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A GENERALIZATION OF THE BORSUK-WHITEHEAD-HANNER THEOREM

D. M. HYMAN

Let A and B be metric spaces and let $f: A \to B$ be a map. Suppose that X and Y are ANR's containing A and B, respectively, as closed subsets, and consider f to be a map from A into Y. One of the results of this paper is that the question as to whether or not the adjunction space $X \bigcup_f Y$ is an absolute neighborhood extensor for metric pairs (or ANR if $X \bigcup_f Y$ is metrizable) depends only on f and not on X and Y; that is, if $X \bigcup_f Y$ is an ANE (metric) and if X and Y are replaced by ANR's X' and Y', respectively, then $X' \bigcup_f Y'$ is an ANE (metric). This result is a consequence of the main theorem: Let B be a strong neighborhood deformation retract of a space Y and suppose that both B and $Y - B$ are ANE (metric). If $Y - B$ has a certain type of covering, then Y is an ANE (metric). This generalizes the known result that if Y is metrizable, then Y is an ANR.

By a pair (X, A) we shall mean a space X together with a closed subset A. If a space Y has the property that for every metric pair (X, A), each map $f: A \to Y$ has a neighborhood extension, then Y is called an absolute neighborhood extensor for metric pairs (abbreviated ANE). In particular, a space is an ANR if and only if it is a metrizable ANE [2].

Let (X, A) be a pair, and let $f: A \to Y$ be a map. It is well known [4, p. 178] that if X, A and Y are ANR's, then the adjunction space $X \bigcup_f Y$ is an ANR provided that it is metrizable. This result was essentially proved in successive stages by Borsuk [1], Whitehead [7], and Hanner [3]. Our purpose is to generalize this theorem.

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2. **The main theorem.** Let (Y, B) be a pair. Generalizing the notion of a canonical cover [2], we say that a collection $\{V_\alpha\}$ of open subsets of Y is a semi-canonical cover of (Y, B) if (1) $\bigcup_\alpha V_\alpha = Y - B$ and (2) for each $b \in B$ and each neighborhood U of b there is a neighborhood W of b such that $V_\alpha \subset U$ whenever V_α meets W.[1] If a semi-canonical cover exists for a pair (Y, B), we call (Y, B) a semi-canonical pair.

For later use, we establish the following simple property of semi-

[1] A semi-canonical cover differs from a canonical cover only in that a semi-canonical cover is not required to be locally finite.

canonical covers.

LEMMA 2.1. *Suppose that* $\{V_\alpha\}$ *is a semi-canonical cover for a pair* (Y, B). *Let* $\{x_\nu\}$ *and* $\{y_\nu\}$ *be two nets in* $Y - B$, *and suppose that for each* ν, x_ν *and* y_ν *lie in a common element* V_ν *of* $\{V_\alpha\}$. *Then* $\{x_\nu\}$ *converges to a point* $b \in B$ *if and only if* $\{y_\nu\}$ *converges to* b.

Proof. Suppose that $\{x_\nu\}$ converges to b. Let U be any neighborhood of b, and let W be a neighborhood of b such that $V_\alpha \subset U$ whenever $V_\alpha \cap W \neq \emptyset$. Since $\{x_\nu\}$ is eventually in W, the sets $\{V_\nu\}$ eventually lie in U, and since $y_\nu \in V_\nu$, it follows that $\{y_\nu\}$ converges to b. The converse is proved similarly.

REMARK. If $\{V_\alpha\}$ is a semi-canonical cover of (Y, B) and if for each $y \in Y - B$ an element—call it V_y—of $\{V_\alpha\}$ containing y is chosen, then the collection $\{V_y\}$, $y \in Y - B$, is a semi-canonical cover of (Y, B).

A closed subset $B \subset Y$ is called a strong neighborhood deformation retract of Y if there exists a neighborhood W of B and a homotopy $h: W \times I \to Y$ such that h_0 is the inclusion, h_1 is a retraction of W onto B, and $h(b, t) = b$ for all $b \in B, t \in I$. h is called a strong deformation retraction of W onto B.

We now establish the main theorem.

THEOREM 2.2. *Let* (Y, B) *be a semi-canonical pair such that* B *is a strong neighborhood deformation retract of* Y. *If both* B *and* $Y - B$ *are* ANE, *then* Y *is an* ANE.

