Pacific Journal of Mathematics

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Vol. 23, No. 1 March 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 1, 1967

# MULTIPLY TRANSITIVE GROUPS OF TRANSFORMATIONS

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**A group $G$ of homeomorphisms of a topological space $X$ onto itself is called $n$-transitive if any set of $n$ points in $X$ can be mapped onto any other set of $n$ points by some member of $G$. In this paper, we investigate the transitivity of $G$ when $X$ is euclidean $m$-space $E^m$ or real projective $m$-space $\Pi^m$, and $G$ properly contains the group $A_m$ of affine transformations or the group $P_m$ of projective transformations, respectively. We show that $G \supset A_1$ implies that $G$ is at least 3-transitive, $G \supset P_1$ implies that $G$ is at least 4-transitive, and, for a fairly wide class of groups, $G$ is $n$-transitive for every $n$. For higher dimensional spaces, our information is considerably more meager. We show that $G \supset A_m$ or $G \supset P_m$ implies that $G$ is at least 3-transitive, and that if some member of $G$ leaves fixed the points of some open set, then $G$ is $n$-transitive for every $n$.**

2. Multiple transitivity. Let $X$ be a topological space and $H(X)$ the group of all homeomorphisms of $X$ onto itself. The identity of $H(X)$ will be denoted by $e$. For each $h \in H(X)$, we set $K(h) = \{x \in X: h(x) = x\}$, and observe that

$$K(h_1h_2) \supset K(h_1) \cap K(h_2) , \qquad K(h_1h_2h_1^{-1}) = h_1(K(h_2)) .$$

For any subgroup $G$ of $H(X)$ and any $x \in X$, we call $G(x) = \{g(x): g \in G\}$ an orbit of $G$ and note that orbits are either coincident or disjoint. When $n$ is a positive integer, we define $G$ to be $n$-transitive if, for any subsets $\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_n\}$ of $n$ distinct points in $X$, we can find $g \in G$ such that $g(x_i) = y_i$ $(i = 1, \cdots, n)$. If $g$ is unique, we call $G$ strictly $n$-transitive. If $G$ is $n$-transitive for every $n$, we will call $G$ $\omega$-transitive. When $X$ is a connected, locally euclidean manifold of dimension $m \geqq 2$, then $H(X)$ is clearly $\omega$-transitive, but $H(E^1)$ is only 2-transitive, and $H(\Pi^1)$ is only 3-transitive under the above definition. To remedy this, we will modify the definition in these two cases by requiring that as $i$ increases from 1 to $n$, $x_i$ should move in the positive sense of orientation, and $y_i$ should move in either the positive or negative sense. Thus $H(X)$ is also $\omega$-transitive when $X = E^1$ or $\Pi^1$. The group $H^+(X)$ of orientation-preserving homeomorphisms of $X$ evidently sends any positively oriented $n$-tuple into any other positively oriented $n$-tuple for every $n$. We will say that a subgroup $G$ of $H^+(X)$ is $n$-transitive relative to $H^+(X)$ if $G$ sends any positively oriented $n$-tuple into any other positively oriented $n$-tuple.

LEMMA 1. *Let $X$ be a topological space and $G$ a subgroup of $H(X)$. Suppose that, for each subset $L$ of $n$ points in $X$ and each $x \in X - L$, the orbit $G_0(x)$ of the group $G_0 = \{g \in G: L \subset K(g)\}$ has a nonempty interior in $X$. Then $G_0(x)$ contains a connected component of $X - L$.*

*Proof.* Let $U \subset G_0(x)$ be an open subset of $X$, and $y \in G_0(x)$ be arbitrary. Then we can find $g_1, g_2 \in G_0$ with the properties $g_1(x) \in U$ and $g_2(x) = y$. Thus $y = g_2(x) \in g_2 g_1^{-1}(U) \subset G_0(x)$, and $y$ lies in the interior of $G_0(x)$, so that $G_0(x)$ is open. The orbits of $G_0$ are either coincident or disjoint, and no two of them can intersect the same connected component of $X - L$ unless they coincide. Since $e \in G_0$, we have $x \in G_0(x)$, and the orbits $G_0(x)$ cover $X - L$. Hence, each of them contains a connected component.

LEMMA 2. *With the same hypotheses as in Lemma 1, suppose $X$ is a connected, locally euclidean manifold of dimension $m \geqq 2$, and $G$ is $n$-transitive for some $n$. Then $G$ is $(n + 1)$-transitive.*

*Proof.* To show that $G$ is $(n + 1)$-transitive, it is evidently sufficient to show that, for any points $x_1, \cdots, x_{n+1}, y_{n+1} \in X$, there is a $g \in G$ satisfying $g(x_i) = x_i$ $(i = 1, \cdots, n)$ and $g(x_{n+1}) = y_{n+1}$. Since $X - \{x_1, \cdots, x_n\}$ is connected, this is precisely the conclusion of Lemma 1.

LEMMA 3. *With the same hypotheses as in Lemma 1, suppose $X = E^1$, $G$ is $n$-transitive for some $n \geqq 2$, and the condition "$x \in X - L$" is replaced by "$x$ lies to the right of $L$". Then $G$ is $(n + 1)$-transitive. If $G \subset H^+(E^1)$ is $n$-transitive $(n \geqq 0)$ relative to $H^+(E^1)$, then $G$ is $(n + 1)$-transitive relative to $H^+(E^1)$.*

*Proof.* Let $x_1 < \cdots < x_{n+1}$ and either (i) $y_1 < \cdots < y_{n+1}$ or (ii) $y_1 > \cdots > y_{n+1}$ be given. In case (i), we choose $g_1 \in G$ so that $g_1(x_i) = y_i$ $(i = 1, \cdots, n)$. Since $g_1$ is order-preserving, we have $g_1(x_{n+1}) > y_n$, and the same argument as in the proof of Lemma 1 shows that the orbit $G_0(g_1(x_{n+1}))$ is the open interval $(y_n, \infty)$, where $G_0 = \{g \in G: \{y_1, \cdots, y_n\} \subset K(g)\}$. Thus we can find $g_2 \in G_0$ satisfying $g_2(g_1(x_{n+1})) = y_{n+1}$, so that $g_2 g_1(x_i) = y_i$ $(i = 1, \cdots, n + 1)$. This also suffices to prove the last statement in the Lemma. In case (ii), we choose $g_3 \in G$ so that $g_3(x_i) = y_i$ $(i = 2, \cdots, n + 1)$. From $n \geqq 2$ we infer that $g_3$ is order-reversing, whence $g_3(x_1) > y_2$, and we can find $g_4 \in G$ satisfying $y_i \in K(g_4)$ $(i = 2, \cdots, n + 1)$ and $g_4(g_3(x_1)) = y_1$. Thus $g_4 g_3(x_i) = y_i$ $(i = 1, \cdots, n + 1)$.

If, in the hypothesis of Lemma 3, "$x$ lies to the right of $L$" is replaced by "$x$ lies to the left of $L$", then an argument similar to the preceding one yields the same conclusions.

3. **Extensions of finite sets.** Let $L$ be a finite subset of an arbitrary subset $M$ of a topological space $X$, and $G$ a subgroup of $H(X)$. We set $M_0 = M$ and, for $i \geqq 0$,

$$M_{i+1} = \bigcup \{g(M_i) \cup g^{-1}(M_i) \colon g \in G \text{ and } g(L) \subset M_i\} .$$

Since $e \in G$ and $L \subset M_0$, we have $M_0 \subset M_1$ and, in general, $M_i \subset M_{i+1}$. Thus $\{M_i\}$ is an increasing family of sets, and we shall call its union $N$ the extension of $M$ with respect to $L$ and $G$. We observe that if $g \in G$ and $g(L) \subset N$, then $g(N) = N$. For $g(L)$ is finite and so is contained in some $M_k$, whence $g(M_i) \subset M_{i+1}$ and $g^{-1}(M_i) \subset M_{i+1}$ for each $i \geqq k$. Hence, $g(N) \subset N$, $g^{-1}(N) \subset N$, and $g(N) = N$.

LEMMA 4. *Suppose $X$ is a Hausdorff space, $L$ has $n$ points, $G$ is $n$-transitive and has the property that, for any net $\{g_k\}$ in $G$ and any $g \in G$, $\lim_k g_k(x) = g(x)$ for all $x \in L$ implies*

$$\lim_k g_k(x) = g(x) , \qquad \lim_k g_k^{-1}(x) = g^{-1}(x) , \qquad x \in X .$$

*Then $g(L) \subset \bar{N}$ implies $g(\bar{N}) = \bar{N}$, where $\bar{N}$ is the closure of $N$.*

*Proof.* If $L = \{x^1, \cdots, x^n\}$ and $g(L) \subset \bar{N}$, then we can find a net $\{(x_k^1, \cdots, x_k^n)\}$ of $n$-tuples in $N$ such that $\lim_k x_k^i = g(x^i)$ $(i = 1, \cdots, n)$. The $n$-transitivity of $G$ implies that there are elements $g_k \in G$ satisfying $g_k(x^i) = x_k^i$ for each $i$ and $k$. Thus

$$\lim_k g_k(x^i) = \lim_k x_k^i = g(x^i) , \qquad i = 1, \cdots, n$$

implies

$$\lim_k g_k(x) = g(x) , \qquad \lim_k g_k^{-1}(x) = g^{-1}(x) , \qquad x \in X .$$

From the remark preceding the lemma, $g_k(L) \subset N$ implies $g_k(x)$, $g_k^{-1}(x) \in N$ for $x \in N$, whence $g(x), g^{-1}(x) \in \bar{N}$ for $x \in N$. Consequently, $g(N) \subset \bar{N}$, $g(\bar{N}) \subset \bar{N}$, $g^{-1}(N) \subset \bar{N}$, $g^{-1}(\bar{N}) \subset \bar{N}$, and $g(\bar{N}) = \bar{N}$.

