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CONGRUENCES ON REGULAR SEMIGROUPS

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For any regular semigroup S the relation θ is defined on the lattice, $\mathcal{A}(S)$, of congruences on S by: $(\rho, \tau) \in \theta$ if and only if ρ and τ induce the same partition of the idempotents of S . Then θ is an equivalence relation on $\mathcal{A}(S)$ such that each equivalence class is a complete modular sublattice of $\mathcal{A}(S)$. If S is an inverse semigroup then θ is a congruence on $\mathcal{A}(S)$, $\mathcal{A}(S)/\theta$ is complete and the natural homomorphism of $\mathcal{A}(S)$ onto $\mathcal{A}(S)/\theta$ is a complete lattice homomorphism.

Any congruence on an inverse semigroup S can be characterized in terms of its kernel, namely, the set of congruence classes containing the idempotents of S . In particular, any congruence on S induces a partition of the set E_S of idempotents of S satisfying certain normality conditions. In this note, those partitions of E_S which are induced by congruences on S and the largest and smallest congruences on S corresponding so such a partition of E_S are characterized.

1. Preliminary results and definitions. We adopt the notation and terminology of Clifford and Preston [2]. A semigroup S is called *regular* if $a \in aSa$, for all $a \in S$. If, for all $a \in S$, there exists an element $b \in S$ such that $aba = a$ and $bab = b$ then we say that b is an *inverse* of a and that (a, b) is a *regular pair* [11]. In a regular semigroup, every element has an inverse. An *inverse semigroup* is a semigroup in which each element has a unique inverse. The elementary properties of regular and inverse semigroups can be found in [2]. In particular, a semigroup S is an inverse semigroup if and only if it is regular and its idempotents commute ([2], Th. 1.17). The inverse of an element a is then denoted by a^{-1} . For any idempotent e , $e^{-1} = e$, and, for any elements a, b of S

$$(a^{-1})^{-1} = a, (ab)^{-1} = b^{-1}a^{-1}.$$

If (a, a') is a regular pair, then aa' and $a'a$ are both idempotents but are not always equal (even in an inverse semigroup).

A *regular (inverse) subsemigroup* T of a semigroup S is just a subsemigroup of S which is a regular (inverse) semigroup in its own right.

For any semigroup S we shall denote by E_S the set of idempotents of S . The set E_S can be partially ordered by defining $e \leq f$ if and only if $ef = fe = e$. Of course, if S is an inverse semigroup then this reduces to $ef = e$ and E_S then becomes a semilattice, with $e \wedge f = ef$, called the *semilattice of idempotents* of S .

We shall call a subset A of a partially ordered set B *convex* if $x \leq y \leq z$, $x, z \in A$ implies that $y \in A$.

If ρ is a congruence on a semigroup S then we shall denote by $\rho|_{E_s}$ the partition of E_s induced by ρ , that is $\rho|_{E_s} = \rho \cap (E_s \times E_s)$, and by $a\rho$ the ρ -class containing the element a . We shall also make use of the fact that, in an inverse semigroup, if $(a, b) \in \rho$ then $(a^{-1}, b^{-1}) \in \rho$ ([4] Corollary 2.3) and, consequently, $(aa^{-1}, bb^{-1}) \in \rho$.

Clearly a homomorphic image of a regular semigroup is regular and it was established in [12] that a homomorphic image of an inverse semigroup is an inverse semigroup.

Finally, two elements of a semigroup S are said to be \mathcal{L} - (\mathcal{R} -) equivalent if they generate the same principal left (right) ideal of S . Clearly \mathcal{L} and \mathcal{R} are equivalence relations on S , as is the relation \mathcal{H} , defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$.

2. Maximal regular subsemigroups. In this section we shall generalize the well known result that for any idempotent e of a semigroup S there is a unique maximum subgroup of S with identity e .

LEMMA 1.1. *Let (a, a') and (b, b') be regular pairs in a semigroup S . Then $a'a$ bb' and bb' $a'a$ are idempotents of S if and only if $(ab, b'a')$ is a regular pair.*

Proof. Let $a'a$ bb' and bb' $a'a$ be idempotents. Then

$$(ab)(b'a')(ab) = aa'abb'a'abb'b = aa'abb'b = ab$$

and

$$(b'a')(ab)(b'a') = b'bb'a'abb'a'aa' = b'bb'a'aa' = b'a'$$

as required.

