

# Pacific Journal of Mathematics

**ON  $w^*$ -SEQUENTIAL CONVERGENCE AND  
QUASI-REFLEXIVITY**

RALPH DAVID MCWILLIAMS

## ON $w^*$ -SEQUENTIAL CONVERGENCE AND QUASI-REFLEXIVITY

R. D. McWILLIAMS

This paper characterizes quasi-reflexive Banach spaces in terms of certain properties of the  $w^*$ -sequential closure of subspaces. A real Banach space  $X$  is quasi-reflexive of order  $n$ , where  $n$  is a nonnegative integer, if and only if the canonical image  $J_X X$  of  $X$  has algebraic codimension  $n$  in the second dual space  $X^{**}$ . The space  $X$  will be said to have property  $P_n$  if and only if every norm-closed subspace  $S$  of  $X^*$  has codimension  $\leq n$  in its  $w^*$ -sequential closure  $K_X(S)$ . By use of a theorem of Singer it is proved that  $X$  is quasi-reflexive of order  $\leq n$  if and only if every norm-closed separable subspace of  $X$  has property  $P_n$ . A certain parameter  $Q^{(n)}(X)$  is shown to have value 1 if  $X$  has property  $P_n$  and to be infinite if  $X$  does not have  $P_n$ . The space  $X$  has  $P_0$  if and only if  $w$ -sequential convergence and  $w^*$ -sequential convergence coincide in  $X^*$ . These results generalize a theorem of Fleming, Retherford, and the author.

2. If  $X$  is a real Banach space,  $S$  a subspace of  $X^*$ , and  $K_X(S)$  the  $w^*$ -sequential closure of  $S$  in  $X^*$ , then  $K_X(S)$  is a Banach space under the norm  $\varphi_S$  defined by

$$\varphi_S(f) = \inf \left\{ \sup_{n \in \omega} \|f_n\| : \{f_n\} \subset S, f_n \xrightarrow{w^*} f \right\}$$

for  $f \in K_X(S)$  [5]. If  $S \subseteq T \subseteq K_X(S)$ , let

$$C_X(S, T) = \sup \{ \varphi_S(f) : f \in T, \|f\| \leq 1 \}.$$

Thus,  $K_X(S)$  is norm-closed in  $(X^*, \|\cdot\|)$  if and only if  $C_X(S, K_X(S))$  is finite [5]. For each integer  $n \geq 0$  let  $\mathcal{F}_n(S)$  be the family of all subspaces  $T$  of  $X^*$  such that  $S \subseteq T \subseteq K_X(S)$  and such that  $K_X(S)$  is the algebraic direct sum of  $T$  and a subspace of dimension  $\leq n$ . Let

$$C_X^{(n)}(S) = \inf \{ C_X(S, T) : T \in \mathcal{F}_n(S) \},$$

and let

$$Q^{(n)}(X) = \sup \{ C_X^{(n)}(S) : S \text{ a subspace of } X^* \}.$$

It will be said that  $X$  has *property*  $P_n$  if and only if  $S \in \mathcal{F}_n(S)$  for every norm-closed subspace  $S$  of  $(X^*, \|\cdot\|)$ .

3. THEOREM 1. *Let  $X$  be a real Banach space and  $n$  a non-*

negative integer. If  $X$  has property  $P_n$ , then  $Q^{(n)}(X) = 1$ . If  $X$  does not have property  $P_n$ , then  $Q^{(n)}(X) = \infty$ .

*Proof.* If  $X$  has property  $P_n$  and  $S_1$  is a norm-closed subspace of  $X^*$ , then  $S_1 \in \mathcal{S}_n(S_1)$  and hence  $C_X^{(n)}(S_1) = 1$ . If  $S$  is an arbitrary subspace of  $X^*$  and  $S_1$  the norm-closure of  $S$ , then  $C_X^{(n)}(S) = C_X^{(n)}(S_1)$  and therefore  $Q^{(n)}(X) = 1$ .