LEMMA 5. *Let $X$ be $m$-dimensional euclidean space $E^m$, $G$ the group $A_m$ of affine transformations defined on $E^m$, $L$ consist of $m + 1$ points which do not lie on any $(m - 1)$-dimensional hyperplane, and $M \supset L$ consist of $m + 2$ points. Then $N$ is dense in $E^m$.*

*Proof.* We recall that the elements $a$ of $A_m$ have the form

$a(x) = t + Tx$, where $t \in E^m$, and $T$ is a nonsingular linear transformation of $E^m$ onto itself. Moreover, $A_m$ is strictly $(m+1)$-transitive on $(m+1)$-tuples which do not lie on any $(m-1)$-dimensional hyperplane. We first consider the case $m = 1$. The hypothesis of Lemma 4 is clearly satisfied with $n = 2$. Let $L = \{x_1, x_2\}$ and $M = \{x_1, x_2, x_3\}$. Evidently we can arrange the indices so that either (i) $x_1 < x_2, x_1 < x_3$ or (ii) $x_1 > x_2, x_1 > x_3$. We will complete the proof for case (i); case (ii) is handled in exactly the same way. Choose $a_1 \in A_1$ so that $a_1(x_1) = x_1$ and $a_1(x_2) = x_3$. Then $a_1(L) \subset N$, and the remark preceding Lemma 4 implies that $a_1(N) = N$. Indeed, $a_1^k(N) = N$ for any integer $k$, where $a_1^k$ is the $k$-th iterate of $a_1$. Now $a_1$ is order-preserving and has just one fixed point at $x_1$, so that $\{a_1^k(x_2): -\infty < k < +\infty\}$ has $x_1$ and $+\infty$ as limit points. In other words, $N$ contains a sequence which converges to $x_1$ from the right and another which converges to $+\infty$. If $\bar{N} \neq E^1$, then $E^1 - \bar{N}$ is the union of disjoint open intervals. Let $I = (\lambda, \mu)$ be one of these, where we allow $\lambda = -\infty$ or $\mu = +\infty$. If $\lambda \neq -\infty$, we can find $a_2 \in A_1$ satisfying $a_2(x_1) = \lambda$ and $\lambda < a_2(x_2) \in N$, whence $a_2$ is order-preserving, $a_2(L) \subset \bar{N}$, $a_2(\bar{N}) = \bar{N}$, and $a_2^{-1}(I) \subset E^1 - \bar{N}$. But $a_2^{-1}(\lambda)$ is the left endpoint of $a_2^{-1}(I)$, while $a_2^{-1}(\lambda) = x_1$ has a sequence in $\bar{N}$ converging to it from the right, so that part of this sequence must lie in $a_2^{-1}(I)$, which is impossible. If $\lambda = -\infty$, then $\mu \leqq x_1$, and we choose $a_3 \in A_1$ so that $a_3(x_2) = x_2, x_1 < a_3(x_1) \in N$, and $a_3(x_1) < x_2$. Thus $a_3$ is order-preserving, $a_3(L) \subset N, a_3(N) = N$, and $a_3(I) \subset E^1 - \bar{N}$. But $a_3(\mu) > \mu$, and $a_3(\mu)$ is the right endpoint of $a_3(I)$, whence $\mu \in a_3(I)$, which is impossible. Therefore, $\bar{N} = E^1$.

We now proceed by induction on $m$. Suppose the lemma has been proved in all dimensions less than a certain $m$,

$$L = \{x_1, \cdots, x_{m+1}\} \subset \{x_0, x_1, \cdots, x_{m+1}\} = M \subset E^m ,$$

and $L$ does not lie on any $(m-1)$-dimensional hyperplane. We can arrange the indices in $L$ so that either (i) $x_0$ lies on the $(m-1)$-dimensional hyperplane $X$ determined by $x_2, \cdots, x_{m+1}$, or (ii) $x_0$ and $x_1$ lie on the same side of $X$. To see this, we set up a coordinate system in $E^m$ in which the points of $L$ are the origin and unit points on the coordinate axes. If each point of $L$ lay on the side opposite $x_0$ of the $(m-1)$-dimensional hyperplane through the remaining points of $L$, then all the coordinates of $x_0$ would be negative, while $x_0$ lay on the side opposite the origin of the hyperplane through the unit points, which is impossible. In case (ii), choose $a_0 \in A_m$ so that $a_0(x_1) = x_0$ and $a_0(x_i) = x_i$ $(i = 2, \cdots, m+1)$. We will show that $x_1, a_0(x_1)$, and $a_0^2(x_1)$ are collinear. Since $K(a_0) = X$, we can refer $a_0(x) = t_0 + T_0x$ to a coordinate system in $E^m$ relative to which $x_1 = (0, \cdots, 0, 1)$, $X$ is the set of points with last coordinate 0, $t_0 = (0, \cdots, 0)$, and $T_0$ has the form

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & \alpha_1 \\ 0 & 1 & 0 & \cdots & \alpha_2 \\ 0 & 0 & 1 & \cdots & \alpha_3 \\ & \cdots & & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_m \end{pmatrix}, \qquad \alpha_m > 0 .$$

Thus we have

$$\begin{aligned} a_0(x_1) &= (\alpha_1, \cdots, \alpha_{m-1}, \alpha_m) , \\ a_0^2(x_1) &= (\alpha_1(1+\alpha_m), \cdots, \alpha_{m-1}(1+\alpha_m), \alpha_m^2) , \\ a_0(x_1) - x_1 &= (\alpha_1, \cdots, \alpha_{m-1}, \alpha_m - 1) , \\ a_0^2(x_1) - a_0(x_1) &= (\alpha_1\alpha_m, \cdots, \alpha_{m-1}\alpha_m, (\alpha_m - 1)\alpha_m) \\ &= \alpha_m(a_0(x_1) - x_1) , \end{aligned}$$

whence $x_1$, $a_0(x_1) = x_0$, and $a_0^2(x_1) = a_0(x_0) = y_0$ are collinear, and $y_0 \neq x_0, x_1$. We will show next that there is a subset $L'$ of $M$ which contains $x_0, x_1$, and $m-1$ of the remaining $m$ points of $L$, but which does not lie on any $(m-1)$-dimensional hyperplane. If $L' = \{x_0, x_1, \cdots, x_m\}$ will not work, then let $k$ be the least integer such that $2 \leqq k \leqq m$ and $\{x_0, x_1, \cdots, x_k\}$ lies on some $(k-1)$-dimensional hyperplane, and set $L' = M - \{x_k\}$. Now if $L'$ lay on an $(m-1)$-dimensional hyperplane $X_{m-1}$, then the unique $(k-1)$-dimensional hyperplane through $\{x_0, x_1, \cdots, x_{k-1}\}$ must contain $x_k$ and lie in $X_{m-1}$, so that $M \subset X_{m-1}$, which is impossible. Hence, $L' = M - \{x_k\}$ satisfies our condition. Let $x_j$ be a fixed element of $L' - \{x_0, x_1\}$, $Y$ be the $(m-1)$-dimensional hyperplane through $L'' = L' - \{x_j\}$, $M'' = L'' \cup \{y_0\}$, and $a_1 \in A_m$ map $L$ onto $L'$. Since $\{y_0, x_0, x_1\}$ is collinear, and $x_0, x_1 \in L''$, we have $M'' \subset Y$. Now $L''$ contains $m$ points, $M''$ contains $m+1$ points, and the group $B$ of elements in $A_m$ which fix $x_j$ and map $Y$ onto itself acts on $Y$ exactly like $A_{m-1}$. By our induction hypothesis, the extension $N''$ of $M''$ with respect to $L''$ and $B$ is dense in $Y$. We will show that $\bigcup M_i'' = N'' \subset N = \bigcup M_i$ by showing inductively that $M_i'' \subset N$. First, $a_0(L) \subset M$ implies $y_0 = a_0(x_0) \in a_0(M) \subset N$, so that $M_0'' = M'' \subset N$. Suppose now that $M_i'' \subset N$ for some $i$, and $b(L'') \subset M_i''$ for some $b \in B$. Then $a_1(L) = L' \subset M$ implies $a_1(N) = N$, and

