Pacific Journal of Mathematics

DISJOINT BASIC SUBGROUPS

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Vol. 23, No. 1 March 1967

DISJOINT BASIC SUBGROUPS

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This paper arose from consideration of the following questions. First, what characterizes those infinite Abelian reduced p-groups which possess disjoint basic subgroups? Second, are there properties that a basic subgroup must possess to insure the existence of a basic subgroup disjoint from it?

We show that a necessary and sufficient condition for an infinite Abelian reduced p-group G to contain disjoint basic subgroups is that |G| = final rank G. Futhermore, in such a group a necessary and sufficient condition for a basic subgroup B to have a basic subgroup disjoint from it is that B is a lower basic subgroup of G.

Throughout this paper the word "group" will mean "Abelian group" and the notation used will be that of L. Fuchs in [1] with the exception that $A \oplus B$ will denote the direct sum of the groups A and B, and A + B will be the, not necessarily direct, sum.

We will use the following theorem:

THEOREM A. (Mitchell and Mitchell in [4]) Let G be an infinite reduced Abelian p-group and B a basic subgroup of G such that $G/B = \sum_{\alpha \in I} (G_{\alpha}/B)$ where $G_{\alpha}/B \cong Z(p^{\infty})$ for all $\alpha \in I$. Then $G = H \oplus K$ and $B = H \oplus L$ where L is a basic subgroup of K such that r(K/L) = r(G/B) = |I| and $|K| = maximum \{ \}, |I| \}$.

We first prove the following lemmas:

LEMMA 1. Let G be a p-group without elements of infinite height, and such that final rank (G) = |G|. Let B be a lower basic subgroup of G. Then there exists a basic subgroup, B', of G which is disjoint from B.

Proof. Let $B = \sum_{\alpha \in I} \langle y_{\alpha} \rangle$, and let $G/B = \sum_{\beta \in J} C_{\beta}$ where each $C_{\beta} \cong Z(p^{\infty})$. Let $\{\{y_{\alpha} \mid \alpha \in I\}, \{c_{\beta,n} \mid \beta \in J, n=1, 2, \cdots\}\}$ be a quasibasis for G. Since B is a lower basic subgroup of G and final rank (G) = |G|, we have $|J| = r(G/B) = |G| \ge |B| \ge |I|$. If indeed we have |J| > |I| a pure subgroup H of G can be chosen such that $H \supset B$, $H^1 = 0$, and final rank (H) = |H| = |I|. We can then prove that there is a basic subgroup B' of H which is disjoint from B and H being pure in G will insure B' is a basic subgroup of H. Thus it suffices to complete the proof when |J| = |I| and we will assume moreover I = J.

Now for each $\alpha \in I$ choose from $\{c_{\alpha,n}\}_{n=1}^{\infty}$ the element $c_{\alpha,2E(y_{\alpha})}$. Define $B' = \langle \{y_{\alpha} - p^{E(y_{\alpha})}c_{\alpha,2E(y_{\alpha})}\}_{\alpha \in I} \rangle$. We now claim that B' is the desired basic subgroup of G which is disjoint from B. To see this we prove the following:

(i) First claim that

$$B' = \sum_{\alpha \in I} \langle y_{\alpha} - p^{E(y_{\alpha})} c_{\alpha, 2E(y_{\alpha})} \rangle$$
.

Suppose that

$$0 = \sum_{i=1}^{n} a_i (y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})$$
,

then

$$\sum_{i=1}^{n} a_{i} y_{\alpha_{i}} = \sum_{i=1}^{n} a_{i} p^{E(y_{\alpha_{i}})} c_{\alpha_{i}, 2E(y_{\alpha_{i}})}$$
 .

