitrary set of subobjects of $A \in \mathscr{C}$. Let $\{u_j\colon A_j \to \Sigma A_j \mid j \in J\}$ be the coproduct of the A_j. Let u be the unique morphism $\Sigma A_j \to A$ whose composition with u_j is s_j for each j. Let $I \to A$ be the image of u. Then we have

$$\begin{tikzcd} & & I \arrow[dr] & \\ A_j \arrow[r, "u_j"] & \Sigma A_j \arrow[ur] \arrow[rr, "u"] & & A \end{tikzcd}$$

so that

$$\begin{tikzcd} A_j \arrow[r, "s_j"] \arrow[d] & A \\ I \arrow[ur] & \end{tikzcd}$$
commutes.

$A_j \to I$ is a monomorphism since s_j is. Hence $I \to A$ is an upper bound.

Suppose $s'\colon A' \to A$ is an upper bound for the s_j. Let s'_j be such that

$$\begin{tikzcd} A_j \arrow[r, "s_j"] \arrow[d, "s'_j"'] & A \\ A' \arrow[ur] & \end{tikzcd}$$
commutes.

Let v be the unique morphism $\Sigma A_j \to A'$ whose composition with u_j is s'_j for each j. Then we have $u_j v s' = u_j u$; therefore $vs' = u$ by definition of coproduct. Hence the image of u = the image of vs' is contained in s' by the preceding lemma. Thus the image of u is the l.u.b.

Let $\{s'_k\colon A'_k \to A \mid k \in K\}$ be the set of monomorphisms $s'\colon A' \to A$ with s' contained in s_j for all $j \in J$. Then there exists s'', the l.u.b. of $\{s'_k \mid k \in K\}$ (as constructed above), and s'' is the g.l.b. of $\{s_j \mid j \in J)$.

Proof of Lemma 0. We have

$$A_1 \xrightarrow{u_1} A_1 + A_2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A_1 \times A_2 \xrightarrow{p_1} A_1 = A_1 \xrightarrow{(1,0)} A_1 \times A_2 \xrightarrow{p_1} A_1$$

and similarly for p_2. Then $u_1\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = (1, 0)$ since the equations hold for both projections. Similarly $u_2\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = (0, 1)$. By the construction of Lemma 2, the l.u.b. of (1, 0) and (0, 1) is image $(A_1 + A_2 \to A_1 \times A_2)$. Hence by definition of product, domain image $(A_1 + A_2 \to A_1 \times A_2)$ is (isomorphic to) $A_1 \times A_2$. Thus

$$\begin{aligned} & A_1 + A_2 \to A_1 \times A_2 \\ &\qquad = \text{coimage}\,(A_1 + A_2 \longrightarrow A_1 \times A_2)(A_1 \times A_2 \longrightarrow A_1 \times A_2) \\ &\qquad = \left(A_1 + A_2 \xrightarrow{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} A_1 \times A_2\right)(A_1 \times A_2 \longrightarrow A_1 \times A_2) \end{aligned}$$

and since $A_1 \times A_2 \to A_1 \times A_2$ is right cancellable,

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \text{coimage}\,(A_1 + A_2 \longrightarrow A_1 \times A_2)$$

and hence it is an epimorphism.

It now remains to show only that every morphism has both a kernel and a cokernel and that every monomorphism is a kernel and that every epimorphism is a cokernel.

LEMMA 2*. *In a category with products and coimages the quotient objects of a given object form a complete lattice.*

Proof. The proof is dual to that of Lemma 2.

LEMMA 3. *If every morphism of a category $\mathscr{C}$ with a zero object and coproducts (products) can be factored into an epimorphism followed by a monomorphism, then every morphism has a kernel (cokernel).*

Proof. We prove the coproducts and kernels case; the other proceeds dually. Let $A \to B$ be a morphism of $\mathscr{C}$. Consider the coproduct ΣA_j of all subobjects of A such that $A_j \to A \to B = 0$. Then $\Sigma A_j \to A \to B = 0$ by definition of coproduct so let $\Sigma A_j \to A = \Sigma A_j \to I \to A$, $\Sigma A_j \to I$ an epimorphism, $I \to A$ a monomorphism, i.e., we have

$$\begin{tikzcd} & & I \arrow[dr] & & \\ A_j \arrow[r] & \Sigma A_j \arrow[ur] \arrow[rr] & & A \arrow[r] & B = 0 \text{ commutative.} \end{tikzcd}$$

Then $\Sigma A_j \to I \to A \to B = 0$ and since $\Sigma A_j \to I$ is an epimorphism, $I \to A \to B = 0$. Moreover, $I \to A$ is an upper bound for the A_j, for there is a map $A_j \to I = A_j \to \Sigma A_j \to I$ such that

$$\begin{tikzcd} I \arrow[r] & A \\ A_j \arrow[u] \arrow[ur] & \end{tikzcd} \quad \text{commutative.}$$

for each A_j. Hence $I \to A$ is the desired kernel.

