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INTEGRAL KERNEL FOR ONE-PART FUNCTION SPACES

HERBERT STANLEY BEAR, JR. AND BERTRAM JOHN WALSH

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H. S. BEAR and BERTRAM WALSH

Let X be a separable compact Hausdorff space, and let B be a linear space of continuous real functions on X , where $1 \in B$ and B separates the points of X . Let Γ denote the Silov boundary of B in X , and assume that $\Delta = X \sim \Gamma \neq \emptyset$. Further assumptions on B are made which are in the nature of axioms for an abstract potential theory. These assumptions are more global than is usual, and in particular a sheaf axiom is not assumed, nor is the existence of a base of regular neighborhoods. Instead the assumptions are concerned with equicontinuity properties of B on Δ , and the consequences of Δ being a single Gleason part of X . With suitable hypotheses on B and Δ there is an integral kernel representation of the following sort: $u(x) = \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta)$, where Q is a jointly measurable function on $\Delta \times \Gamma$ which is "in B " (i.e., abstractly harmonic) as a function of x for each fixed $\theta \in \Gamma$.

2. Topologies on Δ . Let \mathfrak{S} denote the given compact topology of X , usually considered as relativized to Δ . Since X is compact, \mathfrak{S} is the weak topology induced by B . Let $\|x\|_*$ be the norm of B^* transferred to points of Δ by considering them as evaluation functionals. Let \mathfrak{S}_* be the metric topology on Δ obtained from the norm $\| \cdot \|_*$ of B^* . Clearly $\mathfrak{S}_* \supset \mathfrak{S}$. We will later introduce other topologies on Δ which are germane in the presence of additional assumptions on B .

Let ball $B = \{u \in B: \|u\| \leq 1\}$, and let

$$B^+(z) = \{u \mid \Delta: u \in B, u > 0, u(z) = 1\}$$

be the section of B^+ normalized at some $z \in \Delta$. We will be concerned with conditions implying the equicontinuity of ball B and $B^+(z)$. We remark that Loeb and Walsh [7] have recently shown that equicontinuity of $B^+(z)$ can be taken as the convergence axiom of Brelot's axiomatic potential theory.

THEOREM 1. *If $B^+(z)$ is equicontinuous on Δ , then ball B is equicontinuous on Δ . Ball B is equicontinuous on Δ if and only if $\mathfrak{S} = \mathfrak{S}_*$ on Δ .*

Proof. Suppose that $B^+(z)$ is equicontinuous on Δ (with respect to \mathfrak{S}) and that $\|u\| \leq 1$. Then $v = (u + 2)/(u(z) + 2) \in B^+(z)$. Given $\epsilon > 0$ and $x \in \Delta$ there is a neighborhood U of x such that

$|w(y) - w(x)| < \varepsilon$ for all $w \in B^+(z)$ and all $y \in U$. In particular

$$\left| \frac{u(y) + 2}{u(z) + 2} - \frac{u(x) + 2}{u(z) + 2} \right| < \varepsilon$$

for all $y \in U$, and consequently $|u(y) - u(x)| < 3\varepsilon$ if $y \in U$ and $\|u\| \leq 1$. Hence ball B is equicontinuous.

We have already observed that $\mathfrak{F} \subset \mathfrak{F}_*$. If $x_n \rightarrow x$ implies that $u(x_n) \rightarrow u(x)$ uniformly for $\|u\| \leq 1$ (equicontinuity of ball B), then certainly $\|x_n - x\|_* \rightarrow 0$. That is, equicontinuity of ball B implies $\mathfrak{F} = \mathfrak{F}_*$. The converse is clear.

We recall (see [1]) that X is decomposed into parts by the equivalence relation $x \sim y$ if and only if $1/a \leq u(x)/u(y) \leq a$ for some $a \geq 1$ and all positive $u \in B$. If $x \sim y$, let $R(x, y)$ be the infimum of the numbers a which satisfy the inequality. Then $\log R(x, y) = d(x, y)$ is a metric on each part. We call d the "part metric", and let \mathfrak{F}_d be the part metric topology. It will simplify the exposition without any real loss of generality to assume that Δ is a single part. Otherwise the statements below would hold for individual parts within Δ .