Since the

$$E(y_{\alpha}-p^{E(y_{\alpha})}c_{\alpha,2E(y_{\alpha})})=E(y_{\alpha})$$
 ,

we would be finished if $\sum_{i=1}^n a_i y_{\alpha_i} = 0$, so we can assume that $\sum_{i=1}^n a_i y_{\alpha_i} \neq 0$, and without loss of generality a_i is not equal to $0 \mod o(y_{\alpha_i})$. Now the height $h_{\mathcal{G}}(\sum_{i=1}^n a_i y_{\alpha_i}) = r$, where r is the largest positive integer such that p^r divides each a_i , since $\sum_{i=1}^n a_i y_{\alpha_i}$ is an element of $B = \sum_{\alpha \in I} \langle y_{\alpha} \rangle$. But,

$$h_{\mathcal{G}}\!\!\left(\sum_{i=1}^n a_i \, p^{E(y_{\alpha_i})} \, c_{\alpha_i, 2E(y_{\alpha_i})}
ight) \geq \min_{i=1,2,\cdots,n} \left\{r + E(y_{\alpha_i})
ight\} > r$$
 ,

which contradicts the equality

$$\sum\limits_{i=1}^{n}a_{i}y_{\alpha_{i}}=\sum\limits_{i=1}^{n}a_{i}p^{E(y_{\alpha_{i}})}c_{\alpha_{i},2E(y_{\alpha_{i}})}$$
 .

Therefore, we must have that

$$B' = \sum_{\alpha \in I} \langle y_{\alpha} - p^{E(y_{\alpha})} c_{\alpha, 2E(y_{\alpha})} \rangle$$
.

(ii) Next we will show that B' is a pure subgroup of G. Let $z \in B'[p]$, and write

$$z = \sum_{i=1}^{n} a_i \, p^{E(y_{\alpha_i})-1}(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})$$
,

where each a_i is relatively prime to p. Now we have that

$$\begin{array}{l} h_{B'}(z) = \mathop{\rm minimum}_{i=1,\cdots,n} \left\{ h_{B'} [a_i \, p^{B(y_{\alpha_i})-1} (y_{\alpha_i} \, - \, p^{B(y_{\alpha_i})} \, c_{\alpha_i,2B(y_{\alpha_i})})] \right\} \\ = \mathop{\rm minimum}_{i=1,\cdots,n} \left\{ E(y_{\alpha_i}) \, - \, 1 \right\} \, . \end{array}$$

But

$$h_{\rm G}(z) \, = \, h_{\rm G} \bigg[\bigg(\sum_{i=1}^n a_i \, p^{{\rm E}(y_{\alpha_i}) - 1} y_{\alpha_i} \bigg) \, - \, \bigg(\sum_{i=1}^n a_i \, p^{2{\rm E}(y_{\alpha_i}) - 1} c_{\alpha_i, 2{\rm E}(y_{\alpha_i})} \bigg) \bigg] \, ,$$

and

$$\begin{split} h_{\theta}\!\!\left(\sum_{i=1}^n a_i \, p^{2E(y_{\alpha_i})-1} c_{\alpha_i,2E(y_{\alpha_i})}\right) & \geq \text{ minimum } \{2E(y_{\alpha_i})-1\} \\ & > \underset{i=1,\dots,n}{\text{minimum }} \{E(y_{\alpha_i})-1\} \text{ ,} \end{split}$$

and

$$h_{\mathit{G}}\!\!\left(\sum_{i=1}^n a_i\,p^{E(y_{lpha_i})-1}y_{lpha_i}
ight)= \min_{i=1,\cdots,n}\left\{E(y_{lpha_i})-1
ight\}$$
 .

Since the height of the sum of two elements with different heights is just the height of the smaller, we have that

$$h_{\mathcal{G}}(z) = \min_{i=1,\dots,n} \{E(y_{\alpha_i}) - 1\}$$
.

Therefore $h_{\theta}(z) = h_{B'}(z)$ for each element $z \in B'[p]$, and hence by Lemma 7, page 20, in [3], we have that B' is pure.

(iii) We will now complete the proof that B' is a basic subgroup of G by showing that B' cannot be extended to a larger pure direct sum of cyclic groups. Suppose that $B' \oplus \langle z \rangle$ is a pure direct sum of cyclic groups. Since $\{\{y_{\alpha} \mid \alpha \in I\}, \{c_{\alpha,n} \mid \alpha \in I, n = 1, 2, \cdots\}\}$ is a quasibasis for G, we can write

$$z=\sum\limits_{i=1}^{n}a_{i}y_{lpha_{i}}+\sum\limits_{i=1}^{k}s_{j}c_{lpha_{j},r_{j}}$$
 .

