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# KERNEL REPRESENTATIONS OF OPERATORS AND THEIR ADJOINTS

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**If $S$ is a locally compact and Hausdorff space and $A$ is a continuous linear operator from $C_0(S)$ into the space $C(T)$ with the supremum norm topology then the Riesz Representation Theorem yields the formula $[Af](x) = \int_S f(y)\lambda(x, dy)$, where for each $x \in T$ $\lambda(x, \cdot)$ is a complex-valued regular Borel measure on $S$. More generally a study is made of kernel functions $\lambda$ such that $\int_S f(y)\lambda(\cdot, dy) \in C(T)$ for $f$ of compact support on $S$. It is shown that $\lambda(\cdot, E)$ is measurable for each Borel set $E$ and that $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ is a regular measure on $S$ yielding the adjoint formula $A^*\nu = \mu$. Necessary and sufficient conditions are given on $\lambda$ so that $A^{**}(C(S)) \subset C(T)$ and that $A^{**}$ be continuous from $C(S)_\beta$ to $C(T)_\beta$ when $S$ is paracompact. Furthermore, kernel representations of $\beta$-continuous operators are studied with applications to semi-groups of operators in $C_0(S)$ and $C(S)_\beta$ when $S$ is locally compact.**

We point out that as a consequence of our work the condition (1.7) in the paper by Foguel [7] follows from (1.6) when the space is locally compact and Hausdorff. Further the regularity of the above measure yields the more specific vector-valued measure representation of $A$, $\mu(E) = \lambda(\cdot, E)$ in the sense of [5, Th. 2, p. 492].

DEFINITION AND NOTATION. If $X$ is a locally compact Hausdorff space we denote by $C(X), C_0(X)$ and $C_c(X)^+$ the collection of all bounded continuous complex-valued functions on $X$, those vanishing at infinity, and those nonnegative functions of compact support, respectively. The $\sigma$-algebra of Borel sets is the $\sigma$-algebra generated by the open subsets of $X$. We denote by $M(X)$ the space of bounded regular Borel measures on $X$ with variation norm and by $B(X)$ the space of bounded Borel measurable functions on $X$. Let $M(X)^+$ denote the nonnegative measures in $M(X)$. We give $B(X), C_0(X)$ and $C(X)$ the supremum norm topology and $\|f\| = \sup\{|f(x)|: x \in X\}$.

We wish to consider two further topologies on the space $C(X)$. We denote by $C(X)_\beta$ the space $C(X)$ with the locally convex topology defined by the collection of seminorms $P_\phi(f) = \|\phi f\|$, $\phi \in C_0(X)$. Buck [1] has shown that $C(X)_\beta$ has adjoint or dual space $M(X)$. We denote by $C(X)_{\beta'}$ the space $C(X)$ with the locally convex topology whose base of neighborhoods at the origin consists of all convex, balanced, absorbent sets $V$ such that for each $r > 0$ there is a $\beta$ neighborhood of the origin, $W$, such that $W \cap B_r \subset V$ where $B_r = \{f \in C(X): \|f\| \leqq r\}$. In a recently submitted paper Dorroh [4] introduces this topology and shows that $C(X)_{\beta'}$ has dual $M(X)$ and that $\beta = \beta'$ for $X$ a paracompact space. Further results on $C(X)_\beta$ and $C(X)_{\beta'}$ have been recently obtained by Collins and Dorroh in [2]. A set $H \subset M(X)$ is $\beta$-equicontinuous ($\beta'$-equicontinuous) if there is a $\beta(\beta')$ neighborhood of 0, $W$, such that $\left|\int_X f d\mu\right| \leqq 1$ for all $f \in W$ and $\mu \in H$. The $\beta$-equicontinuous sets of $M(X)$ have been characterized by Conway [3] who has shown that $H$ is $\beta$-equicontinuous if and only if $H$ is uniformly bounded and for each $\varepsilon > 0$ there is a compact set $K \subset X$ such that the variation of $\mu$ on $X - K$ is less than $\varepsilon$ for all $\mu \in H$. Since $\beta'$ is a finer topology than $\beta$ any $\beta$-equicontinuous set is $\beta'$-equicontinuous and these are the same when $X$ is paracompact.