Proof. By hypothesis, there exists a strong deformation retraction $h: W \times I \to Y$ onto B. Let $\{V_y\}$, $y \in Y - B$, be a semi-canonical cover for (Y, B) as in the remark above.

To prove that Y is an ANE it is sufficient to show that for any metric pair (X, A), each map $f: A \to W$ has a neighborhood extension $F: U \to Y$. For from this it follows first that $F \mid F^{-1}(W): F^{-1}(W) \to W$ is a neighborhood extension of f, so that W is an ANE; and then Y, being the union of the open ANE subspaces W and $Y - B$, is itself an ANE [4, p. 44]. Given (X, A) and $f: A \to W$, we proceed to construct F.

Let $A_0 = f^{-1}(B)$, $A_1 = A - A_0$ and $X_1 = X - A_0$. Then $f(A_1) \subset Y - B$, and since $Y - B$ is an ANE, there is a neighborhood G_1 of A_1 in X_1 and a map $\phi_1: G_1 \to Y - B$ such that $\phi_1 \mid A_1 = f \mid A_1$. Let d be a metric on X. For each $a \in A_1$, let G_a be the set of points x in G_1 such that

(1) $d(x, A_0) > 1/2\, d(a, A_0)$,

(2) $d(x, a) < d(a, A_0)$,

(3) $x \in \phi_1^{-1}(V_{\phi_1(a)})$, and
(4) $x \in \phi_1^{-1}(W)$.

Let $G_2 = \bigcup \{G_a \mid a \in A_1\}$. G_2 is open in X_1 and contains A_1. Let G be a neighborhood of A_1 in X_1 such that its closure K (in X_1) is contained in G_2, and let $\lambda\colon X_1 \to [0, 1]$ be a map such that $\lambda(A_1) = 0$ and $\lambda(X_1 - G) = 1$. Define $\phi_2\colon K \cup A_0 \to Y$ by

$$\begin{aligned} \phi_2(x) &= h(\phi_1(x), \lambda(x)) && \text{if } x \in K\,, \\ &= f(x) && \text{if } x \in A_0\,. \end{aligned}$$

ϕ_2 is well-defined and extends f. Furthermore, ϕ_2 is clearly continuous except possibly at those points of A_0 which are limit points of $K - A_1$. To prove its continuity at these points also, we suppose $a \in A_0$ is the limit of a sequence $\{x_n\}$ in $K - A_1$ and show that $\{\phi_2(x_n)\}$ converges to $\phi_2(a)$. For each n, choose $a_n \in A_1$ such that $x_n \in G_{a_n}$. Since $\{x_n\}$ converges to $a \in A_0$, it follows from (1) that $\{d(a_n, A_0)\} \to 0$, and from (2) that $d\{(x_n, a_n)\} \to 0$. Therefore $\{a_n\}$ converges to a. Since $\{\phi_1(a_n)\} = \{f(a_n)\}$ converges to $f(a)$, we find by (3) and 2.1 that $\{\phi_1(x_n)\}$ converges to $f(a)$. Given a neighborhood V of $f(a)$ in Y, there is a neighborhood V_1 of $f(a)$ such that $h(V_1 \times I) \subset V$. Since $\{\phi_1(x_n)\}$ converges to $f(a)$, $\{\phi_1(x_n)\}$ is eventually in V_1, and by the definition of ϕ_2, $\{\phi_2(x_n)\}$ is eventually in V. Therefore ϕ_2 is continuous at a, and hence is continuous on $K \cup A_0$.

Since $\lambda = 1$ on the boundary (in X_1) of G, and since h maps $W \times 1$ into B, it follows that ϕ_2 maps the boundary (in X) of $K \cup A_0$ into B. Since B is an ANE, it follows that ϕ_2 has an extension $F\colon U \to Y$ for some open set U in X, and the proof is complete.

3. Applications.

In order to apply Theorem 2.2, it is necessary to have on hand some semi-canonical pairs. For this purpose we establish.

LEMMA 3.1. *Every metric pair (Y, B) is semi-canonical.*

Proof. As in [2], for each $y \in Y - B$ let V_y be the open $\varepsilon/2$ ball centered at y, where ε is the distance from y to B under some fixed metric for Y. The collection $\{V_y\}$ is a semi-canonical cover for (Y, B).