Now we also know that $c_{\alpha_j,r_j}=pc_{\alpha_j,r_j+1}+b_j$ where $b_j\in B=\sum_{\alpha\in I}\langle y_\alpha\rangle$, hence we can write

$$z = \sum\limits_{j=1}^{k} s_{j} p c_{lpha_{j}, r_{j}+1} + \sum\limits_{i=1}^{m} t_{i} y_{lpha_{i}}$$
 .

Now write

$$z = \sum_{j=1}^{k} s_{j} p c_{\alpha_{j}, r_{j}+1} + \sum_{i=1}^{m_{1}} t'_{i} y_{\alpha_{i}} + \sum_{i=1}^{m_{2}} t''_{i} y_{\alpha_{i}}$$

where each t_i' is divisible by p, and each t_i'' , is relatively prime to p. We are assuming that $H = B' \bigoplus \langle z \rangle$ is pure in G, and hence, $h_G(z) = h_H(z) = 0$, and since H is a direct sum of cyclic groups we must also have that $h_H(b' + z) = 0$, for any $b' \in B$. Consider the following element of B',

$$\sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})})$$
.

Now we have

$$\begin{split} z &- \sum_{i=1}^{m_2} t_i''(y_{\alpha_i} - p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})}) \\ &= \sum_{j=1}^k s_j p c_{\alpha_j, r_j+1} + \sum_{i=1}^{m_1} t_i' y_{\alpha_i} + \sum_{i=1}^{m_2} t_i'' p^{E(y_{\alpha_i})} c_{\alpha_i, 2E(y_{\alpha_i})} \;, \end{split}$$

thus

$$h_{\it G}\!\!\left(z - \sum\limits_{i=1}^{m_2} t_i''(y_{lpha_i} - p^{E(y_{lpha_i})} c_{lpha_i, 2E(y_{lpha_i})}
ight) \geqq 1$$
 ,

but this contradicts the assumption that H is a pure subgroup of G. Thus B' is a basic subgroup of G.

To complete the proof of the theorem, we need only show that $B \cap B' = 0$. To see this suppose that

$$\sum_{j=1}^k s_j (y_{\alpha_j} - p^{B(y_{\alpha_j})} c_{\alpha_j, 2E(y_{\alpha_j})}) = \sum_{i=1}^n a_i y_{\alpha_i}.$$

Consider

$$\begin{split} \sum_{i=1}^n a_i y_{\alpha_i} + B &= 0 + B = \sum_{j=1}^k s_j (y_{\alpha_j} - p^{\mathbb{E}(y_{\alpha_j})} c_{\alpha_j, 2\mathbb{E}(y_{\alpha_j})}) + B \\ &= \sum_{j=1}^k s_j p^{\mathbb{E}(y_{\alpha_j})} c_{\alpha_j, 2\mathbb{E}(y_{\alpha_j})} + B \\ &= \sum_{j=1}^k s_j c_{\alpha_j, \mathbb{E}(y_{\alpha_j})} + B \ , \end{split}$$

so that $s_j c_{\alpha_j, E(y_{\alpha_j})} + B = 0 + B$ for $j = 1, \dots, k$ since each is from a different summand of G/B. But this means that s_j is divisible by $p^{E(y_{\alpha_j})}$ for $j = 1, \dots, k$. Thus

$$\sum_{j=1}^{k} s_{j}(y_{\alpha_{j}} - p^{E(y_{\alpha_{j}})} c_{\alpha_{j}, 2E(y_{\alpha_{j}})}) = 0 ,$$

and so $B \cap B' = 0$.

LEMMA 2. Let G be a reduced p-group such that final rank (G) = |G|. Let B be a lower basic subgroup of G. Then there exists a basic subgroup B' of G which is disjoint from B.

Proof. Let H be a high subgroup of G which contains B. By Theorem 5 in [2] H is pure and a basic subgroup of H is a basic of G. Thus rank (G/B) = rank (H/B) + rank (G/H), and we consider the following cases:

Case (i). Suppose that rank $(H/B) = \operatorname{rank} (G/B)$, then we know that final rank $(H) \ge \operatorname{rank} (H/B) = \operatorname{rank} (G/B) = \operatorname{final rank} (G) = |G| \ge |H|$. Thus final rank (H) = |H|, and Lemma 1 completes the proof.