Suppose $S$ and $T$ are locally compact and Hausdorff. Let $\Delta$ denote the collection of open sets in $S$ and $\sigma(\Delta)$ the collection of Borel sets. We consider complex-valued functions $\lambda$ defined on $T \times \sigma(\Delta)$ such that $\lambda(x) = \lambda(x, \cdot) \in M(S)$. For brevity we will denote this by $\lambda: T \to M(S)$. We denote the norm of the measure $\lambda(x)$ by $\|\lambda(x)\|$ and set $\|\lambda\| = \sup\{\|\lambda(x)\|: x \in T\}$. If $f \in B(S)$ we write $\lambda(f)$ for the function defined by $\lambda(f)(x) = \int_S f(y)\lambda(x, dy)$ and $\lambda(\cdot, E)$ is the function whose value at $x$ is $\lambda(x, E)$ for $E \in \sigma(\Delta)$. We let $|\lambda|(x, E)$ be the variation of the measure $\lambda(x, \cdot)$ on the set $E$. We will say that the kernel $\lambda$ satisfies condition $E(E')$ if $\{\lambda(x): x \in K\}$ is $\beta$-equicontinuous ($\beta'$-continuous) for each compact set $K \subset T$.

Finally we take our topology from [8] and topological vector space terminology from [9]. We make use of the Riesz Representation theorem throughout and in particular its corollary:

$$|\mu|(U) = \sup\left\{\left|\int f d\mu\right|: f \in C_c(S),\ \|f\| \leqq 1,\ \text{support}\,(f) \subset U\right\}$$

for each open set $U$.

We prove the following theorems.

THEOREM 1. (1) *If* $\lambda: T \to M(S)^+$ *and* $\lambda(f)$ *is lower semi-continuous for each* $f \in C_c(S)^+$ *then* $\lambda(\cdot, E)$ *is Borel measurable for each* $E \in \sigma(\Delta)$.

(2) *If* $\lambda: T \to M(S)$ *and* $\lambda(f) \in C(T)$ *for all* $f \in C_c(S)$ *then* $\lambda(\cdot, E)$ *and* $|\lambda|(\cdot, E)$ *are measurable for each* $E \in \sigma(\Delta)$.

(3) *If* $\lambda$ *satisfies* (1) *or* (2) *and* $\|\lambda\| < \infty$ *then* $\lambda(f) \in B(T)$ *for* $f \in B(S)$.

THEOREM 2. *If* $\lambda$ *satisfies* (3) *of Theorem* 1 *then for each* $\nu \in M(T)$

*the formula* $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ *defines a regular Borel measure on* $S$ *such that* $|\mu|(E) \leqq \int_T |\lambda|(x, E)|\nu|(dx)$ *and for* $f \in B(S)$ *we have* $\int f d\mu = \int \lambda(f) d\nu$.

THEOREM 3. *Suppose* $A$ *is a continuous linear operator from the space* $X$ *to the space* $Y$ *where* $X$ *denotes* $C_0(S)$, $C(S)_\beta$ *or* $C(S)_{\beta'}$ *and* $Y$ *denotes* $C(T)$, $C(T)_\beta$ *or* $C(S)_{\beta'}$. *Then there is a unique mapping* $\lambda: T \to M(S)$ *such that*

(1) $Af = \lambda(f)$ *for all* $f \in X$ *and*

$$||\lambda|| = \sup\{||Af||: f \in X, ||f|| \leqq 1\} < \infty .$$

(2) *The adjoint of* $A$, $A^*$, *takes* $M(T)$ *into* $M(S)$ *and is given by*

$$(A^*\mu)(E) = \int_T \lambda(x, E)\mu(dx) .$$

(3) *Under the natural imbeddings of* $B(S)$ *and* $B(T)$ *into* $M(S)^*$ *and* $M(T)^*$ *respectively we have for* $f \in B(S)$
$\lambda(f) = A^{**}f$ *where* $A^{**}$ *is the adjoint of* $A^*$ *restricted to* $M(T)$ *Hence* $A^{**}(B(S)) \subset B(T)$ *and* $A^{**}$ *defines a continuous extension of* $A$ *to* $B(S)$ *into* $B(T)$.

THEOREM 4. *Let* $\lambda: T \to M(S)$. *If* $\lambda(f) \in C(T)$ *for all* $f \in C_c(S)$ *and* $\lambda$ *satisfies condition* $E'$ *then* $\lambda(f)$ *is a continuous function on* $T$ *for* $f \in C(S)$. *Conversely, if* $S$ *is paracompact and* $\lambda(f)$ *is continuous for* $f \in C(S)$ *then* $\lambda$ *satisfies condition* $E$.