Combining 3.1 and 2.2, we obtain the following result, which was first proved in [5]:

THEOREM 3.2. (*Kruse-Liebnitz*). *Let (Y, B) be a metric pair such that B is a strong neighborhood deformation retract of Y. If B and $Y - B$ are* ANR's, *then Y is an* ANR.

Given a metric space A, let ANR(A) denote the class of all ANR's that contain A as a closed subset. Let f be a map from A into an ANR Y. Our next result (3.5) states that either the adjunction space $X \bigcup_f Y$ is an ANE for every $X \in \text{ANR}(A)$ or for no $X \in \text{ANR}(A)$. Therefore, given an $X \in \text{ANR}(A)$, the question of whether or not $X \bigcup_f Y$ is an ANE depends only on the map f, and not on the choice of X.

To obtain this result from 2.2, some additional information concerning semi-canonical covers and strong neighborhood deformation retractions will be needed. The necessary facts are supplied by the following lemmas.

For any pair (X, A) and map $f: A \to Y$, let $X + Y$ denote the disjoint union of X and Y, and let $p: X + Y \to X \bigcup_f Y$ be the natural projection.

LEMMA 3.3. *Let (X, A) be a pair and let $f: A \to Y$ be a map. If $\{V_\alpha\}$ is a semi-canonical cover for $(X + Y, A + Y)$, then $\{p(V_\alpha)\}$ is a semi-canonical cover for $(X \bigcup_f Y, p(Y))$.*

Proof. Since p maps $X - A$ homeomorphically onto $X \bigcup_f Y - p(Y)$, it follows that each $p(V_\alpha)$ is open and $\bigcup_\alpha p(V_\alpha) = X \bigcup_f Y - p(Y)$. Let $y \in p(Y)$ and let U be a neighborhood of y. Since $\{V_\alpha\}$ is semi-canonical, for each $x \in p^{-1}(U \cap p(Y))$ there is a neighborhood $W_x \subset p^{-1}(U)$ such that $V_\alpha \subset p^{-1}(U)$ whenever $V_\alpha \cap W_x \neq \varnothing$. Let $W = \bigcup \{W_x \mid x \in p^{-1}(U \cap p(Y))\}$.

From our construction it is clear that $y \in p(W)$ and that $p(V_\alpha) \subset U$ whenever $p(V_\alpha) \cap p(W) \neq \varnothing$. It remains to show that $p(W)$ is open. Since p is an identification, it is sufficient to show that W is saturated, that is, $W = p^{-1}(S)$ for some $S \subset X \bigcup_f Y$. From our construction we have $W \cap p^{-1}(p(Y)) = p^{-1}(U) \cap p^{-1}(p(Y)) = p^{-1}(U \cap p(Y))$. Moreover, since p is one-to-one on $(X + Y) - p^{-1}(p(Y))$ it follows that $W - p^{-1}(p(Y))$ is saturated. Since W is the union of the saturated sets $W \cap p^{-1}(p(Y))$ and $W - p^{-1}(p(Y))$, W itself is saturated, and the lemma is proved.

LEMMA 3.4. *Let X and Y be* ANR'*s, and let $f: A \to Y$ be a map, where A is a closed subset of X. Then $X \bigcup_f Y$ is an* ANE *if and only if $p(Y)$ is a strong neighborhood deformation retract of $X \bigcup_f Y$.*

Proof. Suppose that $X \bigcup_f Y$ is an ANE. Since Y is an ANR, f has an extension $F: \bar{U} \to Y$, where U is some neighborhood of A in X. Define a map $g: X \times \{0\} \cup A \times I \cup \bar{U} \times \{1\} \to X \bigcup_f Y$ by

$$g(x, 0) = p(x) \qquad \text{if } x \in X ;$$
$$g(a, t) = p(a) \qquad \text{if } a \in A , \quad 0 \leqq t \leqq 1 ;$$
$$g(x, 1) = pF(x) \qquad \text{if } x \in \bar{U} .$$

Since $X \cup_f Y$ is an ANE, g has an extension $G: V \to X \cup_f Y$, for some open subset V of $X \times I$. Let W be a neighborhood of A in X such that $W \times I \subset V$. The map $h: p(W + Y) \times I \to X \cup_f Y$ defined by

$$\begin{aligned} h(z, t) &= G((p \mid X)^{-1}(z), t) && \text{if } z \in p(W) , \quad 0 \leqq t \leqq 1 , \\ &= z && \text{if } z \in p(Y) , \quad 0 \leqq t \leqq 1 , \end{aligned}$$

is the desired deformation.