Case (ii). Suppose that $\operatorname{rank}(G/B) > \operatorname{rank}(H/B)$. Since $|G| = \operatorname{final rank}(G)$, and final $\operatorname{rank}(G) = \operatorname{rank}(G/B)$, we know that $\operatorname{rank}(G/B)$ is infinite. But if $\operatorname{rank}(G/B) > \operatorname{rank}(H/B)$, and is infinite, then the $|G^1[p]|$ is infinite, and hence we have $|G^1[p]| = \operatorname{rank}(G/H) > \operatorname{rank}(H/B)$. Now $\operatorname{rank}(G/B) = \operatorname{rank}(H/B) + \operatorname{rank}(G/H) = \operatorname{rank}(H/B) + |G^1[p]|$, and thus $|G^1[p]| = \operatorname{rank}(G/B) = |G|$. So that $|G^1[p]| \ge |B|$, and for the purposes of this proof we can assume that $|G^1[p]| = |B|$. Let $G^1[p] = \sum_{\alpha \in I} \langle y_\alpha \rangle$, and let $B = \sum_{\alpha \in I} \langle x_\alpha \rangle$. For each $\alpha \in I$ choose z_α such that $y_\alpha = p^{B(x_\alpha)-1}z_\alpha$, which can be done since each y_α has infinite height. Now consider the subgroup $B' = \langle \{x_\alpha + z_\alpha\}_{\alpha \in I} \rangle$. We claim that B' is a basic subgroup of G which is disjoint from G. To see this we prove:

(i) First we must show that $B' = \sum_{\alpha \in I} \langle x_{\alpha} + z_{\alpha} \rangle$. Suppose

$$\sum_{i=1}^n a_i(x_{\alpha_i} + z_{\alpha_i}) = 0 ,$$

where $a_i \not\equiv 0 \mod (o(x_{\alpha_i}))$, and $a_i < o(x_{\alpha_i})$. Notice that $o(x_{\alpha}) = o(x_{\alpha} + z_{\alpha})$ since $p(y_{\alpha}) = 0 = p^{E(x_{\alpha})}z_{\alpha}$. Let k_i be the largest positive integer such that p^{k_i} divides a_i . Let $r = \max_{i=1,\dots,n} \{E(x_{\alpha_i}) - k_i\}$, and consider

$$egin{aligned} 0 &= p^{r-1} \Big(\sum\limits_{i=1}^n a_i (x_{lpha_i} + z_{lpha_i}) \Big) = \sum\limits_{i=1}^n a_i p^{r-1} x_{lpha_i} + \sum\limits_{i=1}^n a_i p^{r-1} z_{lpha_i} \ &= \sum\limits_{i=1}^n a_i p^{r-1} x_{lpha_i} + \sum\limits_{i=1}^n a_i' y_{lpha_i} \,. \end{aligned}$$

Hence we have

$$\sum_{i=1}^{n} a_i' y_{\alpha_i} = -\sum_{i=1}^{n} a_i p^{r-1} x_{\alpha_i}$$
,

but this means that an element of infinite height is equal to an element of finite height which is contradiction. Thus $B' = \sum_{\alpha \in I} \langle x_{\alpha} + z_{\alpha} \rangle$.

(ii) We must show that B' is pure. Let $s \in B'[p]$, and write

$$s = \sum_{i=1}^{n} a_{i} p^{E(x_{\alpha_{i}})-1} (x_{\alpha_{i}} + z_{\alpha_{i}})$$

where a_i is relatively prime to p for each i. Since $B' = \sum_{\alpha \in I} \langle x_\alpha + z_\alpha \rangle$, we know that $h_{B'}(s) = \min_{i=1,...,n} \{E(x_{\alpha_i}) - 1\}$. Now consider

$$\begin{split} h_{\mathbf{G}}(s) &= h_{\mathbf{G}} \bigg(\sum_{i=1}^n a_i \, p^{E(x_{\alpha_i})-1} x_{\alpha_i} \, + \, \sum_{i=1}^n a_i \, p^{E(x_{\alpha_i})-1} z_{\alpha_i} \bigg) \\ &= h_{\mathbf{G}} \bigg(\sum_{i=1}^n a_i \, p^{E(x_{\alpha_i})-1} x_{\alpha_i} \, + \, \sum_{i=1}^n a_i \, y_{\alpha_i} \bigg) \\ &= h_{\mathbf{G}} \bigg(\sum_{i=1}^n a_i \, p^{E(x_{\alpha_i})-1} x_{\alpha_i} \bigg) \\ &= \min_{i=1,\dots,n} \left\{ E(x_{\alpha_i}) \, - \, 1 \right\} \, . \end{split}$$

Thus B' is a pure subgroup of G.