THEOREM 2. *If Δ is a part, then $\mathfrak{F}_d \supset \mathfrak{F}_* \supset \mathfrak{F}$, and $B^+(z)$ is equicontinuous if and only if $\mathfrak{F} = \mathfrak{F}_d$.*

Proof. Suppose $x_n, x \in \Delta$ and $d(x_n, x) \rightarrow 0$; i.e., $R(x_n, x) \rightarrow 1$. Given $\varepsilon > 0$ there is N such that

$$\left| \frac{u(x_n)}{u(x)} - 1 \right| = \frac{|u(x_n) - u(x)|}{u(x)} < \varepsilon$$

for all $u > 0$, if $u \geq N$. If $\|v\| \leq 1$, and $u = v + 2$, then $1 \leq u \leq 3$, $u(x_n) - u(x) = v(x_n) - v(x)$, and

$$\left| \frac{v(x_n) - v(x)}{v(x) + 2} \right| < \varepsilon$$

if $n \geq N$. Therefore $|v(x_n) - v(x)| < 3\varepsilon$ if $n \geq N$ and $\|v\| \leq 1$, and $\|x_n - x\|_* \rightarrow 0$ if $d(x_n, x) \rightarrow 0$.

It is shown in [2, Th. 1] that $d(x_n, x) \rightarrow 0$ if and only if $u(x_n) \rightarrow u(x)$ uniformly for all $u \in B^+(z)$. If $B^+(z)$ is equicontinuous on Δ , then by definition we have such convergence uniformly over $B^+(z)$ whenever $x_n \rightarrow x$ (in \mathfrak{F}). Hence $\mathfrak{F} \supset \mathfrak{F}_d$ if $B^+(z)$ is equicontinuous.

We will say that B is a (U) -space if for each $x \in \Delta$ the evaluation functional $e_x \in B^*$ has a unique maximal (in the sense of [9, §§ 4, 6]) representing probability measure μ_x on Γ ; recall that this measure is in an appropriate sense supported by the Choquet boundary bX of X with respect to B . Clearly B is a (U) -space whenever the base

$\{F: F \in B^*, \|F\| = 1 = F(1)\}$ of the positive cone in B^* is a simplex [9, § 9], since that means that every positive linear functional on B has a unique maximal representing measure. It is known (see [6, p. 63, (14b)]) that this occurs if B has the Riesz decomposition property and if and only if its uniform closure does, so B is a (U) -space whenever it is a Dirichlet space [2, p. 294]. If B is a (U) -space and Δ is a part, then the maximal representing measures for the point of Δ are all mutually absolutely continuous with bounded derivatives both ways; for in the argument in [4] in which representing measures are constructed, there would be no loss in generality in taking the measures α and β to be maximal, whence (since the maximal measures form a cone [9, p. 65]) μ_x and μ_y as constructed there would also be maximal—but uniqueness guarantees that those are our μ_x and μ_y . Let $\mu = \mu_z$ represent the point $z \in \Delta$, and write $d\mu_x = g_x d\mu$ for $x \in \Delta$. We then have Δ identified with a subset $\{g_x: x \in \Delta\}$ of $L_\infty(\mu)$ so that $u(x) = \int_r u g_x d\mu$ for all $u \in B$ and all $x \in \Delta$. Let $\|\cdot\|_\infty$ be the $L_\infty(\mu)$ norm, and write $\|x - y\|_\infty = \|g_x - g_y\|_\infty$ to transfer this norm-metric to Δ . Let \mathfrak{F}_∞ be the resulting topology on Δ .