$$ba_1(L) = b(L') = \{x_j\} \cup b(L'') \subset \{x_j\} \cup M_i'' \subset N$$

implies $ba_1(N) = N$. Thus $b(N) = b(a_1(N)) = N$, $b(M_i'') \cup b^{-1}(M_i'') \subset N$, and $M_{i+1}'' \subset N$, so that $N'' \subset N$. Suppose $\{y_1, \cdots, y_{m-1}\}$ is a subset of $N''$ which does not lie in any $(m-3)$-dimensional hyperplane. Since $L''$ does not lie on any $(m-2)$-dimensional hyperplane, we can find an $x_i \in L''$ such that $\{x_i, y_1, \cdots, y_{m-1}\}$ does not lie on any $(m-2)$-dimensional hyperplane. Then $\{x_i, x_j, y_1, \cdots, y_{m-1}\}$ does not lie on any $(m-1)$-dimensional hyperplane, and we can find an $a_2 \in A_m$ which maps $L'$ onto $\{x_i, x_j, y_1, \cdots, y_{m-1}\}$ in such a way that $a_2(x_i) = x_j$ and $a_2(x_j) = x_i$. From $a_2a_1(L) = a_2(L') \subset N$, we infer that $a_2a_1(N) = N$ and $a_2(N) = a_2(a_1(N)) = N$, so that $a_2(N'') \subset N$. Now $a_2(N'')$ is a dense subset of $a_2(Y)$, and $a_2(Y)$ is an $(m-1)$-dimensional hyperplane through $\{x_j\}$ and $\{y_1, \cdots, y_{m-1}\}$. The union of such hyperplanes as $\{y_1, \cdots, y_{m-1}\}$ ranges over $N''$ is clearly dense in $E^m$, whence $N$ is dense in $E^m$, and our main induction step is complete for case (ii). For case (i), the preceding argument becomes considerably simpler. We set

$$L'' = \{x_2, \cdots, x_{m+1}\}, \qquad M'' = \{x_0, x_2, \cdots, x_{m+1}\},$$

and let $B$ be the set of elements in $A_m$ which fix $x_1$ and map $X$ onto itself. Then $N'' \subset N$, and $N''$ is dense in $X$. The last part of the argument with $L' = L$, $Y = X$, and $x_j = x_1$ shows that $N$ is dense in $E^m$ in this case as well.

LEMMA 6. *The conclusion of Lemma 5 remains valid if, in the hypothesis, we set $m = 1$ and replace $A_1$ with the group $A_1^+$ of order-preserving elements in $A_1$.*

*Proof.* We observe that all of the elements in $A_1$ which appear in the proof of Lemma 5 are order-preserving. The only other lemma used in that proof was Lemma 4 which assumes that $G$ is 2-transitive. Although $A_1^+$ is only 2-transitive relative to $H^+(E^1)$, the net $\{g_k\}$ can still be found, if we recall that any pair of points which lies sufficiently close to a positively oriented pair is also positively oriented.

LEMMA 7. *Let $X$ be a topological space, $L$ consist of $n$ points, $L \subset M$, $f \in H(X)$, $G$ and $G'$ be subgroups of $H(X)$, and $G'$ have the property that if $g' \in G'$ and $K(g')$ contains $n$ points, then $g' = e$. Suppose that, for every $g \in G$, there is a $g' \in G'$ such that $fg(x) = g'f(x)$ for all $x \in M$. Then $fg(x) = g'f(x)$ for all $x$ in the extension $N$ of $M$ with respect to $L$ and $G$.*

*Proof.* We will prove the result inductively for the sets $M = M_0, M_1, M_2, \cdots$. Suppose that, for every $g \in G$, there is a $g' \in G'$ such that $fg(x) = g'f(x)$ for all $x \in M_i$, and $g_1(L) \subset M_i$, where $g_1 \in G$. If $y \in L$, then $g_1(y) \in M_i$ and

$$(1) \qquad fg(g_1(y)) = g'f(g_1(y)), \qquad y \in L.$$

We know that there are elements $g_1', g_2' \in G'$ satisfying

$$(2) \qquad fg_1(y) = g_1'f(y)\,, \qquad fgg_1(y) = g_2'f(y)\,, \qquad y \in M_i\,.$$

Combining (1) and (2) and recalling that $L \subset M_i$, we obtain

$$g_2'f(y) = fgg_1(y) = g'fg_1(y) = g'g_1'f(y)\,, \qquad y \in L\,.$$

Thus $f(y) \in K(g_2'^{-1}g'g_1')$, $f(L) \subset K(g_2'^{-1}g'g_1')$, and $f(L)$ contains $n$ points, so that $g_2'^{-1}g'g_1' = e$ and $g_2' = g'g_1'$. From (2) we have

$$fgg_1(y) = g_2'f(y) = g'g_1'f(y) = g'fg_1(y)\,, \qquad y \in M_i\,,$$

that is, $fg(x) = g'f(x)$ for all $x \in g_1(M_i)$. To see that $fg(x) = g'f(x)$ for all $x \in g_1^{-1}(M_i)$, we observe that $L \subset M_i$ implies

$$(3) \qquad fgg_1^{-1}(y) = g'fg_1^{-1}(y)\,, \qquad y \in g_1(L)\,.$$

We can also find elements $g_3', g_4' \in G'$ satisfying

$$(4) \qquad fg_1^{-1}(y) = g_3'f(y)\,, \qquad fgg_1^{-1}(y) = g_4'f(y)\,, \qquad y \in M_i\,.$$

From (3), (4), and $g_1(L) \subset M_i$ we obtain

$$g_4'f(y) = fgg_1^{-1}(y) = g'fg_1^{-1}(y) = g'g_3'f(y)\,, \qquad y \in g_1(L)\,.$$

Thus $fg_1(L) \subset K(g_4'^{-1}g'g_3')$ and $g_4' = g'g_3'$. Finally, from (4) we have

$$fgg_1^{-1}(y) = g_4'f(y) = g'g_3'f(y) = g'fg_1^{-1}(y)\,, \qquad y \in M_i\,,$$

in other words, $fg(x) = g'f(x)$ for all $x \in g_1^{-1}(M_i)$. Therefore, $fg(x) = g'f(x)$ for all $x \in M_{i+1}$, and the induction step is complete.

LEMMA 8. *With the same hypotheses as in Lemma 7, suppose $G = G'$ and $f(x) = x$ for all $x \in M$. Then $f(x) = x$ for all $x \in N$.*

*Proof.* Again we proceed by induction on the sets $M_i$. Suppose $f(x) = x$ for all $x \in M_i$, and $g_1(L) \subset M_i$, where $g_1 \in G$. Then we can find $g_1' \in G$ such that

$$fg_1(x) = g_1'f(x) = g_1'(x)\,, \qquad x \in M_i\,.$$

Since $L, g_1(L) \subset M_i$, we have

$$g_1(y) = fg_1(y) = g_1'(y)\,, \qquad y \in L\,,$$

whence $L \subset K(g_1^{-1}g_1')$ and $g_1 = g_1'$. Thus $fg_1(x) = g_1(x)$ for all $x \in M_i$, that is, $f(z) = z$ for all $z \in g_1(M_i)$. Similarly, there is a $g_2' \in G$ satisfying

$$fg_1^{-1}(x) = g_2'f(x) = g_2'(x)\,, \qquad x \in M_i\,,$$
$$g_1^{-1}(y) = fg_1^{-1}(y) = g_2'(y)\,, \qquad y \in g_1(L)\,,$$

so that $g_1^{-1} = g_2'$ and $fg_1^{-1}(x) = g_1^{-1}(x)$ for all $x \in M_i$. Therefore, $f(z) = z$ for all $z \in M_{i+1}$, and the induction step is complete.

THEOREM 1. *Suppose $X = E^1$, $L$ consists of two points, $M$ of three points, $f \in H^+(E^1)$, and, for every $a \in A_1^+$, there is an $a' \in A_1^+$ such that $fa(x) = a'f(x)$ for all $x \in M$. Then $f \in A_1^+$.*

*Proof.* The hypotheses of Lemma 7 are evidently satisfied when $n = 2$ and $G = G' = A_1^+$, whence $fa(x) = a'f(x)$ for all $x \in N$. By Lemma 6, $N$ is dense in $E^1$, and the continuity of $a$, $a'$, and $f$ implies that $fa = a'f$, that is, $fA_1^+f^{-1} \subset A_1^+$. If we choose $a_1 \in A_1^+$ so that $a_1(0) = f(0)$, $a_1(1) = f(1)$, and set $f_1 = a_1^{-1}f$, then $0, 1 \in K(f_1)$ and $f_1A_1^+f_1^{-1} \subset A_1^+$. In particular, if we define $a_2(x) = 1 + x$ for $x \in E^1$, then $a_3 = f_1a_2f_1^{-1} \in A_1^+$. Now $K(a_3) = f_1(K(a_2)) = f_1(\varnothing) = \varnothing$, so that $a_3$ is also a translation, and $a_3(0) = 1$ implies $a_3 = a_2$. Thus $2 = a_3(1) = f_1a_2f_1^{-1}(1) = f_1(2)$, and $0, 1, 2 \in K(f_1)$. Setting $M = \{0, 1, 2\}$ in Lemmas 6 and 8, we conclude that $f_1 = e$ and $f = a_1 \in A_1^+$.