If  $X$  does not have property  $P_n$ , then  $X^*$  has a norm-closed subspace  $S$  such that  $K_X(S)$  contains an  $(n+1)$ -dimensional subspace  $V$  such that  $S \cap V = \{0\}$ . Now  $V$  has a basis  $\{f_1, \dots, f_{n+1}\}$  of vectors with  $\|f_i\| = 1$ , and there exist  $F_1, \dots, F_{n+1} \in X^{**}$  such that for each  $j \in \{1, \dots, n+1\}$ ,  $F_j(f) = 0$  for every  $f \in S$  and  $F_j(f_i) = \delta_{ij}$  for each  $i \in \{1, \dots, n+1\}$  [7, p. 186]. Let  $\alpha = \max\{\|F_j\| : 1 \leq j \leq n+1\}$ . Further, there exist vectors  $x_1, \dots, x_{n+1} \in X$  such that  $f_i(x_j) = \delta_{ij}$  for  $1 \leq i, j \leq n+1$  [7, p. 138].

Since  $f_1, \dots, f_{n+1} \in K_X(S)$ , the restrictions of  $J_X x_1, \dots, J_X x_{n+1}$  to  $S$  must be linearly independent on  $S$ , and hence for each

$$i \in \{1, \dots, n+1\}$$

there exists  $g_i \in S$  such that  $g_i(x_j) = \delta_{ij}$  for each  $j$  [7, p. 138]. Now for each  $i = 1, \dots, n+1$  there is a sequence  $\{p_{ih}\} \subset S$  such that  $p_{ih} \xrightarrow{w^*} f_i$ . The sequence  $\{p_{ih}\}$  may be chosen so that

$$|p_{ih}(x_j) - \delta_{ij}| < \frac{2^{-h}}{(n+1)\|g_j\|}$$

for each  $j$ . If we let  $f_{ih} = p_{ih} + \sum_{j=1}^{n+1} [\delta_{ij} - p_{ih}(x_j)]g_j$ , then  $f_{ih}(x_j) = \delta_{ij}$  for all  $i, h, j$ , and  $\|f_{ih} - p_{ih}\| < 2^{-h}$ , so that  $f_{ih} \xrightarrow{w^*} f_i$ ; clearly  $\{f_{ih}\} \subset S$ .

For each  $i \in \{1, \dots, n+1\}$  and  $h \in \omega$ , let  $g_{ih} = f_{ih} - f_i$ . Thus  $g_{ih}(x_j) = 0$  and  $F_j(g_{ih}) = -\delta_{ij}$  for all  $i, h, j$ , and  $g_{ih} \xrightarrow{w^*} 0$  for each  $i$ . Generalizing a method of Fleming [3], for each positive number  $N$  we let  $R_N$  be the linear span and  $S_N$  the norm-closed linear span of  $\{f_{ih} + Ng_{ih} : 1 \leq i \leq n+1; h \in \omega\}$ . Note that for each

$$i \in \{1, \dots, n+1\}, f_{ih} + Ng_{ih} \xrightarrow{w^*} f_i;$$

thus  $V \subseteq K_X(R_N)$ . Now let  $f$  be a nonzero element of  $V$  and  $\{v_m\}$  a sequence in  $R_N$  such that  $v_m \xrightarrow{w^*} f$ . Clearly  $f$  has the form

$$f = \sum_{i=1}^{n+1} \alpha_i f_i$$

and each  $v_m$  has the form

$$v_m = \sum_{i=1}^{n+1} \sum_{h=1}^{h_{mi}} \alpha_{mih} (f_{ih} + Ng_{ih}) .$$

For every  $j \in \{1, \dots, n + 1\}$ ,

$$\alpha_j = f(x_j) = \lim_m v_m(x_j) = \lim_m \sum_{h=1}^{h_{mj}} \alpha_{mjh} ,$$