(iii) To complete the proof that B' is a basic subgroup of G, we need only prove that the quotient G/B' is divisible. If every element $s+B'\in (G/B')[p]$ has infinite height then G/B' is divisible. Thus we can assume $h_{G/B'}(s+B')=n$, a finite integer, and we can assume that o(s)=o(s+B'). Now since G/B is divisible we know that s+B has infinite height in G/B. Consider the following cases:

Case (a). Suppose that $s \in B$, then

$$s = \sum_{i=1}^m a_i p^{E(x_{\alpha_i})-1} x_{\alpha_i}$$

where a_i is relatively prime to p for each i. Now define the following element of B', let

$$b' = \sum_{i=1}^m a_i \, p^{E(x_{\alpha_i})-1}(x_{\alpha_i} + z_{\alpha_i})$$
 .

But

$$s-b'=\sum\limits_{i=1}^m a_iy_{\alpha_i}$$
, and $\sum\limits_{i=1}^m a_iy_{\alpha_i}$

has infinite height in G so that $h_{G/B'}(s+B')$ is infinite. Therefore G/B' must be divisible.

Case (b). Suppose that $s \notin B$, then there exists an element $\sum_{i=1}^{m} a_i x_{\alpha_i} \in B$, such that

$$s + \sum_{i=1}^m a_i x_{\alpha_i} = p^{n+1} g$$

since $h_{G/B}(s+B)$ is infinite. Now write

$$\sum_{i=1}^{m} a_i x_{\alpha_i} = \sum_{j=1}^{r} c_j x_{\alpha_j} + \sum_{k=1}^{t} d_k x_{\alpha_k} ,$$

where c_j is divisible by $p^{\mathbb{E}(x_{\alpha_j})^{-1}}$, and d_k is not divisible by $p^{\mathbb{E}(x_{\alpha_k})^{-1}}$. Thus

$$s + \sum_{j=1}^{r} c_{j} x_{\alpha_{j}} + \sum_{k=1}^{t} d_{k} x_{\alpha_{k}} = p^{n+1} g$$

and so multiplication by p yields

$$\sum\limits_{k=1}^{t}pd_{k}x_{lpha_{k}}=p^{n+2}g$$
 .

By choice of the x_{α_k} 's we know $pd_k x_{\alpha_k} \neq 0$. Thus we must have

$$h_{\mathcal{G}}\!\!\left(\sum_{k=1}^t d_k x_{lpha_k}
ight) \geqq n+1$$
 .

Therefore by letting $c_j' = c_j/p^{{\scriptscriptstyle E}(x_{\alpha_j})-1}$ we have that

$$s + \sum_{j=1}^{r} p^{E(x_{\alpha_j})-1} c'_j x_{\alpha_j} = p^{n+1} g'$$
.

Consider the element $b' \in B'$ such that

$$b' = \sum_{j=1}^r c_j' p^{E(x_{\alpha_j})-1} (x_{\alpha_j} + z_{\alpha_j}) = \sum_{j=1}^r c_j' p^{E(x_{\alpha_j})-1} x_{\alpha_j} + \sum_{j=1}^r c_j' y_{\alpha_j}$$
 .

Then

$$s-b'=p^{n+1}g'-\sum_{j=1}^r c_j'y_{\alpha_j}=p^{n+1}g''$$

since $\sum_{j=1}^{r} c'_j y_{\alpha_j}$ has infinite height in G. But this implies that

$$h_{G/B'}(s+B') \geq n+1$$

which contradicts the assumption that $h_{G/B'}(s+B')=n$. Thus G/B' must be divisible.