The converse is an immediate consequence of 3.3 and 2.2.

We now obtain the main result of this section.

THEOREM 3.5. *Let f be a map from an arbitrary metric space A into an* ANR *Y. If $X_0 \cup_f Y$ is an* ANE *for some $X_0 \in$* ANR*(A), then $X \cup_f Y$ is an* ANE *for every $X \in$* ANR*(A).*

Proof. Given $X \in \mathrm{ANR}(A)$, let $p: X + Y \to X \cup_f Y$ and $q: X_0 + Y \to X_0 \cup_f Y$ be the natural projections. To prove that $X \cup_f Y$ is an ANE it is sufficient, by 3.4, to show that $p(Y)$ is a strong neighborhood deformation retract of $X \cup_f Y$.

Since X is an ANR, there exists a neighborhood G of A in X_0 and a map $\phi: G \to X$ such that $\phi \mid A$ is the identity map. By 3.4, there is a neighborhood W of $q(Y)$ in $X_0 \cup_f Y$ and a strong deformation retraction h of W onto $q(Y)$ over $q(G + Y)$. Since $q^{-1}(W) \cap X_0$ is open in X_0, $q^{-1}(W) \cap X_0$ is an ANR; therefore there exists a neighborhood U of A in X and a map $\psi: U \to q^{-1}(W) \cap X_0$ such that $\psi \mid A$ is the identity map. Since U is open in X, U is an ANR; and it follows that there exists a neighborhood V of A in U and a deformation $j: V \times I \to U$ such that $j(a, t) = a$, for all $a \in A$, $0 \leqq t \leqq 1$, and such that $j_1 = \phi\psi \mid V$. Letting $\phi + 1_Y: G + Y \to X + Y$ be the map defined by ϕ and the identity on Y, define a map $k: p(V + Y) \times I \to X \cup_f Y$ by

$$\begin{aligned} k_t(z) &= pj_{2t}(p \mid X)^{-1}(z) && \text{if } z \in p(V) , \ 0 \leqq t \leqq 1/2 , \\ &= p(\phi + 1_Y)q^{-1}h_{2t-1}q\psi(p \mid X)^{-1}(z) && \text{if } z \in p(V) , \ 1/2 \leqq t \leqq 1 , \\ &= z && \text{if } z \in p(Y) , \ 0 \leqq t \leqq 1 . \end{aligned}$$

It is easily verified that k is a strong deformation retraction of $p(V + Y)$ onto $p(Y)$, and the proof is complete.

An application of 3.5 gives a direct generalization of the BWH theorem:

COROLLARY 3.6. *Let* (X, A) *be a pair, and let* $f: A \to Y$ *be a map. If* X, A *and* Y *are* ANR's, *then* $X \bigcup_f Y$ *is an* ANE.

Proof. This result can be obtained as a consequence of 3.3 and 2.2, but it also follows quite simply from 3.5: Taking $X_0 = A$, we see that $X_0 \bigcup_f Y$ is an ANR, since it is homeomorphic to Y. Therefore by 3.5, $X \bigcup_f Y$ is an ANE.

If we take Y in 3.5 to be a single point, we obtain

COROLLARY 3.7. *If* A *is a metric space, then either* X/A *is an* ANE *for every* $X \in \mathrm{ANR}(A)$ *or for no* $X \in \mathrm{ANR}(A)$.

If A is a compact subset of a metric space X, then X/A is metrizable [6]. Therefore we have from 3.7

COROLLARY 3.8. *If* A *is a compact metric space, then either* X/A *is an* ANR *for every* $X \in \mathrm{ANR}(A)$ *or for no* $X \in \mathrm{ANR}(A)$.

We have seen that for a map $f: A \to Y$, the question of whether or not $X \bigcup_f Y$ is an ANE is independent of the choice of $X \in \mathrm{ANR}(A)$. Our final result, which slightly generalizes 3.5, shows that this question is also independent of Y. Precisely, we have

THEOREM 3.9. *Let* A *and* B *be metric spaces and let* $f: A \to B$ *be a map. Either* $X \bigcup_f Y$ *is an* ANE *for every* $X \in \mathrm{ANR}(A)$ *and* $Y \in \mathrm{ANR}(B)$ *or for no* $X \in \mathrm{ANR}(A)$ *and* $Y \in \mathrm{ANR}(B)$.