Conversely, if $(ab, b'a')$ is a regular pair, then

$$(a'abb')(a'abb') = a'(ab)(b'a')(ab)b' = a'abb'$$

and similarly $bb'a'a$ is an idempotent.

LEMMA 1.2. ([5] Lemma 1.1) *If e and f are idempotents in a regular semigroup S then, for some idempotent g in S , (g, ef) is a regular pair.*

LEMMA 1.3. *For a regular semigroup S the following are equivalent:*

- (1) $E_s E_s \subseteq E_s$;
- (2) $e \in E_s$, (e, x) a regular pair implies that $x \in E_s$;

(3) (a, a') and (b, b') regular pairs implies that $(ab, b'a')$ is a regular pair.

Proof. (1) implies (2). (Cf. [11] Th. 1.) Let $e \in E_s$ and (e, x) be a regular pair. Then $x = xex = xeex \in E_s E_s \subseteq E_s$. (2) implies (1). Let $e, f \in E_s$, then by Lemma 1.2, for some idempotent $g, (g, ef)$ is a regular pairs and so $ef \in E_s$.

(1) implies (3) by Lemma 1.1.

(3) implies (1). Let $e, f \in E_s$. Then, since (e, e) and (f, f) are regular pairs, (ef, fe) is also a regular pair. Then $ef = (ef)(fe)(ef) = (ef)(ef)$ and $ef \in E_s$.

LEMMA 1.4. *Let S be a regular semigroup such that E_s is a subsemigroup. Then, for any regular pair (a, a')*

$$aE_s a' \subseteq E_s .$$

Proof. Let (a, a') be a regular pair and $e \in E_s$. Since

$$a'ae \in E_s E_s \subseteq E_s ,$$

we have

$$(aea')^2 = aea'aea' = aa'aea'aea' = aa'aea' = aea' .$$

THEOREM 1.5. *Let E be an idempotent subsemigroup of a semigroup S . Then $E^c = \{x \in S: \text{for some } x', (x, x') \text{ is a regular pair, } xx', x'x \in E, xEx' \subseteq E \text{ and } x'Ex \subseteq E\}$ is the largest regular subsemigroup of with E as its set of idempotents.*

Proof. Let $a, b \in E^c$ and let a', b' be elements of S such that a, a' and b, b' satisfy the conditions of membership for a and b , respectively. Then clearly $a', b' \in E^c$. Moreover, since $a'a, bb' \in E$, we have $a'abb'$ and $bb'a'a$ contained in E and so, by Lemma 1.1, $(ab, b'a')$ is a regular pair. Also

$$abb'a' \in aEa' \subseteq E, b'a'ab \in b'Eb \subseteq E$$

and

$$abEb'a' \subseteq aEa' \subseteq E, b'a'Eab \subseteq b'Eb \subseteq E .$$

Thus $ab \in E^c$ and E^c is a subsemigroup of S . Since, as pointed out above, for each $a \in E^c$ some inverse of a is also in E^c , E^c is a regular subsemigroup of S .

To show that $E_{E^c} = E$, let a be an idempotent in E^c and let (a, a') be a regular pair with $a, a' \in E^c$. Then

$$a' = a'aa' = a'aaa' \in EE \subseteq E$$

and so

$$a = aa'a = aa'a'a \in EE \subseteq E.$$

Hence $E_{E^c} = E$, since clearly $E \subseteq E_{E^c}$.

Now suppose that T is a regular subsemigroup of S such that $E_T = E$. Then, for any $x \in T$, there exists an $x' \in T$ such that (x, x') is a regular pair in T and so $xx', x'x \in E_T = E$. Then, since $EE \subseteq E$, we have, by Lemma 1.4,

$$xE'x' \subseteq E \text{ and } x'Ex \subseteq E.$$

Thus $x \in E^c$.

COROLLARY 1.6. *Let E be a subsemigroup of commuting idempotents of a semigroup S . Then E^c (defined as in Theorem 1.5) is the largest inverse subsemigroup of S with E as its set of idempotents.*

Proof. By Theorem 1.5, E^c is the largest regular subsemigroup with E as its set of idempotents. Since the elements of E commute, E^c is, in fact, an inverse subsemigroup and so is clearly the largest such.

3. The lattice of congruences on a regular semigroup. For any semigroup S we denote by $\mathcal{A}(S)$ the lattice of congruences on S .