(iv) To complete the proof of the theorem, we need only show that $B \cap B' = 0$. To see this, suppose that

$$\sum_{i=1}^n a_i x_{\alpha_i} = \sum_{j=1}^k s_j (x_{\alpha_j} + z_{\alpha_j}) \neq 0$$

so

$$\sum_{i=1}^{n} a_{i} x_{\alpha_{i}} - \sum_{j=1}^{k} s_{j} x_{\alpha_{j}} = \sum_{j=1}^{k} s_{j} z_{\alpha_{j}} \neq 0$$
 ,

and by multiplying both sides of this equation by an appropriate power of p we get an element of infinite height on one side and an element of finite height on the other side, which is a contradiction. Thus $B \cap B' = 0$, and the proof is finished.

The following theorem gives a sufficient condition for a group G to possess disjoint basic subgroups.

THEOREM 3. Let G be a reduced Abelian p-group. If final rank (G) = |G|, then G contains two disjoint basic subgroups.

Proof. Since every p-group has a lower basic subgroup, then Lemma 2 will complete the proof.

The next corollary shows that the restriction final rank (G) = |G|, can be removed if instead of disjoint basic subgroups, one is seeking two basic subgroups whose intersection is bounded.

COROLLARY 4. Let G be a reduced Abelian p-group. Then there exists two basic subgroups of G whose intersection is bounded.

Proof. By Theorem 31.5, page 106, in [1], we can write $G = H \bigoplus K$, where K is bounded direct sum of cyclic groups, and final rank (H) = |H|. Now by Theorem 3 there exists A and B which are disjoint basic subgroups of H. Now $A \bigoplus K$ and $B \bigoplus K$ are basic subgroups of G whose intersection is bounded.

THEOREM 5. Let G be a reduced Abelian p-group, and suppose that A and B are two disjoint basic subgroups of G. Then rank (G/A) = rank (G/B) = |G|.

Proof. Suppose that rank (G/A) < |G|, then by Lagrange's Theorem and since basic subgroups are isomorphic we know that |G| = |B| = |A|. By Theorem A we have $G = L \oplus F$ and $A = A' \oplus F$, where $|L| = \max \{ \bigoplus_{i=1}^{n} a_i \}$ since A and B are disjoint basic subgroup of G we know G cannot be bounded. Now $(G/A)[p] \supset [(A \oplus B)/A][p]$ and $|[(A \oplus B)/A][p]| = |B|$ which must be at least $\bigoplus_{i=1}^{n} a_i$. Thus rank $(G/A) \supseteq \bigoplus_{i=1}^{n} a_i$, and therefore

$$|L| = \operatorname{rank}(G/A) < |G|$$
.

We can write each $x \in B$ as $x = y_x + f_x$, where $y_x \in L$ and $f_x \in F$. Since |B| = |A| = |G| > |L| and B is a subgroup, there must exist some $y \in B$ such that $y \in F$, but $F \subset A$ which contradicts $A \cap B = 0$. Thus rank (G/A) = |G|, and similarity rank (G/B) = |G|.

We are now in a position to state the results of the original questions in Theorem 6 and Theorem 7.

THEOREM 6. A necessary and sufficient condition for a reduced Abelian p-group to possess disjoint basic subgroups is that final rank(G) = |G|.

Proof. If final rank (G) = |G| then Theorem 3 completes the proof. If A and B are disjoint basic subgroups of G then by Theorem 5 we have r(G/A) = r(G/B) = |G|. But final rank $(G) \ge \operatorname{rank}(G/M)$ for any basic subgroup M of G. Thus final rank $(G) \ge \operatorname{rank}(G/A) = |G|$, and since $|G| \ge \operatorname{final} \operatorname{rank}(G)$ we have final rank (G) = |G|.

THEOREM 7. If G is a reduced Abelian p-group such that final rank (G) = |G|, and A is a basic subgroup of G, then there is a basic subgroup of G which is disjoint from A if and only if A is a lower basic subgroup of G.

Proof. If A is a lower basic subgroup then Lemma 2 assures the existence of a disjoint basic subgroup. If G possesses a basic

subgroup B disjoint from A then by Theorem 5 we have rank (G/A) = |G| and by hypothesis final rank (G) = |G| thus rank (G/A) = |G| rank (G) and A is a lower basic subgroup.

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Received July 14, 1966.

PACIFIC JOURNAL OF MATHEMATICS

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Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

Pacific Journal of Mathematics

Vol. 23, No. 1

March, 1967

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