LEMMA 4. *In a category $\mathscr{C}$ with kernels and cokernels in which every epimorphism is a cokernel, if $A \to B$ factors through an epimorphism $A \to C$ and a monomorphism $C \to B$, this factorization is unique up to equivalence.*

Proof. Suppose $A \to C' \to B$ and $A \to C \to B$ are two factorizations of $A \to B$ into an epimorphism followed by a monomorphism. Let $K \to A$ be the kernel of $A \to C$; then $A \to C$ is the cokernel of $K \to A$ and similarly for $K' \to A$ and $A \to C'$. Then $K \to A \to C' \to B = 0$

and $K \to A \to C' = 0$ since $C' \to B$ is right cancellable. Hence $K \to A$ is contained in $K' \to A$ and hence $A \to C$ contains $A \to C'$. Similarly $A \to C'$ contains $A \to C$. Now we have

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d, shift left] & B \\ & C' \arrow[u, shift left] \arrow[ur] & \end{tikzcd}$$

where both triangles commute.

Since $A \to C'$ is an epimorphism, $C' \to C \to B = C' \to B$ and similarly $C \to C' \to B = C \to B$. Hence $C' \to B$ and $C \to B$ are also equivalent.

LEMMA 5. *In a category as in Lemma* 0 *if* $f\colon A \to B$ *is an epimorphism and* $g\colon B \to C$, *then image of* $fg =$ *image of* g.

Proof. Let $I \to C$ be the image of $B \to C$. Then $A \to I$ is the composition of epimorphisms

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[r, "g"] \arrow[d] & C \\ & I \arrow[ur] & \end{tikzcd}$$

and hence an epimorphism. Thus by Lemma 4 it is the coimage of $A \to C$ and $I \to C$ is the image of $A \to C$.

LEMMA 6. *In a category such as in Lemma* 0, *if* $m_1\colon A_1 \to A$, $m_2\colon A_2 \to A$ *are monomorphisms and* $f\colon A \to C$, *then*

$$\text{image}\,((\text{l.u.b. }\{m_1, m_2\})f) = \text{image}\,(\text{l.u.b. }\{\text{image } m_1 f, \text{image } m_2 f\})\,.$$

Proof. Let $u_i\colon A_i \to A_1 + A_2$, $u_i'\colon A_i' \to A_1' + A_2'$, where $A_i' \to C$ is the image of $m_i f$. Then we have

$$\begin{aligned} & A_i \xrightarrow{u_i} A_1 + A_2 \xrightarrow{\begin{pmatrix}\text{coimage }(m_1 f)u_1' \\ \text{coimage }(m_2 f)u_2'\end{pmatrix}} A_1' + A_2' \xrightarrow{\begin{pmatrix}\text{image }(m_1 f) \\ \text{image }(m_2 f)\end{pmatrix}} C \\ &= A_i \xrightarrow{\text{coimage }(m_i f)} A_i' \longrightarrow A_1' + A_2' \xrightarrow{\begin{pmatrix}\text{image }(m_1 f) \\ \text{image }(m_2 f)\end{pmatrix}} C \\ &= A_i \xrightarrow{\text{coimage }(m_i f)} A_i' \xrightarrow{(\text{image } m_i f)} C \\ &= A_i \xrightarrow{u_i} A_1 + A_2 \xrightarrow{\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}} A \xrightarrow{f} C\,. \end{aligned}$$

Since these equations hold for u_1 and u_2, $\begin{pmatrix}\text{coimage}\,(m_1 f)u_1' \\ \text{coimage}\,(m_2 f)u_2'\end{pmatrix}\begin{pmatrix}\text{image}\,(m_1 f) \\ \text{image}\,(m_2 f)\end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} f$. Then image$\left(A_i \xrightarrow{u_i} A_1 + A_2 \to A_1' + A_2'\right)$ is contained in the image of $A_1 + A_2 \to A_1' + A_2'$. But by the factorization above and the fact that $A_1 + A_2$ is a coproduct, the image of $A_i \to A_1 + A_2 \to A_1' + A_2'$ is u_i'. Thus since the l.u.b. of the u_i's is $A_1' + A_2' \to A_1' + A_2'$, this identity is the image of $A_1 + A_2 \to A_1' + A_2'$ and $A_1 + A_2 \to A_1' + A_2'$ is its own coimage and hence an epimorpism. Then the image of $\begin{pmatrix} \text{coimage } (m_1 f)u_1' \\ \text{coimage } (m_2 f)u_2' \end{pmatrix} \begin{pmatrix} \text{image } (m_1 f) \\ \text{image } (m_2 f) \end{pmatrix}$ is the image of the second map by Lemma 5.