REMARK. For $Y \in \mathrm{ANR}(B)$, we consider f to be not only a map from A into B but also from A into Y. This justifies the symbol $X \bigcup_f Y$.

Proof of Theorem. Suppose that $X \bigcup_f Y_0$ is an ANE for some $X \in \mathrm{ANR}(A)$ and some $Y_0 \in \mathrm{ANR}(B)$. In view of 3.5, we need only to show that if $Y \in \mathrm{ANR}(B)$ then $X \bigcup_f Y$ is an ANE.

Since Y is an ANR, there is a neighborhood U of B in Y_0 and a map $\phi: U \to Y$ such that $\phi(b) = b$ for all $b \in B$.

Letting $p: X + Y \to X \bigcup_f Y$ and $q: X + U \to X \bigcup_f U$ be the natural projections, define a map $\psi: X \bigcup_f U \to X \bigcup_f Y$ by

$$\begin{aligned} \psi(z) &= p(q \mid X)^{-1}(z) \qquad \text{if } z \in q(X)\,, \\ &= p\phi(q \mid U)^{-1}(z) \qquad \text{if } z \in q(U)\,. \end{aligned}$$

$X \bigcup_f U$ is open in $X \bigcup_f Y_0$, and therefore $X \bigcup_f U$ is an ANE. By 3.4 there is a strong deformation retraction h of an open set W onto $q(U)$ in $X \bigcup_f U$. Define a homotopy $k_t: \psi(W) \cup p(Y) \to X \bigcup_f Y$ by

$$\begin{aligned} k_t(z) &= \psi h_t \psi^{-1}(z) && \text{if } z \in \psi(W)\,, \\ &= z && \text{if } z \in p(Y)\,. \end{aligned}$$

It follows from the equation $\psi(W) \cup p(Y) = p((q \mid X)^{-1}(W) + Y)$ that $\psi(W) \cup p(Y)$ is an open subset of $X \bigcup_f Y$, and it is easily verified that k is a strong deformation retraction of $\psi(W) \cup p(Y)$ onto $p(Y)$. The result now follows from 3.4.

4. **Results for AR's.** In this section we establish results for AR's and AE's analogous to Theorems 2.2 and 3.9. A space Y is called an absolute extensor for metric pairs (abbreviated AE) if for every metric pair (X, A) each map $f: A \to Y$ has an extension $F: X \to Y$. A link between AE's and ANE's is provided by the following

LEMMA 4.1. *If Y is an* ANE *and if Y can be deformed into an* AE *subspace, then Y is an* AE.

Proof. Let $B \subset Y$ be an AE and let $h: Y \times I \to Y$ be a deformation such that $h_1(Y) \subset B$. Suppose that (X, A) is a metric pair and let $f: A \to Y$ be a map. Since Y is an ANE, there is a neighborhood U of A in X and an extension $F: \bar{U} \to Y$ of f. Let $g: X \to [0, 1]$ be a map such that $g(A) = 0$ and $g(X - U) = 1$. Since B is an AE, there is a map $G: X - U \to B$ such that $G \mid \text{bdry } U = h_1 F \mid \text{bdry } U$. Define a map $\phi: X \to Y$ by

$$\begin{aligned} \phi(x) &= h(F(x), g(x)) && \text{if } x \in \bar{U}\,, \\ &= G(x) && \text{if } x \in X - U\,. \end{aligned}$$

ϕ extends f, and the lemma is proved.

We now establish the analog of 2.2.

THEOREM 4.2. *Let (Y, B) be a semi-canonical pair such that B is a strong deformation retract of Y. If B is an* AE *and if $Y - B$ is an* ANE, *then Y is an* AE.

Proof. By 2.2, Y is an ANE. Since by hypothesis Y is deformable into B, Y is an AE by 4.1.

In order to obtain the analog of 3.9, we will need the analog of 3.4.