THEOREM 3. *If Δ is a part and B is a (U) -space, then $\mathfrak{F}_\infty = \mathfrak{F}_\Delta \supset \mathfrak{F}_* \supset \mathfrak{F}$. If in addition $B^+(z)$ is equicontinuous on Δ , then $\mathfrak{F} = \mathfrak{F}_\Delta = \mathfrak{F}_* = \mathfrak{F}_\infty = \mathfrak{F}_*$.*

Proof. If $u \in B^+(z)$, then

$$\begin{aligned} |u(x_n) - u(x)| &= \left| \int u(x_n - x) d\mu \right| \\ &\leq \|x_n - x\|_\infty \int u d\mu \\ &= \|x_n - x\|_\infty u(z). \end{aligned}$$

Since $u(z) = 1$ for $u \in B^+(z)$, $u(x_n) \rightarrow u(x)$ uniformly for $u \in B^+(z)$ if $\|x_n - x\|_\infty \rightarrow 0$. Hence $d(x_n, x) \rightarrow 0$ if $\|x_n - x\|_\infty \rightarrow 0$ by Theorem 1 of [2].

Now we show that d -convergence implies L_∞ convergence. Since B is a (U) -space and $R(x, y)$ is the infimum of the constants c usable in the proof of [4, Th. 1], any two Radon-Nikodým derivatives g_x, g_y must satisfy

$$\frac{1}{R(x, y)} \leq \frac{g_x}{g_y} \leq R(x, y)$$

almost everywhere with respect to μ . We also have, comparing g_x and $g_z \equiv 1$ in the inequality above, that $0 \leq g_x \leq R(x, z) = \exp d(x, z)$ holds a.e. μ . For $x, y \in \Delta$, we have

$$|g_x - g_y| \leq g_y[R(x, y) - 1] \leq R(y, z)[R(x, y) - 1]$$

holding a.e. μ . Since $d(x, y) \rightarrow 0$ is equivalent to $R(x, y) \rightarrow 1$, and $R(y, z)$ is fixed, we have that $\|g_x - g_y\|_\infty \rightarrow 0$ if $x \rightarrow y$ in \mathfrak{S}_d . Hence $\mathfrak{S}_d = \mathfrak{S}_\infty$ on Δ .

The final statement of the theorem follows from the equivalence of $\mathfrak{S} = \mathfrak{S}_d$ with equicontinuity of $B^+(z)$.

3. An integral kernel for B . The second half of the proof of Theorem 3 is a modification of that used by Nakai [8] in the case that B consists of all harmonic functions on a Riemann surface with an ideal boundary which makes B a Dirichlet space. The results below include those obtained by Nakai, and the proof of Theorem 4 is essentially a modification of Nakai's technique to our general situation.

THEOREM 4. *If Δ is a part, $B^+(z)$ is equicontinuous, and B is a (U) -space, then there is a positive measure $\mu = \mu_z$ and a jointly measurable function $Q(x, \theta)$ on $\Delta \times \Gamma$ such that $Q(\cdot, \theta)$ is continuous on Δ for each $\theta \in \Gamma$, $0 \leq Q(x, \theta) \leq R(x, z)$ for all $(x, \theta) \in \Delta \times \Gamma$, and*

$$u(x) = \int_\Gamma u(\theta)Q(x, \theta)d\mu(\theta)$$

for all $u \in B$, all $x \in \Delta$.

Proof. Let μ represent z and let D be a countable dense subset of Δ containing z . For each fixed $x \in D$ pick a measurable function $Q(x, \cdot)$ on Γ such that $Q(x, \cdot)d\mu(\cdot)$ represents x . Then the inequalities

$$|Q(x, \cdot) - Q(y, \cdot)| \leq R(y, z)[R(x, y) - 1]$$

and

$$0 \leq Q(x, \cdot) \leq R(x, z)$$

hold a.e. μ for all $x, y \in D$. Let E be the union of the countably many μ -null subsets of Γ where the inequalities above fail. Then $\mu(E) = 0$ and