**4. 3-transitive groups containing $A_m$ and $P_m$.** We are now ready to investigate the transitivity of groups of homeomorphisms of euclidean $m$-space $E^m$ or real projective $m$-space $\Pi^m$ which contain the affine group $A_m$ or the projective group $P_m$, respectively, as a proper subgroup. The groups which we will consider are all obtained by adjoining some homeomorphism to $A_m$ or $P_m$ and generating the smallest group containing them. Any larger group will obviously have at least as high a degree of transitivity. In the case $m = 1$, we will obtain slightly sharper results by adjoining an element of $H^+(E^1)$ or $H^+(\Pi^1)$ to $A_1^+$ or $P_1^+$, respectively, and considering transitivity relative to $H^+(E^1)$ or $H^+(\Pi^1)$. Then if an orientation-reversing element of $A_1$ or $P_1$ is added, the resulting group will clearly have the same degree of transitivity relative to $H(E^1)$ or $H(\Pi^1)$, respectively.

THEOREM 2. *If $f \in H^+(E^1) - A_1$, then the group $G$ generated by $f$ and $A_1^+$ is 3-transitive relative to $H^+(E^1)$.*

*Proof.* Given any three points $x_1 < x_2 < x_3$ in $E^1$, let $L = \{x_1, x_2\}$ and $M = \{x_1, x_2, x_3\}$. For each $a \in A_1^+$, we can find $a' \in A_1^+$ satisfying $a'(f(x_i)) = fa(x_i)$ $(i = 1, 2)$. If $a(x) = \alpha + \beta x$ and $a'(x) = \alpha' + \beta' x$, then $\alpha'$ and $\beta'$ must satisfy the equations

$$\alpha' + \beta' f(x_1) = f(\alpha + \beta x_1)\ ,$$
$$\alpha' + \beta' f(x_2) = f(\alpha + \beta x_2)\ ,$$

so that $\alpha'$ and $\beta'$ are continuous functions of $\alpha$ and $\beta$. We can identify

$A_1^+$ with the set of pairs $(\alpha, \beta)$ of real numbers, where $\beta > 0$. If we give $A_1^+$ the euclidean topology of a half-plane and hold $x \in E^1$ fixed, then the mapping $a \to a(x)$ or $(\alpha, \beta) \to \alpha + \beta x$ from $A_1^+$ into $E^1$ is evidently continuous. Since $f$ and $f^{-1}$ are continuous, so also is the mapping $a \to \varphi(a) = f^{-1}a'^{-1}fa(x_3)$ from $A_1^+$ into $E^1$. From Theorem 1, we know that there is at least one $a_0 \in A_1^+$ such that $a_0'f(x_3) \neq fa_0(x_3)$, for otherwise $f \in A_1^+$, contrary to our hypothesis. Thus $\varphi(a_0) \neq x_3$ while $\varphi(e) = x_3$. From the connectedness of $A_1^+$ we infer that $\varphi(A_1^+)$ is a nondegenerate interval and so contains an open set. Moreover, $f^{-1}a'^{-1}fa \in G$ and $x_1, x_2 \in K(f^{-1}a'^{-1}fa)$. By Lemma 3, $G$ is 3-transitive relative to $H^+(E^1)$.

THEOREM 3. *If $m \geqq 2$ and $f \in H(E^m) - A_m$, then the group $G$ generated by $f$ and $A_m$ is 3-transitive.*

*Proof.* We know that $A_m$ maps any noncollinear triple onto any other noncollinear triple. If we can show that $G$ maps every collinear triple onto some noncollinear triple, then we will have established that $G$ is 3-transitive. Let $M$ be a collinear triple, $L \subset M$ consist of two points, $X$ be the line through $M$, and suppose that, for every $a \in A_m$, $fa(M)$ is a collinear triple. The group $B$ of all those elements in $A_m$ which map $X$ onto itself behaves exactly like $A_1$ on $X$. By Lemma 5, the extension $N$ of $M$ with respect to $L$ and $B$ is dense in $X$. We will show by induction on the sets $M_i$ that, for every $a \in A_m$, $fa(N)$ is a collinear set. Suppose $fa(M_i)$ is a collinear set for each $a \in A_m$, and $b(L) \subset M_i$ for some $b \in B$. Then $fa(b(M_i)) = fab(M_i)$ and $fa(b^{-1}(M_i)) = fab^{-1}(M_i)$ are each collinear, and

$$\begin{aligned} &fa(M_i) \cap fa(b(M_i)) \supset fa(b(L)) \, , \\ &fa(M_i) \cap fa(b^{-1}(M_i)) \supset fa(L) \, . \end{aligned} \tag{5}$$

Since $fa(b(L))$ and $fa(L)$ each contain two points, the sets $fa(M_i)$, $fa(b(M_i))$, and $fa(b^{-1}(M_i))$ all lie on the same line, so that $fa(M_{i+1})$ is collinear, and the induction step is complete. From $\bar{N} = X$ we infer that $fa(X)$ is collinear for each $a \in A_m$. If $Y$ is any line in $E^m$, then we can choose $a_0 \in A_m$ such that $a_0(X) = Y$, whence $f(Y) = fa_0(X)$ is also collinear. Since $Y$ is closed, connected, and separated by each of its points, the same must also be true of $f(Y)$ so that $f(Y)$ is a line. Let $Y_1$, $Y_2$ be parallel lines and $Z$ a line which meets them both. Then $Y_1 \cap Y_2 = \varnothing$, and any line which meets $Z$ and $Y_1$ in distinct points mush also meet $Y_2$. Since $f$ preserves these incidence relations, we conclude that $f(Y_1)$ and $f(Y_2)$ are parallel. Let $L'$ consist of the origin and the $m$ unit points in a coordinate system for $E^m$, and let $M'$ be the set of $2^m$ vertices of the unit cube determined by $L'$.

Then $fa(M')$ is the set of vertices of a parallelotope for each $a \in A_m$, and we can find $a' \in A_m$ satisfying $fa(x) = a'f(x)$ for all $x \in M'$. If we select $a_1 \in A_m$ so that $a_1(x) = f(x)$ for all $x \in M'$ and set $f_1 = a_1^{-1}f$, then $M' \subset K(f_1)$ and

$$f_1a(x) = a_1^{-1}fa(x) = a_1^{-1}a'f(x) = a_1^{-1}a'a_1f_1(x)\ , \qquad x \in M' .$$

We infer from Lemmas 5 and 8 that $f_1 = e$ and $f = a_1$, which contradicts the hypothesis of our theorem. Hence, $fa(M)$ is not collinear for some $a \in A_m$.

The conclusion of Theorem 3 seems especially weak in view of the fact that $A_m$ itself is $(m + 1)$-transitive on subsets which do not lie on any $(m - 1)$-dimensional hyperplane. The difficulty in extending our method to higher transitivity comes from (5). If we knew, for example, that $fa(b(L))$ and $fa(L)$ each contained three points, it would not follow that these triples were noncollinear, and we could not conclude that $fa(M_i)$, $fa(b(M_i))$, and $fa(b^{-1}(M_i))$ were coplanar.

LEMMA 9. *Suppose the group $F$ generated by $A_1^+$ and $f \in H^+(E^1)$ is $n$-transitive relative to $H^+(E^1)$. If we extend $f$ to an element $\bar{f}$ of $H^+(\Pi^1)$ by making $\bar{f}$ fix the point at infinity, then the group $G$ generated by $P_1^+$ and $\bar{f}$ is $(n + 1)$-transitive relative to $H^+(\Pi^1)$.*

*Proof.* An element $p \in P_1^+ = P_1 \cap H^+(\Pi^1)$ has the form $p(x) = (\alpha x + \beta)/(\gamma x + \delta)$, where $\alpha\delta - \beta\gamma > 0$. We can identify $A_1^+$ with the subgroup of $P_1^+$ which leaves fixed the point $\infty$ at infinity. Suppose that $\{x_1, \cdots, x_{n+1}\}$ and $\{y_1, \cdots, y_{n+1}\}$ are given such that, as $i$ increases from 1 to $n + 1$, $x_i$ and $y_i$ each move in the positive sense of orientation. Choose $p_0, p_1 \in P_1^+$ so that $p_0(x_1) = \infty$ and $p_1(y_1) = \infty$. Then $\{p_0(x_2), \cdots, p_0(x_{n+1})\}, \{p_1(y_2), \cdots, p_1(y_{n+1})\} \subset \Pi^1 - \{\infty\}$, and the points in each set increase with $i$. Thus we can find $g_0 \in F$ satisfying $g_0(p_0(x_i)) = p_1(y_i)$ $(i = 2, \cdots, n + 1)$, and $g_1 = p_1^{-1}\bar{g}_0p_0 \in G$ must satisfy $g_1(x_i) = y_i$ $(i = 1, \cdots, n + 1)$.