and since  $F_j(f_{ih} + Ng_{ih}) = -N\delta_{ij}$ , it follows that

$$F_j(v_m) = -N \sum_{h=1}^{h_{mj}} \alpha_{mjh} .$$

Thus  $\lim_m F_j(v_m)$  exists and is equal to  $-N\alpha_j$ . Now

$$\|v_m\| \geq \frac{|F_j(v_m)|}{\|F_j\|} ,$$

and hence  $\liminf_m \|v_m\| \geq N|\alpha_j|/\|F_j\| \geq N|\alpha_j|/\alpha$ . Since  $j$  is arbitrary,  $\liminf_m \|v_m\| \geq (N/\alpha) \max |\alpha_j|$ . From the definition of  $\varphi_{S_N}$ , it follows that  $\varphi_{R_N}(f) = \varphi_{S_N}(f) \geq N/\alpha \max_j |\alpha_j| \geq N\|f\|/\alpha(n+1)$ . If  $T \in \mathcal{S}_n(S_N)$ , then  $T$  must contain some nonzero  $f \in V$  since  $V$  is  $(n+1)$ -dimensional, and hence  $C_X(S_N, T) \geq N/\alpha(n+1)$ . Therefore  $C_X^{(n)}(S_N) \geq N/\alpha(n+1)$ . Since  $N$  is arbitrary and  $\alpha(n+1)$  is independent of  $N$ , it follows that  $Q^{(n)}(X) = +\infty$ .

**THEOREM 2.** *Let  $X$  be a real Banach space and  $n$  a nonnegative integer. If  $X$  is quasi-reflexive of order  $\leq n$ , then  $X$  has property  $P_n$ . If  $X$  is separable and has property  $P_n$ , then  $X$  is quasi-reflexive of order  $\leq n$ .*

*Proof.* If  $X$  is quasi-reflexive of order  $m \leq n$  and  $S$  is a norm-closed subspace of  $X^*$ , then it can be seen from the proofs of Theorems 5 and 6 of [4] that  $K_X(S)$  is the direct sum of  $S$  with a subspace of  $X^*$  of dimension  $\leq m$ . Hence  $S \in \mathcal{S}_m(S)$ , and consequently  $X$  has property  $P_n$ .

On the other hand, let  $X$  be separable and suppose that  $X$  has property  $P_n$ . Let  $F_1, \dots, F_{n+1}$  be linearly independent elements of  $X^{**}$  and  $S = \bigcap_{i=1}^{n+1} \{f \in X^* : F_i(f) = 0\}$ . Thus  $S$  is a norm-closed subspace of  $X^*$  of codimension  $n+1$ , and hence, by property  $P_n$ ,  $K_X(S)$  has codimension  $m$  for some  $m \in \{1, \dots, n+1\}$ . There exists a subspace  $U$  of  $X^*$  of codimension 1 such that  $K_X(S) \subseteq U$ . Thus  $U = S \oplus V$  for some subspace  $V$  of  $X^*$  of dimension  $n$ . Now  $U = K_X(U)$ . Indeed, if  $\{g_i\} \subset U$  and  $g_i \xrightarrow{w^*} g$ , and if  $P$  is the projection of  $U$  onto

$V$  along  $S$ , then as in the proof of Theorem 5 of [4],  $P$  is bounded and  $\{g_i\}$  is bounded, so that  $\{Pg_i\}$  is bounded and hence has a subsequence  $\{Pg_{i_j}\}$  which converges inner  $m$  to some  $v$  in the finite-dimensional subspace  $V$ . It follows that  $g_{i_j} - Pg_{i_j} \xrightarrow{w^*} g - v \in K_x(S)$  and hence that  $g \in K_x(S) + V = U$ .

Since  $U = K_x(U)$  and  $X$  is separable, it follows, by an argument involving the  $bw^*$ -topology of  $X^*$  [3], that  $U$  is  $w^*$ -closed. If  $n = 0$ , let  $F = F_1$ . If  $n > 0$ , there exist linearly independent vectors  $f_1, \dots, f_n$  spanning  $V$ , and there exist scalars  $\alpha_1, \dots, \alpha_{n+1}$ , not all of which are zero, such that  $\sum_{i=1}^{n+1} \alpha_i F_i(f_j) = 0$  for  $1 \leq j \leq n$ ; indeed, the  $(n + 1)$  vectors

$$\begin{bmatrix} F_i(f_1) \\ \vdots \\ F_i(f_n) \end{bmatrix} \quad (i = 1, \dots, n + 1)$$

in  $n$ -dimensional Euclidean space must be linearly dependent. Let  $F = \sum_{i=1}^{n+1} \alpha_i F_i$ . Thus, for  $n \geq 0$ ,  $F \neq 0$  and  $U = \{f \in X^* : F(f) = 0\}$ . Since  $U$  is  $w^*$ -closed,  $F$  is  $w^*$ -continuous on  $X^*$  [7, p. 139], and hence  $F \in J_x X$ . Thus every  $(n + 1)$ -dimensional subspace of  $X^{**}$  contains a nonzero element of  $J_x X$ , which means that  $X$  is quasi-reflexive of order  $\leq n$ .