$$\begin{aligned} |Q(x, \theta) - Q(y, \theta)| &\leq R(y, z)[R(x, y) - 1], \\ 0 &\leq Q(x, \theta) \leq R(x, z), \end{aligned}$$

hold for all $x, y \in D$ and all $\theta \in \Gamma \sim E$. If $\{x_n\}, \{x'_n\}$ are two sequences in D both approaching $x \in \Delta$, then $|Q(x_n, \cdot) - Q(x'_n, \cdot)|$ converges uniformly to zero on $\Gamma \sim E$. For any $x \in \Delta$, pick any sequence $x_n \in D$ with $x_n \rightarrow x$, and define $Q(x, \theta) = \lim Q(x_n, \theta)$ for $\theta \notin E$, and $Q(x, \theta) \equiv 1$ for $\theta \in E$. The function Q is well defined on $\Delta \times \Gamma$ and satisfies the

desired inequalities. Moreover, Q is measurable in θ and continuous in x by its definition. Therefore (see [5, p. 285]) Q is jointly measurable. By the bounded convergence theorem, if $u \in B$ then

$$\begin{aligned} u(x) &= \lim u(x_n) \\ &= \lim \int_{\Gamma} u(\theta)Q(x_n, \theta)d\mu(\theta) \\ &= \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta), \end{aligned}$$

and hence $Q(x, \cdot)d\mu(\cdot)$ represents x .

The kernel obtained by Nakai [8] by the sort of argument above is harmonic in x for each fixed θ . Walsh and Loeb [10] have a generalization of this result in the setting of the abstract potential theory of Brelot. Nakai's result can also be obtained by specializing the results of [3]. We show below that our kernel can be taken to be "in B " as a function of x with no local hypotheses whatsoever.

Let \hat{B} denote the closure, in the topology of uniform convergence on compact subsets of Δ , of $B|\Delta$. This space \hat{B} is our abstract replacement for the space of all harmonic functions on the open set Δ .

LEMMA 5. *If Δ is a part, $B^+(z)$ is equicontinuous, and B is a (U) -space, then the mapping $T: B|\Gamma \rightarrow B|\Delta$ given by*

$$T(u)(x) = \int_{\Gamma} u(\theta)Q(x, \theta)d\mu(\theta)$$

extends to a mapping $T: L_1(\mu) \rightarrow \hat{B}$ which is continuous with respect to the L_1 norm and the u.c.c. topology of \hat{B} .

Proof. $Q(x, \theta)$ is uniformly bounded on $K \times \Gamma$ for each compact $K \subseteq \Delta$. The uniqueness of the maximal representing measure $\mu_z = \mu$ implies that $B|\Gamma$ is dense in $L_1(\mu)$, for if $g \in L_{\infty}(\mu)$ has the property that $g \cdot \mu$ annihilates $B|\Gamma$, then (assuming without loss of generality that $\|g\|_{\infty} < 1$) the measure $(1 + g) \cdot \mu$ is also maximal (since by [9, p. 65] the cone of maximal measures is hereditary) and also represents z , so that $g = 0$. Thus the mapping T can be extended by denseness and continuity to all of $L_1(\mu)$, and the images will remain in \hat{B} .

LEMMA 6. *If Δ is a part and $B^+(z)$ is equicontinuous, then Δ is σ -compact.*

Proof. Since Δ is open in X and X is separable, Δ is also separable. Since $\mathfrak{S} = \mathfrak{S}_\Delta$ with our hypotheses, Δ is a metric space.

Let $\{y_k\}$ be a countable dense subset of \mathcal{A} , and let

$$R_k = \sup \{r: \overline{S(y_k, r)} \cap \Gamma = \emptyset\}$$

where $S(y_k, r)$ is the r -sphere about y_k . If some $R_k = \infty$, then the sets $\overline{S(y_k, n)}$ are compact subsets of \mathcal{A} whose union is all of \mathcal{A} , and we are done. Otherwise each $R_k < \infty$ and the spheres $\overline{S(y_k, r)}$, where r runs through all rationals $< R_k$, exhaust \mathcal{A} . To see this, notice that for any $x \in \mathcal{A}$ there is a rational $\rho > 0$ such that $\overline{S(x, \rho)} \subset \mathcal{A}$. If $y_k \in S(x, \rho/2)$, then $x \in \overline{S(y_k, \rho/2)} \subset \mathcal{A}$ and $\rho/2 < R_k$.