THEOREM 5. *Let* $\lambda: T \to M(S)$ *and* $A$ *the linear operator on* $C(S)$ *defined by* $Af = \lambda(f)$. *Then* $A$ *is a continuous operator from* $C(S)_{\beta'}$ *into* $C(T)_{\beta'}$ *or* $C(T)_\beta$ *if and only if* $||\lambda|| < \infty$, $\lambda(f) \in C(T)$ *for* $f \in C_c(S)$ *and* $\lambda$ *satisfies condition* $E'$.

COROLLARY 1. *Let* $A: C_0(S) \to Y$ *where* $Y$ *is as in Theorem* 3. *Then* $A^{**}$ *is a continuous operator from* $C(S)_{\beta'}$ *into* $C(T)_{\beta'}$ *if and only if the kernel* $\lambda$ *satisfies condition* $E'$. *Moreover* $A^{**}$ *is the only extension of* $A$ *to* $C(S)$ *given by a kernel and consequently is the only* $\beta$ *or* $\beta'$ *continuous extension of* $A$ *to* $C(S)$.

*Proof of Theorem* 1. Let $U$ be an open subset of $S$ and let $\chi$ denote its characteristic function. Since $\lambda(x)$ is regular it follows that $\lambda(x, U) = \sup\{\lambda(f)(x): 0 \leqq f \leqq \chi, f \in C_c(S)^+\}$. Since $\lambda(f)$ is lower semi-continuous for each $f \in C_c(S)^+$, then $\lambda(\cdot, U)$ is lower semi-continuous and hence Borel-measurable. Let $\Sigma$ denote the class of Borel sets $E$

for which $\lambda(\cdot, E)$ is measurable. Then $\Sigma$ contains all open sets and is closed under countable unions of mutually disjoint sets $E \in \Sigma$ and, if $A, B \in \Sigma$ and $A \supset B$ then $A - B \in \Sigma$. It now follows from [6, p. 2] that $\Sigma = \sigma(\Delta)$ and (1) is proven.

We now prove (2). If $U$ is an open set then as a consequence of the Riesz Representation Theorem we have

$$|\lambda|(x, U) = \sup\{|\lambda(f)(x)| : f \in C_c(S), \|f\| = 1 \text{ and support } (f) \subset U\} \quad \text{for each } x \in T.$$

This means that $|\lambda|(\cdot, U)$ is lower semi-continuous and as in the proof of (1) that $|\lambda|(\cdot, E)$ is measurable for each Borel set $E$.

We can suppose for the remainder of the proof that $\lambda(x)$ is a real signed measure for each $x \in T$ and we then have [5, p. 123] that $\lambda(x) = \lambda(x)^+ - \lambda(x)^-$ where $\lambda(x)^+, \lambda(x)^- \in M(S)^+$ and $|\lambda(x)| = \lambda(x)^+ + \lambda(x)^-$ for all $x \in T$. We show that $\lambda^+, \lambda^-$ satisfy condition (1).

Let $f \in C_c(S)^+$ and set $\mu(x, E) = \int_E f(y)\lambda(x, dy)$. Then for each $x$, $\mu(x) \in M(S)$ and for

$$g \in C_c(S),\ \mu(g) = \int_S g(y)f(y)\lambda(x, dy) = \lambda(gf).$$

Hence $\mu(g)$ is continuous for each $g \in C_c(S)$ and therefore from what we have just shown $|\mu|(\cdot, S)$ is lower-semicontinuous since $S$ is open. But $|\mu|(x, S) = \int_S f(y)|\lambda|(x, dy)$ and therefore $|\lambda|(f)$ is lower semi-continuous for each $f \in C_c(S)^+$. Since $|\lambda|(x) = \lambda^+(x) + \lambda^-(x)$ and $\lambda(x) = \lambda^+(x) - \lambda^-(x)$ it now follows that for $f \in C_c(S)^+$, $\lambda^+(f)$ and $\lambda^-(f)$ are lower semi-continuous. But then it follows from (1) that $\lambda^+(\cdot, E)$, $\lambda^-(\cdot, E)$ and hence $\lambda(\cdot, E)$ are measurable for each Borel set $E$.