THEOREM 4. *If $f \in H^+(\Pi^1) - P_1^+$, then the group $G$ generated by $f$ and $P_1^+$ is 4-transitive relative to $H^+(\Pi^1)$.*

*Proof.* Let $f(\infty) = x_0$, and choose $p_0 \in P_1^+$ so that $p_0(x_0) = \infty$. Then $p_0f(\infty) = \infty$, and the restriction $f_0$ of $p_0f$ to $\Pi^1 - \{\infty\} = E^1$ belongs to $H^+(E^1)$. Theorem 2 says that the group $F$ generated by $f_0$ and the set $A_1^+$ of those elements of $P_1^+$ which fix $\infty$ is 3-transitive relative to $H^+(E^1)$, and Lemma 9 gives the desired result.

THEOREM 5. *If $m \geqq 2$ and $f \in H(\Pi^m) - P_m$, then the group $G$ generated by $f$ and $P_m$ is 3-transitive.*

*Proof.* Since $P_m$ maps any noncollinear triple onto any other noncollinear triple, our result will be proved if we can show that, for any collinear triple $M$, there is a $p \in P_m$ such that $fp(M)$ is noncollinear. Suppose that, for some collinear triple $M = \{x_1, x_2, x_3\}$ and every $p \in P_m$, $fp(M)$ is collinear. Let $X$ be a projective line in $\Pi^m$, $p_0 \in P_m$ map $M$ into $X$, and $Q$ be the subgroup of $P_m$ which maps $X$ onto itself. We know that $Q$ acts like $P_1$ on $X$ and is, therefore, 3-transitive without regard to orientation. Let $x \in X - \{p_0(x_1), p_0(x_2)\}$ be arbitrary, and choose $q \in Q$ so that $\{p_0(x_1), p_0(x_2)\} \subset K(q)$ and $q(p_0(x_3)) = x$. Then $fq(p_0(M))$ and $f(p_0(M))$ are each collinear and have two points in common, so that $f(x)$ lies on the projective line $Y$ through $f(p_0(M))$, and $f(X) \subset Y$. Since $f$ is a homeomorphism, and $X$, $Y$ are topological circles, we must have $f(X) = Y$. If $Z$ denotes the $(m-1)$-dimensional projective hyperplane at infinity, then any projective line which meets $Z$ in two points must lie in $Z$. Moreover, $f(Z)$ must have the same property, for $f$ preserves incidence relations. Hence, $f(Z)$ is a projective hyperplane, and $f(Z)$ has dimension $m-1$. If we choose $p_1 \in P_m$ so that $p_1(Z) = f(Z)$ and set $f_1 = p_1^{-1}f$, then $f_1(Z) = Z$, and the restriction $f_1^*$ of $f_1$ to $\Pi^m - Z = E^m$ maps lines onto lines. Following the argument in the proof of Theorem 3, we infer that $f_1^*$ is affine, $f_1 \in P_m$, and $f \in P_m$, which contradicts the hypothesis of our theorem. Therefore, $fp(M)$ is noncollinear for some $p \in P_m$.

5. **$\omega$-transitive groups.** So far, we have not exhibited any $f$ such that the group generated by $f$ and $A_m$ is $\omega$-transitive. This we will now do. As before, the results for the case $m = 1$ are much stronger than those for $m > 1$, and this seems to be due to the fact that a nondegenerate connected subset of $E^1$ has a nonempty interior. The conditions which we shall impose on $f$ all have to do with its fixed point set and require, at the very least, that this should have a nonempty interior.

THEOREM 6. *Suppose $f \in H^+(E^1)$, $f \neq e$, and $K(f)$ contains a half-line. Then the group $G$ generated by $f$ and the set $B$ of all translations in $A_1^+$ is $\omega$-transitive relative to $H^+(E^1)$.*

*Proof.* Let $x_1 < \cdots < x_{n+1}$ be arbitrary points of $E^1$, and suppose $(-\infty, x_0]$ is a connected component of $K(f)$. The case $[x_0, +\infty) \subset K(f)$ is handled in the same way. Choose $b_0 \in B$ so that $b_0(x_0) = x_{n+1}$. If we set $f_0 = b_0 f b_0^{-1}$, then $K(f_0) = b_0(K(f))$ has $(-\infty, x_{n+1}]$ as a connected component. The elements of $B$ have the form $b(x) = \beta + x$, and if we give to $B$ the topology induced by the euclidean topology for $\beta$, then the mapping $\varphi(b) = bf_0b^{-1}(x_{n+1})$ from $B$ into $E^1$ becomes continuous. Now $\varphi(e) = f_0(x_{n+1}) = x_{n+1}$, and we can find a connected neighborhood

$U \subset B$ of $e$ so that $b \in U$ implies $b(x_{n+1}) \in (x_n, +\infty)$. Since $x_{n+1}$ is a boundary point of $K(f_0)$, we can find $b \in U$ with the property that $b^{-1}(x_{n+1}) \in E^1 - K(f_0)$, whence $\varphi(b) \neq x_{n+1}$. From the connectedness of $U$ we infer that $\varphi(U)$ is a nondegenerate interval which must have a nonempty interior. If we set $G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\}$, then $b \in U$ implies

$$K(bf_0b^{-1}) \supset b((-\infty, x_{n+1}]) = (-\infty, b(x_{n+1})] \supset (-\infty, x_n]\ ,$$

so that $bf_0b^{-1} \in G_0$ and $\varphi(U) \subset G_0(x_{n+1})$. Lemma 3 tells us that if we know $G$ to be $n$-transitive relative to $H^+(E^1)$, then $G$ is $(n+1)$-transitive. Since $G$ is clearly 0-transitive, a simple induction argument shows that $G$ is $\omega$-transitive.

Clearly the group $G_2$ generated by $f$ and any conjugate $hBh^{-1}$ of $B$, where $h \in H(E^1)$, is also $\omega$-transitive relative to $H^+(E^1)$. For the fixed point set of $f_1 = h^{-1}fh$ is homeomorphic to that of $f$, so that the group $G_1$ generated by $f_1$ and $B$ is $\omega$-transitive by Theorem 6, and $G_2 = hG_1h^{-1}$. Similar remarks apply to the other theorems in this section. We also observe that some groups generated by $f \in H^+(E^1) - A_1^+$ and $B$ are not even 2-transitive. Choose $b_0 \in B$ and $f \in H^+(E^1) - A_1^+$ so that $b_0(x) = \beta_0 + x$, where $\beta_0 \neq 0$, and $f$ has period $\beta_0$ in the sense that $f(\beta_0 + x) = \beta_0 + f(x)$, or $b_0fb_0^{-1} = f$. Now $f$ and each element of $B$ commutes with $b_0$, so every element of the group $G$ generated by $f$ and $B$ commutes with $b_0$. If any such element maps $x$ into $y$, then it maps $x + \beta_0$ into $y + \beta_0$, and $G$ is not 2-transitive.

THEOREM 7. *Suppose $\{f_1, f_2, \cdots\} \subset H^+(E^1)$, and, for every compact subset $C$ of $E^1$, there is an $f_m$ satisfying $E^1 \neq K(f_m) \supset C$. Then the group $G$ generated by $\{f_1, f_2, \cdots\}$ and $B$ is $\omega$-transitive relative to $H^+(E^1)$.*

*Proof.* Let $x_1 < \cdots < x_{n+1}$ be arbitrary points in $E^1$, and $f_m$ have the property that $E^1 \neq K(f_m) \supset [x_1 - 1, x_{n+1}]$. If $K(f_m)$ contains a half-line, then our result follows from Theorem 6. We will assume, therefore, that the connected component $[y_0, y_1]$ of $K(f_m)$ which contains $[x_1 - 1, x_{n+1}]$ is bounded. Choose $b_0 \in B$ so that $b_0(y_1) = x_{n+1}$, set $g_0 = b_0f_mb_0^{-1}$, and let $\varphi(b) = bg_0b^{-1}(x_{n+1})$ for each $b \in B$. Then $K(g_0)$ has $[y_2, x_{n+1}]$ as a connected component, where $y_2 = b_0(y_0) \leqq x_1 - 1$. As in the proof of Theorem 6, $\varphi$ is continuous, $\varphi(e) = x_{n+1}$, and we can find a connected neighborhood $U \subset B$ of $e$ such that $b \in U$ implies $b(x_{n+1}) \in (x_n, +\infty)$ and $b(y_2) \in (-\infty, x_1)$. Again there is a $b \in U$ such that $\varphi(b) \neq x_{n+1}$, and if we define $G_0$ as before, then $b \in U$ implies

$$K(bg_0b^{-1}) \supset b([y_2, x_{n+1}]) \supset [x_1, x_n]\ ,$$

so that $bg_0b^{-1} \in G_0$ and $\varphi(U) \subset G_0(x_{n+1})$. The rest of the proof follows that of Theorem 6.