REMARK. Theorems 1 and 2 contain a generalization of Fleming's theorem [3] that if  $X$  is a separable Banach space, then  $X$  is reflexive if and only if  $Q(X) = 1$ . The following theorem generalizes a theorem of [3] and [4].

**THEOREM 3.** *A real Banach space  $X$  is quasi-reflexive of order  $\leq n$ , where  $n \geq 0$ , if and only if every norm-closed separable subspace  $Y$  of  $X$  has the property  $P_n$ .*

*Proof.* If  $X$  is quasi-reflexive of order  $\leq n$  and  $Y$  is a closed subspace of  $X$ , then  $Y$  is also quasi-reflexive of order  $\leq n$  [1] and hence  $Y$  has property  $P_n$  by Theorem 2. Conversely, if every norm-closed separable subspace  $Y$  of  $X$  has property  $P_n$ , then every such  $Y$  is quasi-reflexive of order  $\leq n$  by Theorem 2, and hence  $X$  is quasi-reflexive of order  $\leq n$  by a theorem of Singer [6].

REMARK. In Theorem 3 the word "separable" can be deleted. By virtue of Theorem 1, Theorem 3 is also true if "property  $P_n$ " is replaced with "property that  $Q^{(n)}(Y) = 1$ ". Since a space  $X$  is quasi-reflexive of order  $n$  if and only if  $X$  is quasi-reflexive of order  $\leq n$  but not of order  $\leq (n - 1)$ , Theorem 3 can easily be reworded in such

a way as to give a necessary and sufficient condition that  $X$  be quasi-reflexive of order exactly  $n$ .

4. THEOREM 4. *If  $X$  is a real Banach space, then  $Q^{(0)}(X) = 1$  if and only if  $w$ -sequential convergence and  $w^*$ -sequential convergence coincide in  $X^*$ .*

*Proof.* Suppose the two kinds of sequential convergence coincide and  $S$  is a subspace of  $X^*$ . If  $\{f_i\} \subset S$  and  $f_i \xrightarrow{w^*} f$ , then  $f_i \xrightarrow{w} f$  and hence some sequence of averages far out in  $\{f_i\}$  converges in norm to  $f$  [2, p. 40]; thus  $f \in S_1$ , the norm-closure of  $S$ , and hence  $\varphi_S(f) = \|f\|$ . Therefore,  $C_X^{(0)}(S) = 1$  and  $Q^{(0)}(X) = 1$ .

Conversely, suppose there are a sequence  $\{f_i\}$  in  $X^*$  and an  $f_0 \in X^*$  such that  $f_i \xrightarrow{w^*} f_0$  but  $f_i \not\xrightarrow{w} f_0$ . Then there exists an  $F \in X^{**}$  such that  $F(f_i) \not\xrightarrow{w} F(f_0)$ . The sequence  $\{F(f_i)\}$  is bounded and hence contains a subsequence  $\{F(f_{i_j})\}$  such that the limit  $\alpha = \lim_j F(f_{i_j})$  exists, but  $\alpha \neq F(f_0)$ . Since  $F \neq 0$ , there exists  $g \in X^*$  such that  $F(g) \neq 0$ . Let  $g_j = f_{i_j} - (F(f_{i_j})/F(g))g$  for each  $j \in \omega$  and

$$g_0 = f_0 - \frac{\alpha}{F(g)}g.$$

Then  $F(g_j) = 0$  for each  $j \in \omega$ , but  $F(g_0) \neq 0$ . For every  $x \in X$ ,

$$g_j(x) \rightarrow f_0(x) - \frac{\alpha}{F(g)}g(x) = g_0(x),$$

so that  $g_j \xrightarrow{w^*} g_0$ . Let  $S$  be the norm-closed subspace of  $X^*$  spanned by  $\{g_j : j \in \omega\}$ . Then  $g_0 \in K_X(S)$ , but  $g_0 \notin S$ , since  $F(g_0) \neq 0$  whereas  $F(f) = 0$  for all  $f \in S$ . Thus  $S \notin \mathcal{S}_0(S)$ , and hence  $X$  does not have property  $P_0$ , so that  $Q^{(0)}(X) = \infty$  by Theorem 1.

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