LEMMA 4.3. *Let X and Y be* AR'*s, and let $f: A \to Y$ be a map, where A is a closed subset of X. Then $X \bigcup_f Y$ is an* AE *if and only if $p(Y)$ is a strong deformation retract of $X \bigcup_f Y$.*

Proof. Suppose that $X \bigcup_f Y$ is an AE. Since Y is an AR, f has an extension $F: X \to Y$. Since $X \bigcup_f Y$ is an AE, the map

$$g: X \times \{0\} \cup A \times I \cup X \times \{1\} \to X \bigcup_f Y$$

defined by

$$\begin{aligned} g(x, 0) &= p(x) && \text{if } x \in X, \\ g(a, t) &= p(a) && \text{if } a \in A, \quad 0 \leqq t \leqq 1, \\ g(x, 1) &= pF(x) && \text{if } x \in X, \end{aligned}$$

has an extension $G: X \times I \to X \bigcup_f Y$. The map $h: X \bigcup_f Y \times I \to X \bigcup_f Y$ defined by

$$\begin{aligned} h(z, t) &= G((p \mid X)^{-1}(z), t) && \text{if } z \in p(X), \quad 0 \leqq t \leqq 1 \\ &= z && \text{if } z \in p(Y), \quad 0 \leqq t \leqq 1 \end{aligned}$$

is the desired deformation.

Conversely, if $p(Y)$ is a strong deformation retract of $X \bigcup_f Y$, then $X \bigcup_f Y$ is an ANE by 3.4 and an AE by 4.1.

We now establish the analog of 3.9.

THEOREM 4.4. *Let A and B be metric spaces and let $f: A \to B$ be a map. Either $X \bigcup_f Y$ is an* AE *for every $X \in$* AR(A) *and $Y \in$* AR(B) *or for no $X \in$* AR(A) *and $Y \in$* AR(B).

Proof. Suppose $X_0 \bigcup_f Y_0$ is an AE for some $X_0 \in$ AR(A) and $Y_0 \in$ AR(B), and suppose $X \in$ AR(A) and $Y \in$ AR(B). Let $p: X + Y \to X \bigcup_f Y$ and $q: X_0 + Y_0 \to X_0 \bigcup_f Y_0$ be the natural projections.

By 3.9, $X \bigcup_f Y$ is an ANE; to prove that it is an AE it is sufficient, by 4.3, to show that $X \bigcup_f Y$ can be deformed into $p(Y)$. Since X and X_0 are AR's, there are maps $\phi: X \to X_0$ and $\phi_0: X_0 \to X$, each extending the identity on A, and a deformation j_t on X leaving A pointwise fixed and such that $j_1 = \phi_0\phi$. Similarly, there are maps $\psi: Y \to Y_0$ and $\psi_0: Y_0 \to Y$, each extending the identity on B, and a deformation k_t on Y leaving B pointwise fixed and such that $k_1 = \psi_0\psi$. By 4.3, there is a strong deformation retraction h_t of $X_0 \bigcup_f Y_0$ onto $q(Y_0)$. Define a deformation g_t on $X \bigcup_f Y$ by

$$\begin{aligned} g_t(z) &= pj_{2t}(p \mid X)^{-1}(z) && \text{if } z \in p(X), \quad 0 \leqq t \leqq 1/2, \\ &= pk_{2t}(p \mid Y)^{-1}(z) && \text{if } z \in p(Y), \quad 0 \leqq t \leqq 1/2, \\ &= p(\phi_0 + \psi_0)q^{-1}h_{2t-1}q\phi(p \mid X)^{-1}(z) && \text{if } z \in p(X), \quad 1/2 \leqq t \leqq 1, \\ &= p(\phi_0 + \psi_0)q^{-1}h_{2t-1}q\psi(p \mid Y)^{-1}(z) && \text{if } z \in p(Y), \quad 1/2 \leqq t \leqq 1, \end{aligned}$$

where $\phi_0 + \psi_0: X_0 + Y_0 \to X + Y$ is the map defined by ϕ_0 and ψ_0. g

deforms $X \bigcup_f Y$ into $p(Y)$, and the proof is complete.

By taking B to be a single point, we obtain

COROLLARY 4.5. *If A is a metric space, then either X/A is an* AE *for every $X \in \mathrm{AR}(A)$ or for no $X \in \mathrm{AR}(A)$.*

COROLLARY 4.6. *If A is a compact metric space, then either X/A is an* AR *for every $X \in \mathrm{AR}(A)$ or for no $X \in \mathrm{AR}(A)$.*

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