THEOREM 8. *Suppose $f, g \in H^+(E^1)$, $E^1 \neq K(f)$ has a nonempty interior, and $K(g) = \{y_0\}$. Then the group $G$ generated by $f, g$ and $B$ is $\omega$-transitive relative to $H^+(E^1)$.*

*Proof.* Choose $y_1, y_2 \in E^1$ and $b_0 \in B$ so that $[y_1, y_2] \subset K(f)$ and $b_0(y_0) = y_1$. If we set $g_0 = b_0 g b_0^{-1}$, then $K(g_0) = \{y_1\}$, and if we define $g_1 = g_0$ in case $g_0(y_2) > y_2$ and $g_1 = g_0^{-1}$ in case $g_0^{-1}(y_2) > y_2$, then $g_1^m(y_2) \to +\infty$ as $m \to +\infty$. Finally, let $b_m(x) = \beta_m + x$ and

$$f_m = b_m^{-1} g_1^m f g_1^{-m} b_m \, .$$

Then

$$\begin{aligned} K(f_m) &= b_m^{-1} g_1^m(K(f)) \supset b_m^{-1}([y_1, g_1^m(y_2)]) \\ &= [-\beta_m + y_1, -\beta_m + g_1^m(y_2)] \, . \end{aligned}$$

If we choose $\beta_m = g_1^m(y_2)/2$, then any compact subset of $E^1$ will eventually lie in some $K(f_m)$, and our result follows from Theorem 7.

COROLLARY. *With the same hypotheses as in Theorem 8, the group generated by $f$ and $A_1^+$ is $\omega$-transitive relative to $H^+(E^1)$.*

THEOREM 9. *Suppose $\{f_1, f_2, \cdots\} \subset H^+(\Pi^1)$, and there is a point $y_0 \in \Pi^1$ such that, for every neighborhood $U$ of $y_0$, we can find an $f_m$ satisfying $\Pi^1 \neq K(f_m) \supset \Pi^1 - U$. Then the group $G$ generated by $\{f_1, f_2, \cdots\}$ and $Q$ is $\omega$-transitive relative to $H^+(\Pi^1)$, where $Q$ is the group of "rotations" $q \in P_1^+$ of the form $q(x) = (\alpha x - \beta)/(\beta x + \alpha)$ with $\alpha, \beta$ real and not both $0$.*

*Proof.* The name "rotation" for an element of $Q$ is suggested by the fact that $Q$ is strictly 1-transitive, so that $e$ is the only one of its elements with fixed points. We can identify $Q$ with the set of ordered pairs $(\alpha, \beta)$, excluding $(0, 0)$, but we must also identify $(\alpha, \beta)$ with $(\lambda\alpha, \lambda\beta)$ for each real $\lambda \neq 0$. Thus $Q$ is topologically equivalent to $\Pi^1$, that is, a circle. The action of $Q$ on $\Pi^1$ is, therefore, the same as that of the group of real numbers modulo $2\pi$ on itself by means of left translation. We will show, first of all, that the group $G_1$ of those elements in $G$ which fix $\infty$ is $\omega$-transitive relative to $H^+(E^1)$. Let $x_1 < \cdots < x_{n+1} \in E^1 \subset \Pi^1$ be arbitrary, $q_0 \in Q$ map $y_0$ into $x_{n+1} + 1$, and $f_m$ have the property that

$$\Pi^1 \neq K(f_m) \supset \Pi^1 - q_0^{-1}((x_{n+1}, x_{n+1} + 2)) \, .$$

Setting $f = q_0 f_m q_0^{-1}$, we have $\Pi^1 \neq K(f) \supset \Pi^1 - (x_{n+1}, x_{n+1} + 2)$. Let $y_1$ be the right-hand endpoint of the connected component $D$ of $K(f)$ which contains $\Pi^1 - (x_{n+1}, x_{n+1} + 2)$, where $\Pi^1$ is oriented so as to agree with the ordering of $E^1$. If we choose $q_1 \in Q$ so that $q_1(y_1) = x_{n+1}$ and set $g_1 = q_1 f q_1^{-1}$, then $q_1(D)$ is a connected component of $K(g_1)$ which contains $\Pi^1 - (x_{n+1}, x_{n+1} + 2)$. We define $\varphi(q) = q g_1 q^{-1}(x_{n+1})$ for each $q \in Q$, and observe that $\varphi$ is continuous, $\varphi(e) = x_{n+1}$, and there is a connected neighborhood $V \subset Q$ of $e$ such that $q \in V$ implies $q((x_{n+1}, x_{n+1}+2)) \subset (x_n, +\infty)$. As before, $\varphi(V)$ has a nonempty interior, and $q \in V$ implies

$$K(qg_1q^{-1}) \supset q(\Pi^1 - (x_{n+1}, x_{n+1} + 2)) \supset \Pi^1 - (x_n, +\infty) ,$$

so that $qg_1q^{-1} \in G_1$. If we set $G_0 = \{g \in G_1 \colon \{x_1, \cdots, x_n\} \subset K(g)\}$, then $G_0(x_{n+1})$ has a nonempty interior, and Lemma 3 implies that $G_1$ is $\omega$-transitive relative to $H^+(E^1)$. To show that $G$ is $\omega$-transitive relative to $H^+(\Pi^1)$, we can apply the argument in the proof of Lemma 9 with $P_1^+$ replaced by $Q$, for only the 1-transitivity of $P_1^+$ was used in that case.

THEOREM 10. *Suppose $f, g \in H^+(\Pi^1)$, $\Pi^1 \neq K(f)$ has a nonempty interior, and $K(g) = \{y_0\}$. Then the group $G$ generated by $f, g$ and $Q$ is $\omega$-transitive relative to $H^+(\Pi^1)$.*

*Proof.* Choose $y_1 < y_2$ in $E^1 \subset \Pi^1$ and $q_0, q_1 \in Q$ so that $[y_1, y_2] \subset K(f)$, $q_0(y_0) = \infty$, and $q_1(y_1) = \infty$. Then $g_0 = q_0 g q_0^{-1}$ has only one fixed point at $\infty$, and $f_0 = q_1 f q_1^{-1}$ leaves fixed the points of $[-\infty, y_3]$, where $y_3 = q_1(y_2)$ and, for the sake of our interval notation, we identify $-\infty$ and $+\infty$ with $\infty$. Now $\{g_0^k(y_3) \colon -\infty < k < +\infty\}$ has $+\infty$ as a limit point, and, for every neighborhood $U$ of $\infty$, we can find an integer $k$ satisfying

$$\Pi^1 - U \subset [-\infty, g_0^k(y_3)] \subset K(g_0^k f_0 g_0^{-k}) .$$

Our result now follows from Theorem 9.

COROLLARY. *With the same hypotheses as in Theorem 10, the group generated by $f$ and $P_1^+$ is $\omega$-transitive relative to $H^+(\Pi_1)$.*

THEOREM 11. *Suppose $X$ is a locally compact, locally connected metric space which can not be separated by any finite set,*

$$\{f_1, f_2, \cdots\} \subset H(X) ,$$

*and $y_0 \in X$ has the property that $\{X - K(f_k)\}$ is a base for the neighborhoods of $y_0$. Let $R \subset H(X)$ be a 1-transitive group of* *isometries of* $X$, *and* $R_0 = \{r \in R: r(y_0) = y_0\}$. *Suppose there is a continuous mapping* $\sigma$ *from* $[0,1]$ *into* $R$ *with the topology of uniform convergence on compact sets such that* $\sigma(0) \in R_0$, $\sigma(1) \in R - R_0$, *and, for each* $y \in X$, $R_0(y)$ *is the sphere containing* $y$ *with center at* $y_0$. *Then the group* $G$ *generated by* $\{f_1, f_2, \cdots\}$ *and* $R$ *is* $\omega$*-transitive.*

*Proof.* Let $x_1, \cdots, x_{n+1} \in X$ be given, and

$$G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\} .$$

If we can show that $G_0(x_{n+1})$ has a nonempty interior, then our result will follow by induction from Lemma 1. Since $G$ is 1-transitive, we may assume that $x_{n+1} = y_0$. For let $g_0 \in G$ map $x_{n+1}$ into $y_0$, and

$$G_0' = \{g' \in G: \{g_0(x_1), \cdots, g_0(x_n)\} \subset K(g')\} .$$

Then $g \in G_0$ implies $g_0 g g_0^{-1} \in G_0'$, and $g' \in G_0'$ implies $g_0^{-1} g' g_0 \in G_0$, whence $g_0^{-1} G_0' g_0 = G_0$. If we know that $G_0'(y_0)$ has a nonempty interior, then

$$G_0(x_{n+1}) = g_0^{-1} G_0' g_0(x_{n+1}) = g_0^{-1}(G_0'(y_0))$$

also has a nonempty interior. Hence, we can assume that $x_{n+1} = y_0$. If we set $\sigma(t) = r_t$ for $t \in [0,1]$, then $\alpha = \rho(r_1(y_0), y_0) > 0$, where $\rho$ is the metric for $X$. Let $\beta$ be the shortest distance from $y_0$ to $\{x_1, \cdots, x_n\}$, $U_\varepsilon$ the open ball with center $y_0$ and radius $\varepsilon = \min(\alpha, \beta/2)$, and $f_k$ such that $y_0 \in X - K(f_k) \subset U_\varepsilon$. Since $\varepsilon \leqq \alpha$, and $\rho(r_t(y_0), y_0)$ is a continuous function of $t$, we can find $\delta \in [0,1]$ satisfying $\rho(r_t(y_0), y_0) \leqq \varepsilon$ for $t \in [0, \delta]$ and $\rho(r_\delta(y_0), y_0) = \varepsilon$. This also implies that $\rho(y_0, r_t^{-1}(y_0)) \leqq \varepsilon$ for $t \in [0, \delta]$. If we set

$$G_1 = \{s r_t^{-1} f_k r_t s^{-1}: t \in [0, \delta], s \in R_0\} ,$$

then $G_1 \subset G_0$. For

$$\begin{aligned} K(s r_t^{-1} f_k r_t s^{-1}) &= s r_t^{-1}(K(f_k)) \supset X - s r_t^{-1}(U_\varepsilon) \supset X - s(U_{2\varepsilon}) \\ &= X - U_{2\varepsilon} \supset \{x_1, \cdots, x_n\} . \end{aligned}$$