Also we have

$$\text{image}\left[\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} f\right] = \text{image}\left[\left(\text{image}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right) f\right]$$

since the coimage of $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ is an epimorphism. We have

$$\begin{tikzcd}
 & & X & \\
A_1 + A_2 \arrow[urr, "\text{coimage}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} f"] \arrow[rr] \arrow[dr, "\text{coimage}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}"'] & & A \arrow[r, "f"] & C \arrow[ul, "\text{image}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} f"'] \\
 & I \arrow[ur, "\text{image}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = x"] \arrow[rr] & & Y \arrow[u, "\text{image } xf"']
\end{tikzcd}$$

where everything commutes.

Then

$$\begin{aligned} \text{image}\left[\left(\text{image}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right) f\right] &= \text{image } ((\text{l.u.b. } \{m_1, m_2\}) f) \\ &= \text{image } (\text{l.u.b. } \{\text{image } (m_1 f), \text{image } (m_2 f)\}) \end{aligned}$$

since we get from the above that

$$\begin{aligned} \text{image}\left[\begin{pmatrix} \text{coimage } (m_1 f)u_1' \\ \text{coimage } (m_2 f)u_2' \end{pmatrix} \begin{pmatrix} \text{image } (m_1 f) \\ \text{image } (m_2 f) \end{pmatrix}\right] &= \text{image}\left[\begin{pmatrix} \text{image } (m_1 f) \\ \text{image } (m_2 f) \end{pmatrix}\right] \\ = \text{image}\left[\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} f\right] &= \text{image}\left[\left(\text{image}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right)\right] f, \end{aligned}$$

which proves the lemma.

We now show that any subobject of an abelian object is an abelian object. If in particular the subobject is the kernel in $\mathscr{S}$ of a morphism of $\mathscr{A}$, then it is in $\mathscr{A}$ and clearly is the kernel in $\mathscr{A}$. Suppose $k: K \to A$ is a subobject of an abelian object A. Let $K \times K$ be the product of K with itself, p_i its projection morphisms, p_i' the projection morphisms for $A \times A$. Let x be the morphism $A \times A \to A$ such that $A_i \to A \times A \xrightarrow{x} A = 1_A$, $i = 1, 2$. Let $y = (p_1 k, p_2 k)$ so that $K_i \to K \times K \xrightarrow{y} A \times A \xrightarrow{x} A = k$ as in Theorem 2. $K \times K \to K \times K$ is the l.u.b. of $K_1 \to K \times K$ and $K_2 \to K \times K$ so

$$\text{image}\,((\text{l.u.b.}\,\{K_1 \longrightarrow K \times K,\, K_2 \longrightarrow K \times K\})yx) = \text{image}\, yx\ .$$

Moreover,

$$\text{l.u.b.}\left\{\text{image}\left(K_1 \longrightarrow K \times K \xrightarrow{yx} A\right), \text{image}\left(K_2 \longrightarrow K \times K \xrightarrow{yx} A\right)\right\}$$
$$= \text{image}\, k$$

and by Lemma 6, image yx = image k.

Now we let $x'\colon K \times K \to K$ be the coimage of yx. Then $(1_K, 0)x'k = (1_K, 0)(\text{coimage}\,(yx))(\text{image}\,(yx)) = (1_K, 0)yx = k(1_A, 0)x = k$(by definition of x) and similarly for $(0, 1_K)$. Then k is right cancellable so $(1_K, 0)x' = 1_K$ and $(0, 1_K)\, x' = 1_K$. Hence x' is the desired morphism and $K \in \mathscr{A}$.

Dually to the above, any quotient object of an abelian object is abelian, and in particular the cokernel of a morphism of $\mathscr{A}$ is in $\mathscr{A}$.