Condition (3) easily follows for we can approximate $\lambda(f)$ uniformly by means of measurable functions of the form $\sum_{i=1}^n a_i\lambda(\cdot, E_i)$.

REMARK 1. $T$ need not be Hausdorff or locally compact in Theorem 1.

*Proof of Theorem* 2. It is well known that $\mu(E) = \int_T \lambda(x, E)\nu(dx)$ defines a measure on $S$ such that $\int_S f d\mu = \int_T \lambda(f) d\nu$ for $f \in B(S)$. Hence we will only show that $\mu$ is regular.

We can assume that $\nu$ is real and $\|\nu\| = 1$. Further we can suppose that $\lambda(x) \in M(S)^+$ for each $x \in T$. For we can first assume that $\lambda(x)$ is a real signed measure, and writing $\lambda(x) = \lambda(x)^+ - \lambda(x)^-$, the proof of Theorem 1 shows that for $f \in C_c(S)^+, \lambda^+(f)$ and $\lambda^-(f)$ are lower semi-continuous. Hence we have the condition (1) of Theorem 1 and additionally, $\|\lambda\| = \sup\{\|\lambda(x)\| : x \in S\} < \infty$.

LEMMA 1. *Let $U$ be an open set in $S$, $\chi$ its characteristic function. Let $X = \{f \in C_c(S): 0 \leqq f \leqq \chi\}$, $Y = \{g \in C_c(T): 0 \leqq g \leqq \lambda(\cdot, U)\}$. Then*

$$\sup\left\{\int_T g d\nu : g \in Y\right\} \leqq \sup\left\{\int_T \lambda(f) d\nu : f \in X\right\}.$$

*Proof.* Let $g \in Y$, $\varepsilon > 0$ and let $g$ vanish outside the compact set $K$ and fix $x \in K$.

Since $g \in Y$ then $g(x) - \varepsilon/2 < \lambda(x, U)$ and hence there is a function $f \in X$ such that $g(x) - \varepsilon/2 < \lambda(f)(x)$. Since $\lambda(f)$ is lower semicontinuous there is a neighborhood $V$ of $x$ such that for $t \in V$ one has $g(x) - \varepsilon/2 < \lambda(f)(t)$. But also there is a neighborhood $V'$ of $x$ such that if $t \in V'$ then $g(t) - \varepsilon < g(x) - \varepsilon/2$. Hence there is a neighborhood $W$ of $x$ such that for $t \in W$, $g(t) - \varepsilon < \lambda(f)(t)$. We extract a finite cover of sets $W$ of $K$ with associated functions $f \in X$. If we let $h$ be the pointwise maximum of the corresponding functions $f$ then $h \in X$ and for $t \in K$ we have

$$g(t) - \varepsilon < \lambda(h)(t).$$

Hence $\int_T g d\nu - \varepsilon < \int_T \lambda(h) d\nu$ and the proof is complete.

LEMMA 2. $\int_T \lambda(x, U)\nu(dx) \leqq \sup\left\{\int_T g d\nu : g \in Y\right\}$.

*Proof.* Let $\varepsilon > 0$ and $n$ be an integer such that $n\varepsilon > \|\lambda\| \geqq (n-1)\varepsilon$. Then set

$$E_k = \{x \in T: k\varepsilon < \lambda(x, U) \leqq (k+1)\varepsilon\} \quad \text{for } k = 0, 1, \cdots, n-1.$$

Then $\{E_k\}$ is a partition of $T$ by Borel sets and

$$0 \leqq \int_T \lambda(x, U)\nu(dx) - \sum_{k=0}^{n-1} k\varepsilon\nu(E_k) < \varepsilon. \tag{1}$$