Moreover,

$$r_t^{-1} f_k r_t(y_0) \in r_t^{-1} f_k(\bar{U}_\varepsilon) \subset r_t^{-1}(\bar{U}_\varepsilon) \subset \bar{U}_{2\varepsilon} ,$$

and if we hold $t$ fixed and let $s$ vary, then

$$s r_t^{-1} f_k r_t s^{-1}(y_0) = s(r_t^{-1} f_k r_t(y_0))$$

is a sphere with center $y_0$ and radius

$$\theta(t) = \rho(y_0, r_t^{-1} f_k r_t(y_0)) , \qquad t \in [0, \delta] .$$

Since $r_\delta(y_0)$ lies on the boundary of $\bar{U}_\varepsilon$, we have $r_\delta^{-1} f_k r_\delta(y_0) = y_0$, and since $r_0(y_0) = y_0 \in X - K(f_k)$, we have $r_0^{-1}f_k r_0(y_0) \neq y_0$. Thus $\theta(0) \neq 0$, and $\theta(\delta) = 0$. Now the local compactness and local connectedness of $X$ implies that the mapping $h \to h^{-1}$ is continuous, and $(h, x) \to h(x)$ is jointly continuous in the topology of uniform convergence on compact sets [1], so that $\theta: [0, \delta] \to E^1$ is continuous, and $\theta([0, \delta])$ is a nondegenerate interval. Hence, $G_1(y_0)$ contains all spheres with center $y_0$ and radius less than some positive number, and $G_1(y_0) \subset G_0(y_0)$ has a nonempty interior.

COROLLARY 1. *With the same hypotheses as in Theorem* 11, *suppose that we have* $f, g \in H(X)$ *with the property that* $\{g^k(X - K(f)): k \geqq 0\}$ *is a base for the neighborhoods of* $y_0$. *Then the group generated by* $f, g$, *and* $R$ *is* $\omega$-*transitive.*

*Proof.* We set $f_k = g^k f g^{-k}$ and apply Theorem 11.

COROLLARY 2. *Suppose* $X = E^m$ $(m \geqq 2)$, $R$ *is the group of rigid motions of* $E^m$, $y_0 \in E^m$, *and* $\{f_1, f_2, \cdots\}$ *is as in the hypothesis of Theorem* 11. *Then* $G$ *is* $\omega$-*transitive.*

*Proof.* For the mapping $\sigma$, we set $r_t(x) = tx_0 + x$, where $x_0 \neq 0$ is a fixed point of $E^m$.

COROLLARY 3. *Suppose* $X = \Pi^m$ $(m \geqq 2)$, $R$ *is the set of elements in* $P_m$ *which can be represented by* $(m+1)$-*th order unitary matrices,* $y_0 \in \Pi^m$, *and* $\{f_1, f_2, \cdots\}$ *is as in the hypothesis of Theorem* 11. *Then* $G$ *is* $\omega$-*transitive.*

*Proof.* If we regard $\Pi^m$ as the unit sphere in $E^{m+1}$ with antipodal points identified and the metric induced by $E^{m+1}$, then the elements of $R$ are isometries of $\Pi^m$. For the mapping $\sigma$, we choose a one-parameter subgroup of rotations about some axis which does not pass through $y_0$.

LEMMA 10. *Let* $X$ *be a topological space,* $G$ *a subgroup of* $H(X)$, $\varphi$ *a homeomorphism from* $E^1$ *onto a closed subset* $Y$ *of* $X$, *and* $F = \{g \in G: g(Y) = Y\}$. *Suppose* $\varphi^{-1}F\varphi$ *contains* $A_1$, *and there is a* $g_0 \in G$ *with the properties* $K(g_0) \supset \varphi([0, 1])$ *and* $g_0(Y) - Y \neq \emptyset$. *Then for any interval* $I = [\alpha, \beta]$ *in* $E^1$ *and any* $y \in Y - \varphi(I)$, *we can find a* $g \in G$ *such that* $K(g) \supset \varphi(I)$ *and* $g(y) \in X - Y$.

*Proof.* Let $G_0 = \{g \in G: \varphi(I) \subset K(g)\}$ and $Y_0 = \{y \in Y: G_0(y) - Y \neq \emptyset\}$. Clearly $Y_0$ is open in $Y$. If $a \in A_1$ and $a(I) \supset I$, then we will show that $a\varphi^{-1}(Y_0) \subset \varphi^{-1}(Y_0)$. We first choose $f \in F$ so that $\varphi^{-1}f\varphi = a$. For each

$t \in \varphi^{-1}(Y_0)$, there is a $g \in G_0$ satisfying $g\varphi(t) \in X - Y$. Then

$$K(fgf^{-1}) = f(K(g)) \supset f\varphi(I) = \varphi a(I) \supset \varphi(I)$$

implies that $fgf^{-1} \in G_0$. From

$$fgf^{-1}(\varphi a(t)) = fgf^{-1}(f\varphi(t)) = fg\varphi(t) \in f(X - Y) = X - Y$$

we infer that $a(t) \in \varphi^{-1}(Y_0)$ and $a\varphi^{-1}(Y_0) \subset \varphi^{-1}(Y_0)$. Since we can always find an $a \in A_1$ such that $a(I) \supset I$, and $a$ maps any point in $E^1 - I$ into any other point further away from $I$, it follows that if $\varphi^{-1}(Y_0) \neq \varnothing$, then $\varphi^{-1}(Y_0)$ is the union of two half-lines, that is, $E^1 - \varphi^{-1}(Y_0) = [\gamma, \delta] \supset [\alpha, \beta] = I$. We will show that $\varphi^{-1}(Y_0) \neq \varnothing$ and $[\alpha, \beta] = [\gamma, \delta]$ by deriving a contradiction from the assumption $\gamma < \alpha$. The case $\delta > \beta$ is handled in a similar manner. Let $C$ be the connected component of $\varphi^{-1}(K(g_0))$ which contains $[0, 1]$. Then $C$ is a closed interval with at least one endpoint $\varepsilon$, and we can find an $a_0 \in A_1$ so that $a_0(C) \supset I$ and $a_0(\varepsilon) = \alpha$. If $f_0 \in H(X)$ is an extension of $\varphi a_0 \varphi^{-1}$, and $g_1 = f_0 g_0 f_0^{-1}$, then

$$K(g_1) = f_0(K(g_0)) \supset \varphi a_0 \varphi^{-1}(K(g_0)) \supset \varphi a_0(C) \supset \varphi(I)$$

implies that $g_1 \in G_0$ and $a_0(C)$ is a connected component of $\varphi^{-1}(K(g_1))$. Choose $y_0 \in Y$ so that $g_0(y_0) \in X - Y$. From $g_1(f_0(y_0)) = f_0 g_0(y_0) \in X - Y$ we infer that $Y_0 \neq \varnothing$ and $\varphi^{-1}(Y_0) \neq \varnothing$. Evidently $t \in [\gamma, \alpha]$ implies $g_1\varphi(t) \in Y$, and we can find $t_0 \in (\gamma, \alpha)$ so close to $\alpha$ that $t_0 \neq \varphi^{-1} g_1 \varphi(t_0) \in (\gamma, \alpha)$. We may assume, in fact, that $\varphi^{-1} g_1 \varphi(t_0) < t_0$; for if $\varphi^{-1} g_1 \varphi(t_0) > t_0$, then we would work with $g_1^{-1}$. Choose $a_1 \in A_1^+$ so that $a_1(I) \supset I$, $a_1(t_0) = \gamma$, and let $f_1 \in H(X)$ be an extension of $\varphi a_1 \varphi^{-1}$. As we have already seen, $g_2 = f_1 g_1 f_1^{-1} \in G_0$. Now

$$\begin{aligned}\varphi^{-1} g_2 \varphi(\gamma) &= \varphi^{-1} f_1 g_1 f_1^{-1} \varphi(\gamma) = a_1 \varphi^{-1} g_1 \varphi a_1^{-1}(\gamma) \\ &= a_1 \varphi^{-1} g_1 \varphi(t_0) < a_1(t_0) = \gamma\end{aligned}$$

implies that $g_2\varphi(\gamma) \in Y_0$, and we can find $g_3 \in G_0$ satisfying $g_3(g_2\varphi(\gamma)) \in X - Y$. Since $g_3 g_2 \in G_0$, we conclude that $\varphi(\gamma) \in Y_0$ which contradicts our hypothesis. Hence, $\gamma = \alpha$, and our result is proved.