We now show that all monomorphisms of $\mathscr{A}$ are kernels. Suppose $f\colon A \to B$ is a monomorphism of $\mathscr{A}$. Let $B \times B \xrightarrow{p_i} B_i$, $A \times B \xrightarrow{p'_1} A$, $A \times B \xrightarrow{p'_2} B$ be products. Then we have $(p'_1 f, p'_2)\colon A \times B \to B \times B$ and $A \xrightarrow{(1,0)} A \times B \to B \times B = A \to B \xrightarrow{(1,0)} B \times B$ since followed by either p_i they are equal. Moreover, $B \xrightarrow{(0,1)} A \times B \to B \times B = B \xrightarrow{(0,1)} B \times B$. Let j be the morphism such that $(1_B, 0)j = 1_B = (0, 1_B)j$. Then $B \to A \times B \to B \times B \xrightarrow{j} B = B \xrightarrow{(0,1)} B \times B \xrightarrow{j} B = 1_B$; hence $(p'_1 f, p'_2)j$ is an epimorphism since 1_B is. Then

$$A \longrightarrow A \times B \longrightarrow B \times B \xrightarrow{j} B$$
$$= A \xrightarrow{f} B \xrightarrow{(1,0)} B \times B \xrightarrow{j} B = A \xrightarrow{f} B\ .$$

Now $A \to A \times B$ is a kernel of $A \times B \to B$ and since $A \times B \to B \times B \xrightarrow{j} B$ is an epimorphism, $A \to A \times B \to B \times B \xrightarrow{j} B = A \to B = \text{image}\,(A \to B)$ (since $A \to B$ is a monomorphism) is a kernel by condition (1).

If $f\colon A \to B$ is an epimorphism in $\mathscr{S}$ we form its kernel as above and it is the cokernel of its kernel. It remains to show that if f is an epimorphism of $\mathscr{A}$, it is an epimorphism of $\mathscr{S}$.

Suppose $f\colon A \to B$ is an epimorphism of $\mathscr{A}$. Then suppose $B \to I$ is the cokernel of $A \to B$. Since I is abelian and $A \to B$ is left cancellable in $\mathscr{A}$, $B \to I = 0$, i.e., the cokernel of f is zero. Then its kernel is the image of f, which is then equivalent to $B \to B$, i.e., $A \to B$ is its own coimage and hence an epimorphism.

Thus $\mathscr{A}$ is abelian, completing the proof of Theorem 3.

4. *H*-spaces. In the category $\mathscr{T}$ of topological spaces with base points and continuous maps taking base points into base points, we call a map $\mu: X \times X \rightarrow X$ (Cartesian product) a *continuous multiplication*. We denote $(a, b)\mu$ by ab. The correspondences $x \rightarrow ax$ and $x \rightarrow xa$ for a given $a \in X$ determine the maps $L_a: X \rightarrow X$, $R_a: X \rightarrow X$. A base point $a \in X$ is a *homotopy unit* if a is idempotent and L_a and R_a are homotopic to the identity map relative to a. R_a and L_a are continuous by definition and take base points into base points since a is idempotent. X is an *H-space* if it has a continuous multiplication with homotopy unit.

Clearly R_a factors through $X \times X$ (which is obviously a product in this category) as $X \xrightarrow{(1,0)} X \times X \xrightarrow{\mu} X$, and similarlyf or L_a. If a is a homotopy unit,

$$X \xrightarrow{(1,0)} X \times X \xrightarrow{\mu} X = R_a \simeq l_X$$
$$X \xrightarrow{(0,1)} X \times X \xrightarrow{\mu} X = L_a \simeq l_X \,.$$

Now consider the functor π_1 from the category $\mathscr{T}$ to the category $\mathscr{G}$ of groups and group homomorphisms which assigns to each object of $\mathscr{T}$ its fundamental group. We know that $(X \times X)\pi_1 = (X)\pi_1 \times (X)\pi_1$ (group direct product) so we have

$$(X)\pi_1 \xrightarrow{(1,0)\pi_1} (X)\pi_1 \times (X)\pi_1 \xrightarrow{(\mu)\pi_1} (X)\pi_1 = (R_a)\pi_1 = (1_X)\pi_1$$

(since $R_a \simeq 1_X) = 1_{(X)\pi_1}$. Moreover, $(1, 0)\pi_1 = (1_{(X)\pi_1}, 0)$ and similarly for $(0, 1)\pi_1$ by definition of product and functor. Hence $(\mu)\pi_1$ is the required map in the definition of abelian objects. Thus we obtain the well-known result that the fundamental group of an *H*-space is abelian.

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