Let

$$U_k = \{x: \lambda(x, U) > k\varepsilon\}.$$

Then $U_k$ is an open set and $E_k = U_k - U_{k+1}$. Since $\nu$ is regular then for each $k$ there is a compact set $K_k \subset E_k$ such that $\nu(E_k - K_k) < \varepsilon/n^2$. We can then find for each $k$ an open set $V_k$ with compact closure contained in $U_k$ and containing $K_k$. Further there exist functions $f_k \in C_c(T)^+$ for $k = 0, \cdots, n-1$ such that $f_k(x) = k\varepsilon$ for $x \in K_k$, $f_k(x) = 0$ for $x \in T - V_k$ and $0 \leqq f_k(x) \leqq k\varepsilon$ for all $x \in T$. Therefore $f_k(x) \leqq k\varepsilon < \lambda(x, U)$ for $x \in U_k$ and hence $f_k \in Y$. We let

$$f(x) = \max\{f_k(x): 0 \leqq k \leqq n-1\}.$$

It follows that $f \in Y$ and

$$f(x) \leqq \sum_{k=0}^{n-1} k\varepsilon\chi_k(x),$$

where $\chi_k$ denotes the characteristic function of the set $E_k$. We then have

$$\begin{aligned} 0 &\leqq \int_T \sum_0^{n-1} k\varepsilon\chi_k d\nu - \int_T f d\nu \\ &\leqq \sum_0^{n-1} \int_{E_k} (k\varepsilon - f_k) d\nu \\ &= \sum_0^{n-1} \int_{E_k - K_k} (k\varepsilon - f_k) d\nu \\ &\leqq \sum_0^{n-1} \int_{E_k - K_k} k\varepsilon d\nu \\ &\leqq \sum_0^{n-1} k\varepsilon^2/n^2 \leqq \varepsilon^2 . \end{aligned}$$

But

$$\int_T \sum_0^{n-1} k\varepsilon\chi_k d\nu = \sum_0^{n-1} k\varepsilon\nu(E_k)$$

and applying (1) we have

$$0 \leqq \int_T \lambda(x, U)\nu(dx) - \int_T f d\nu \leqq \varepsilon^2 + \varepsilon$$

completing the proof.

LEMMA 3. $\mu(U) = \sup\left\{\int_S f d\mu : f \in X\right\}$ *and* $\mu$ *is regular.*

*Proof.* Combining Lemma 1 and Lemma 2 we have

$$\mu(U) \leqq \sup\left\{\int_T \lambda(f) d\nu : f \in X\right\}.$$

But $\int_S f d\mu = \int_T \lambda(f) d\nu$ and therefore

$$\mu(U) \leqq \sup\left\{\int_S f d\mu ; f \in X\right\} \leqq \mu(U).$$

Now the mapping $f \rightarrow \int_S f d\mu$ defines a bounded linear form on the space $C_0(S)$ and hence there is a measure $\omega \in M(S)^+$ such that $\int_S f d\mu = \int_S f d\omega$ for all $f \in C_0(S)$ and since $\omega$ is regular

$$\omega(U) = \sup\left\{\int_S f d\omega : f \in X\right\} = \mu(U)\,.$$

This means the collection $\Sigma$ of all Borel sets $E$ for which $\omega(E) = \mu(E)$ contains all open sets and it follows from [6, p. 2] as in the proof of (1) Theorem 1 that $\Sigma$ is the class of all Borel sets. Hence $\mu$ is the regular measure $\omega$. It is easily seen that $|u|(E) \leqq \int_T |\lambda|(x, E)|\nu|(dx)$ and the proof is complete.

*Proof of Theorem* 3. From [1], [4] and the Riesz Representation Theorem, $X^* = M(S)$ and $Y^* \supset M(T)$. From [9, pp. 38–39]

$$A^*(M(T)) \supset M(S)$$

and the formula $\lambda(x) = A^*\mathring{x}$, where $\mathring{x}(E) = 1$ if $x \in E$, $0$ if $x \notin E$, defines a map $\lambda\colon T \to M(S)$ satisfying (3) of Theorem 1 since $\|\lambda\| = \sup\{\|Af\|: \|f\| \leqq 1, f \in C_0(S)\} < \infty$ because the norm, $\beta$ and $\beta'$ bounded sets are the same (see [1] and [4]) and from [9, p. 45] $A$ takes bounded sets into bounded sets. Furthermore $Af = \lambda(f)$ for $f \in X$ and if $\nu(E) = \int_T \lambda(x, E)\mu(dx)$ then

$$\int_S f d\nu = \int_T \lambda(f) d\mu = \int_T Af d\mu = \int_S f d(A^*\mu)$$

for all $f \in X$ and consequently $A^*\mu = \nu$ since $\nu$ is regular. Finally if $A^{**}$ is the adjoint of $A^*$ restricted to $M(T)$ then for $\mu \in M(T)$ and

$$f \in B(S)\ [A^{**}f](\mu) = f(A^*\mu) = \int_S f d(A^*\mu) = \int_T \lambda(f) d\mu = [\lambda(f)](\mu)$$

since $\lambda(f) \in B(T)$. This holds for all $u \in M(T)$ and consequently $A^{**}f = \lambda(f)$. Hence $A^{**}(B(S)) \subset B(T)$ and $\|A^{**}\| = \|\lambda\|$.