THEOREM 12. *Let $X$ be a topological space which can not be separated by any finite subset, $R$ a subgroup of $H(X)$, $f \in H(X)$, $\varphi$ a homeomorphism from $E^1$ onto a closed subset $Y$ of $X$, and $S = \{g \in R: g(Y) = Y\}$. Suppose $\varphi^{-1} S \varphi \supset A_1$, $S_0 = \{g \in G: Y \subset K(g)\}$ is 1-transitive on $X - Y$, $K(f) \supset \varphi([0, 1])$, and $f(Y) - Y \neq \varnothing$. Then the group $G$ generated by $f$ and $R$ is $\omega$-transitive.*

*Proof.* We proceed by induction on the transitivity and assume that $G$ is $n$-transitive for some $n \geqq 0$. If $x_1, \cdots, x_{n+1} \in X$ are given,

$G_0 = \{g \in G: \{x_1, \cdots, x_n\} \subset K(g)\}$, and we can show that $G_0(x_{n+1})$ is an open subset of $X$, then Lemma 1 will imply that $G$ is $(n+1)$-transitive, and our induction step will be complete. By hypothesis, there is a $g_0 \in G$ which maps $\{x_2, \cdots, x_n\}$ into $\varphi((0, 1))$ and $x_{n+1}$ into $\varphi(1)$. We consider three cases for the position of $g_0(x_1)$. In case (i), $g_0(x_1) \in Y$ and $\varphi^{-1}g_0(x_1) < 1$. Then we can find an interval $I = [\alpha, \beta]$ which contains $\varphi^{-1}g_0(x_1), \cdots, \varphi^{-1}g_0(x_n)$ but not $\varphi^{-1}g_0(x_{n+1})$, and Lemma 10 gives us a $g_1 \in G$ with the properties $\varphi(I) \subset K(g_1)$ and $g_1(g_0(x_{n+1})) \in X - Y$. Since $S_0(g_1g_0(x_{n+1})) = X - Y$ is open in $X$, it follows that

$$g_0^{-1}g_1^{-1}S_0g_1g_0(x_{n+1}) = g_0^{-1}g_1^{-1}(X - Y)$$

is open in $X$. From $g \in S_0$ we infer that

$$K(g_0^{-1}g_1^{-1}gg_1g_0) \supset g_0^{-1}g_1^{-1}(Y) \supset g_0^{-1}g_1^{-1}\varphi(I) = g_0^{-1}\varphi(I) \supset \{x_1, \cdots, x_n\} ,$$

whence $g_0^{-1}g_1^{-1}S_0g_1g_0 \subset G_0$, and our induction step is complete in case (i). In case (ii), $g_0(x_1) \in X - Y$. Now Lemma 10 gives us a $g_2 \in G$ with the properties $K(g_2) \supset \{g_0(x_2), \cdots, g_0(x_{n+1})\}$ and $g_2\varphi(0) \in X - Y$. We can also find $g_3 \in S_0$ satisfying $g_3(g_2\varphi(0)) = g_0(x_1)$. Setting $g_4 = g_2^{-1}g_3^{-1}g_0$, we have

$$\begin{aligned} g_4(x_i) &= g_2^{-1}g_3^{-1}g_0(x_i) = g_0(x_i) , \qquad && 2 \leqq i \leqq n+1 , \\ g_4(x_1) &= g_2^{-1}g_3^{-1}g_0(x_1) = \varphi(0) . \end{aligned}$$

Thus case (ii) can be reduced to case (i) with $g_0$ replaced by $g_4$. In case (iii), $g_0(x_1) \in Y$ and $\varphi^{-1}g_0(x_1) > 1$. Again Lemma 10 gives us a $g_5 \in G$ such that $K(g_5) \supset \{g_0(x_2), \cdots, g_0(x_{n+1})\}$ and $g_5(g_0(x_1)) \in X - Y$. Setting $g_6 = g_5g_0$, we have

$$\begin{aligned} g_6(x_i) &= g_5g_0(x_i) = g_0(x_i) , \qquad && 2 \leqq i \leqq n+1 , \\ g_6(x_1) &= g_5g_0(x_1) \in X - Y . \end{aligned}$$

Thus case (iii) can be reduced to case (ii) with $g_0$ replaced by $g_6$, and all the cases relating to the position of $g_0(x_1)$ have been disposed of.

THEOREM 13. *The conclusion of Theorem* 12 *remains valid if we replace* $E^1$ *by* $\Pi^1$, *that is, a circle, and* $A_1$ *by* $P_1$.

*Proof.* The proof of Theorem 12 up to the definition of $g_0$ can be carried over unchanged. This time, however, we choose $g_0$ so as to map $\{x_1, \cdots, x_n\}$ into $\varphi((0, 1))$ and consider two cases for the position of $g_0(x_{n+1})$. In case (i), $g_0(x_{n+1}) \in X - Y$. As we have already seen in the proof of Theorem 12, this implies that $G_0(x_{n+1})$ is open in $X$, and our induction step is complete in this case. In case (ii), $g_0(x_{n+1}) \in Y$. By hypothesis, these is some point $y_0 \in Y - \varphi([0, 1])$

satisfying $f(y_0) \in X - Y$. We choose $p_1 \in P_1$ and a neighborhood $U$ of $\varphi^{-1}g_0(x_{n+1})$ so that $U \subset \Pi^1 - \{\varphi^{-1}g_0(x_1), \cdots, \varphi^{-1}g_0(x_n)\}$, $p_1(\varphi^{-1}(y_0)) = \varphi^{-1}g_0(x_{n+1})$, and $p_1(\Pi^1 - [0, 1]) \subset U$. Let $g_1 \in S$ be an extension of $\varphi p_1 \varphi^{-1}$, and $g_2 = g_1 f g_1^{-1}$. Then

$$\begin{aligned} K(g_2) &= g_1(K(f)) \supset g_1\varphi([0, 1]) \\ &= \varphi p_1([0, 1]) \supset \varphi(\Pi^1 - U) \supset \{g_0(x_1), \cdots, g_0(x_n)\}\ , \\ g_2(g_0(x_{n+1})) &= g_1 f g_1^{-1}(g_0(x_{n+1})) \\ &= g_1 f \varphi p_1^{-1} \varphi^{-1} g_0(x_{n+1}) = g_1 f \varphi \varphi^{-1}(y_0) \in g_1(X - Y) = X - Y\ . \end{aligned}$$

If we set $g_3 = g_2 g_0$, then

$$\begin{aligned} g_3(x_i) &= g_2 g_0(x_i) = g_0(x_i)\ , \qquad 1 \leqq i \leqq n\ , \\ g_3(x_{n+1}) &= g_2 g_0(x_{n+1}) \in X - Y\ , \end{aligned}$$

and case (ii) can be reduced to case (i) with $g_0$ replaced by $g_3$. Thus all the cases relating to the positions of $g_0(x_{n+1})$ have been disposed of.

COROLLARY. *Suppose $R$ is a subgroup of $H(X)$, $f \in H(X)$, $X \neq K(f)$ has a nonempty interior, and either* (i) *$X = E^m$ and $R = A_m$, or* (ii) *$X = \Pi^m$ and $R = P_m$. Then the group $G$ generated by $f$ and $R$ is $\omega$-transitive.*

*Proof.* The case $m = 1$ has already been verified in Theorems 8 and 10, so we will assume that $m \geqq 2$. We first consider case (i) and choose points $x_0 \in \operatorname{int} K(f)$ and $x_1 \in E^m - K(f)$. If $f(x_1)$ does not lie on the line $Y$ through $x_0$ and $x_1$, then our result follows from Theorem 12, since $K(f) \cap Y$ contains a nondegenerate interval. If $f(x_1) \in Y$, then we choose a rotation $a_1 \in A_m$ about the point $x_1$ through such a small positive angle that $K(f) \cap a_1^{-1}(Y)$ contains a nondegenerate interval I. Setting $f_1 = a_1 f a_1^{-1}$, we have

$$\begin{aligned} K(f_1) \cap Y &= a_1(K(f)) \cap Y = a_1(K(f) \cap a_1^{-1}(Y)) \supset a_1(I)\ , \\ f_1(x_1) &= a_1 f a_1^{-1}(x_1) = a_1 f(x_1) \in X - Y\ , \end{aligned}$$

and our result again follows from Theorem 12 with $f$ replaced by $f_1$. Case (ii) is handled in exactly the same way, for we can identify $E^m$ with the finite part of $\Pi^m$, and $a_1$ can be extended to an element of $P_m$.

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Received April 11, 1966.

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# PACIFIC JOURNAL OF MATHEMATICS





Printed in Japan by International Academic Printing Co., Ltd., Tokyo, Japan

**Pacific Journal of Mathematics**
Vol. 23, No. 1 March, 1967