THEOREM 7. *If \mathcal{A} is a part, $B^+(z)$ is equicontinuous, and B is a (U) -space, then there is a function $Q(x, \theta)$ as in Theorem 4 such that $Q(\cdot, \theta) \in \hat{B}$ for each $\theta \in \Gamma$.*

Proof. We give $C(\mathcal{A})$ the locally convex topology of uniform convergence on compact sets. Since \mathcal{A} is σ -compact, $C(\mathcal{A})$ is metrizable. If $\mathcal{A} = \bigcup K_n$ where each K_n is a compact (and metric) subset of \mathcal{A} , then $C(K_n)$ is separable in the uniform topology, and hence $C(\mathcal{A})$ is separable in the u.c.c. topology. Since $C(\mathcal{A})$ has a countable base of convex open sets, the open set $C(\mathcal{A}) \sim \hat{B}$ can be written as a countable union of open convex sets, and we can take each such set U to have its closure disjoint from \hat{B} .

If $E = \{\theta \in \Gamma: Q(\cdot, \theta) \notin \hat{B}\}$ has zero μ -measure, then we can redefine Q to be one on $\mathcal{A} \times E$ and the resulting function will still satisfy Theorem 4 and will be in \hat{B} as a function of x for each $\theta \in \Gamma$. Assume on the contrary that $\mu(E) > 0$. By the countable additivity of μ , there is some U such that $E_U = \{\theta \in \Gamma: Q(\cdot, \theta) \in U\}$ has positive μ -measure, provided these sets are μ -measurable subsets of Γ . To show the measurability of E_U , it suffices to consider E_U for a basic open set $U = \{g \in C(\mathcal{A}): |g(x) - v(x)| < \varepsilon \text{ for } x \in K\}$ where $\varepsilon > 0$ and K is compact. If $\{x_n\}$ is a dense sequence in K , and θ is a fixed point of Γ , then $|Q(x, \theta) - v(x)| \leq \varepsilon'$ for all $x \in K$ if and only if $|Q(x_n, \theta) - v(x_n)| \leq \varepsilon'$ for all n , since Q is continuous in x . The set $\{\theta: |Q(x_n, \theta) - v(x_n)| \leq \varepsilon'\}$ is measurable since Q is measurable in θ , and hence the intersection $\{\theta: |Q(x, \theta) - v(x)| \leq \varepsilon' \text{ all } x \in K\}$ is measurable. Finally, $\{\theta: |Q(x, \theta) - v(x)| < \varepsilon\}$ is a countable union of sets corresponding to values of $\varepsilon' < \varepsilon$.

By the Hahn-Banach theorem we can separate U from the closed subspace \hat{B} , and there is a functional $F \in C(\mathcal{A})^*$ such that $F = 0$ on \hat{B} and $F(u) > 0$ for $u \in U$. In particular, $F(Q(\cdot, \theta)) > 0$ for $\theta \in E_U$. For some $\varepsilon > 0$, the set $S = \{\theta: F(Q(\cdot, \theta)) \geq \varepsilon\}$ must have positive μ -measure. The dual space of $C(\mathcal{A})$ can be represented by the space of regular Borel measures with compact support in \mathcal{A} , and we let λ

be the measure corresponding to F . Define v on \mathcal{A} by

$$v(x) = \int_{\mathcal{I}} \chi_s(\theta) Q(x, \theta) d\mu(\theta) .$$

By Lemma 5, $v \in \hat{B}$ and hence $F(v) = 0$:

$$\begin{aligned} 0 = F(v) &= \int_{\mathcal{A}} v(x) d\lambda(x) \\ &= \int_{\mathcal{A}} \int_{\mathcal{I}} \chi_s(\theta) Q(x, \theta) d\mu(\theta) d\lambda(x) \\ &= \int_{\mathcal{I}} \int_{\mathcal{A}} Q(x, \theta) d\lambda(x) \chi_s(\theta) d\mu(\theta) \\ &= \int_{\mathcal{I}} F(Q(\cdot, \theta)) \chi_s(\theta) d\mu(\theta) \\ &\geq \varepsilon \mu(S) > 0 . \end{aligned}$$

The interchange of integrals is justified because Q is jointly measurable and bounded for x in the compact support λ . The contradiction completes the proof.

REFERENCES

1. H. S. Bear, *A Geometric characterization of Gleason parts*, Proc. Amer. Math. Soc. **16** (1965), 407-412.
2. ———, *Function Algebras*, edited by Frank T. Birtel, Scott Foresman and Co., Chicago, 1966.
3. H. S. Bear and A. M. Gleason, *A global integral representation for abstract harmonic functions*, J. Math. Mech. (to appear)
4. Errett Bishop, *Representing measures for points in a uniform algebra*, Bull. Amer. Math. Soc. **70** (1964), 121-122.
5. C. Kuratowski, *Topologie*, vol. 1 (quatrième édition), Monografie Matematyczne, Warsaw, 1958.
6. Joram Lindenstrauss, *Extensions of compact operators*, Mem. Amer. Math. Soc. **48**, pp. 112.
7. P. A. Loeb and B. Walsh, *The equivalence of Harnack's principle and Harnack's inequality in the axiomatic potential theory of Brelot*, Ann. Inst. Fourier (Grenoble) **15** (1965), 597-600.
8. Mitsuru Nakai, *Radon-Nikodym densities between harmonic measures on the ideal boundary of an open Riemann surface*, Nagoya Math. J. **27** (1965), 71-76.
9. Robert R. Phelps, *Lectures on Choquet's Theorem*, Van Nostrand, Princeton, 1966.
10. Bertram Walsh and Peter A. Loeb, *Nuclearity in axiomatic potential theory*, Bull. Amer. Math. Soc. **72** (1966), 685-689.

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Herbert Stanley Bear, Jr. and Bertram John Walsh, <i>Integral kernel for one-part function spaces</i>	209
Mario Borelli, <i>Some results on ampleness and divisorial schemes</i>	217
John A. Erdos, <i>Unitary invariants for nests</i>	229
Nathaniel Grossman, <i>The volume of a totally-geodesic hypersurface in a pinched manifold</i>	257
D. M. Hyman, <i>A generalization of the Borsuk-Whitehead-Hanner theorem</i>	263
I. Martin (Irving) Isaacs, <i>Finite groups with small character degrees and large prime divisors</i>	273
I. Martin (Irving) Isaacs, <i>Two solvability theorems</i>	281
William Lee Johnson, <i>The characteristic function of a harmonic function in a locally Euclidean space</i>	291
Ralph David Kopperman, <i>Application of infinitary languages to metric spaces</i>	299
John Lauchlin MacDonald, <i>Relative functor representability</i>	311
Mahendra Ganpatrao Nadkarni, <i>A class of measures on the Bohr group</i>	321
Keith Lowell Phillips, <i>Hilbert transforms for the p-adic and p-series fields</i>	329
Norman R. Reilly and Herman Edward Scheiblich, <i>Congruences on regular semigroups</i>	349
Neil William Rickert, <i>Measures whose range is a ball</i>	361
Gideon Schwarz, <i>Variations on vector measures</i>	373
Ronald Cameron Riddell, <i>Spectral concentration for self-adjoint operators</i>	377
Haskell Paul Rosenthal, <i>A characterization of restrictions of Fourier-Stieltjes transforms</i>	403