REMARK 2. If for each $t \in [0, \infty]$, $T(t)$ is a continuous operator from $X$ to $X$ and $T(t + u) = T(t)T(u)$ then $T(t + u)^{**} = T(t)^{**}T(u)^{**}$. If we then write $[T(t)f](x) = \int_S f(y)\lambda_t(x, dy)$, then by the above theorem $\lambda_t(f) = T(t)^{**}f$ for $f \in B(S)$. If $\chi$ is the characteristic function of the Borel set $E$ we have

$$\lambda_{t+u}(\chi) = \lambda_t(\lambda_u(\chi))$$

or the Chapmann-Kolmogorov equation

$$\lambda_{t+u}(x, E) = \int_S \lambda_u(y, E)\lambda_t(x, dy)\,.$$

Consequently a transition function $\lambda_t(x, \cdot)$ can be obtained for a semigroup of $\beta$ or $\beta'$ continuous operators on the space $C(S)$ when $S$ is locally compact.

REMARK 3. One can obtain a kernel $\lambda$ satisfying (1) under the weaker condition that $A$ have range $B(T)$ and domain $C_0(S)$. For the set of linear mappings $f \to \lambda(f)(x)$ for $x \in T$ is pointwise bounded and hence uniformly bounded since $C_0(S)$ is a Banach space.

*Proof of Theorem* 4. For each compact set $K \subset S$ there is a function $\varphi_K \in C_c(S)$ such that $\varphi_K \equiv 1$ on $K$. If $f \in C(S)$ then the net $\{\varphi_K f\} \subset C_c(S)$ converges $\beta'$ to $f$ since it is uniformly bounded and $\beta$ convergent to $f$. Consequently $C_c(S)$ is $\beta'$ dense in $C(S)$. If $x \in T$ and $U$ is a neighborhood of $x$ with compact closure then $\{\lambda(x_\alpha) : x_\alpha \in U\}$ is a $\beta'$-equicontinuous set of linear functionals on $C(S)$ for any net $\{x_\alpha\} \subset U$ converging to $x$. By hypothesis $\lambda(x_\alpha) \to \lambda(x)$ on $C_c(S)$. Since $C_c(S)$ is $\beta'$ dense and $\{\lambda(x_\alpha)\}$ is $\beta'$-equicontinuous, $\lambda(x_\alpha) \to \lambda(x)$ on $C(S)$. Hence $\lambda(f)$ is continuous at $x$ for all $f \in C(S)$.

Conversely if $\lambda(f) \in C(T)$ for $f \in C(S)$ then for any compact set $K \subset T$ $\{\lambda(x): x \in K\}$ is weak-* compact as as ubset of the dual of $C(S)_\beta$ and, as Conway [3] has shown, must be $\beta$-equicontinuous.

*Proof of Theorem* 5. Suppose that $A$ is continuous from $C(S)_{\beta'}$ to $C(T)_{\beta'}$ or $C(T)_\beta$. Then $\|\lambda\| < \infty$ by Theorem 3 and if $K$ is a compact set in $T$ and $V$ is the $\beta$ neighborhood of 0 defined by some function $\varphi \in C_0(T)$ identically 1 on $K$ there is a $\beta'$ neighborhood of 0, $U$, such that $A(U) \subset V$. That is, $|\lambda(f)(x)| \leqq 1$ for all $f \in U$ and $x \in K$. Consequently $\lambda$ satisfies condition $E'$.

Conversely, let us show $A$ is continuous from $C(S)_{\beta'}$ into $C(T)_{\beta'}$ Let $V$ be a $\beta'$ neighborhood of 0 in $C(T)$ and $r > 0$. We show there is a $\beta$ neighborhood $U$ of 0 in $C(S)$ such that $A^{-1}(V) \supset B_r \cap U$ thus showing that $A^{-1}(V)$ is a $\beta'$ neighborhood.

Let $p = r\|\lambda\|$. There is a $\phi \in C_0(T)$ such that

$$V \supset B_p \cap \{g: P_\phi(g) \leqq 1\} \quad \text{and} \quad \phi \geqq 0 .$$

Let $K = \{t: |\phi(t)| \geqq 1/(p+1)\}$. Since $\lambda$ satisfies condition $E'$ there is a $\beta'$ neighborhood $U_0$ in $C(S)$ such that $|\lambda(f)(x)| \leqq 1$ for all $f \in U_0$ and $x \in K$. Let $W = \{f \in C(S): \|\phi\| f \in U\}$. Then $A^{-1}(V) \supset B_r \cap W$ for if $f \in B_r \cap W$ then $Af \in B_p$ and $|\phi(x)[Af](x)| < p/(p+1)$ for $x \notin K$ while for $x \in K$, $|\phi(x)[Af](x)| \leqq \|\phi\| \, |[Af](x)| \leqq 1$ since $\|\phi\| f \in U_0$. Hence

$$A^{-1}(V) \supset A^{-1}(B_p) \cap A^{-1}\{g: P_\phi(g) \leqq 1\} \supset B_r \cap (B_r \cap W) = B_r \cap W .$$

We then choose a $\beta$ neighborhood $U$ such that $W \supset B_r \cap U$ completing the proof.

REMARK 4. If $A$ is continuous from $C(S)_\beta$ into $C(T)_{\beta'}$ it follows that $\lambda$ satisfies $E$.

The proof of Corollary 1 is almost immediate. As a consequence of Theorem 3 and Theorem 5 continuity from $C(S)_{\beta'}$ to $C(T)_{\beta'}$ is equivalent to condition $E'$. If $A'$ is an extension of $A$ to $C(S)$ into $C(T)$ given by a kernel $\mu$ then $\mu = \lambda$ on $C_0(S)$ and consequently $\mu = \lambda$ on $C(S)$ and $A = A'$. Since by Theorem 3 any $\beta$ or $\beta'$ continuous extension is given by a kernel this shows that $A^{**}$ is unique.

It should be noted that if $S$ is paracompact and $A$ is any operator on $C(S)$ into $C(T)$ given by a bounded kernel $\lambda$ then by Theorems 4 and 5, $A$ is continuous from $C(S)_\beta$ to $C(T)_{\beta'}$.

We conclude with a brief remark on operators from $M(T)$ into $M(S)$. Suppose $B$ is such a linear operator and $B^*$ its adjoint on $B(S)$. Define $\lambda$: $T \rightarrow M(S)$ by $\lambda(x) = B\mathring{x}$ where $\mathring{x}$ is the measure defined in the proof of Theorem 3. If $B$ is bounded and $B^*(C_c(S)) \subset C(T)$ then $B^*(B(S)) \subset B(T)$ by Theorem 1. By Theorem 2,

$$(B\mu)(E) = \int_T (B\mathring{x})(E)\mu(dx) .$$

If $\lambda$ satisfies condition $E'$ then by Theorem 5 $B$ is the adjoint of the continuous operator $B^*$ from $C(S)_{\beta'}$ to $C(T)_{\beta'}$. Thus $B$ is completely determined by its action on the point measures $\{\mathring{x}: x \in T\}$.

REMARK 5 (added January 13, 1967). One can amplify Remark 4 by observing that if, moreover, $\lambda$ satisfies $E$ then Theorem 5 remains true with $\beta'$ replaced by $\beta$. For then $A$ is continuous from $C(S)_{\beta'}$ to $C(T)_\beta$ and using condition $E$, [3], part (2) of Theorem 3 and [9, p. 39] it follows that $A^*$ takes $\beta$-equicontinuous sets of $M(T)$ into $\beta$-equicontinuous sets of $M(S)$ making $A$ continuous on $C(S)_\beta$ into $C(T)_\beta$.

REMARK 6. It has recently come to the author's attention that a version of Theorem 2 can be found on page 176 of the recent book by P. A. Meyer, *Probability and Potentials*, Blaisdell, Waltham, Massachusetts, 1966, under the conditions that $S$ be $\sigma$-compact, $\lambda: S \rightarrow M(S)^+$, $\lambda(f)$ be continuous for all $f \in C_c(S)^+$ and that $\nu$ have compact support.

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