If σ and ρ are congruences on a semigroup S such that $\sigma \subseteq \rho$ then the relation ρ/σ on S/σ defined by

$$\rho/\sigma = \{(x\sigma, y\sigma) \in S/\sigma \times S/\sigma : (x, y) \in \rho\}$$

is a congruence on S/σ . Moreover, the mapping $\rho \rightarrow \rho/\sigma$ is a one-to-one order preserving mapping of the congruences ρ on S containing σ onto the congruences on S/σ , as is easy to show.

It is straight forward to verify that if $\sigma, \rho, \tau \in \mathcal{A}(S)$ and $\sigma \subseteq \rho, \tau$ then

$$(\rho \cap \tau)/\sigma = \rho/\sigma \cap \tau/\sigma \text{ and } (\rho \vee \tau)\sigma = (\rho/\sigma) \vee (\tau/\sigma).$$

It is convenient to point out here that, for any semigroup S , $\mathcal{A}(S)$ is complete ([2] p. 24) and that if C is a nonvoid subset of $\mathcal{A}(S)$ then $\bigvee_{\rho \in C} \rho$ may be characterized as $\{(x, y) \in S \times S : \text{there exist } x_1, \dots, x_n \in S \text{ and } \rho_1, \dots, \rho_{n+1} \in C \text{ (not necessarily all distinct) such that } (x, x_1) \in \rho_1, (x_1, x_2) \in \rho_2, \dots, (x_n, y) \in \rho_{n+1}\}$.

In order to show that a sublattice R of $\mathcal{A}(S)$ is a modular sublattice, it follows from ([1] Th. 3, p. 86) that it suffices to show that the congruences in R commute; that is, that $\rho, \tau \in R$ implies that $\rho \circ \tau = \tau \circ \rho$.

We shall call a sublattice R of $\mathcal{A}(S)$ a *complete sublattice* if, for $C \subseteq R$, $\bigvee_{\rho \in C} \rho$ and $\bigcap_{\rho \in C} \rho$ not only exist in $\mathcal{A}(S)$, but also belong to R .

LEMMA 3.1. *Let S be a semigroup and $R = \{\rho_i : i \in I\}$ be a subset of $\mathcal{A}(S)$ such that $\sigma = \bigcap_{i \in I} \rho_i \in R$. If $R/\sigma = \{\rho_i/\sigma : i \in I\}$ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\mathcal{A}(S/\sigma)$ then R is a sublattice (sublattice of commuting congruences, complete sublattice) of $\mathcal{A}(S)$.*

Proof. Let $\rho_1, \rho_2 \in R$. Then $\rho_1/\sigma \cap \rho_2/\sigma$ and $\rho_1/\sigma \vee \rho_2/\sigma$ belong to R/σ , as R/σ is a sublattice of $\mathcal{A}(S/\sigma)$. Now,

$$\rho_1/\sigma \cap \rho_2/\sigma = (\rho_1 \cap \rho_2)/\sigma \quad \text{and} \quad \rho_1/\sigma \vee \rho_2/\sigma = (\rho_1 \vee \rho_2)/\sigma$$

and so, by the one-to-one nature of the mapping $\rho \rightarrow \rho/\sigma$, it follows that $\rho_1 \cap \rho_2$ and $\rho_1 \vee \rho_2$ belong to R . Thus R is a sublattice of $\mathcal{A}(S)$.

Now let R/σ be a sublattice of commuting congruences of $\mathcal{A}(S/\sigma)$, let $\rho, \tau \in R$ and let $(a, b) \in \rho \circ \tau$. Then for some $c \in S$, $(a, c) \in \rho$ and $(c, b) \in \tau$. Hence $(a\sigma, c\sigma) \in \rho/\sigma$ and $(c\sigma, b\sigma) \in \tau/\sigma$. Since the elements of R/σ commute it follows that $\rho/\sigma \circ \tau/\sigma = \tau/\sigma \circ \rho/\sigma$ and, consequently, that there exists a $d\sigma \in S/\sigma$ such that $(a\sigma, d\sigma) \in \tau/\sigma$ and $(d\sigma, b\sigma) \in \rho/\sigma$. Hence $(a, d) \in \tau$ and $(d, b) \in \rho$; that is, $(a, b) \in \tau \circ \rho$. Thus $\rho \circ \tau \subseteq \tau \circ \rho$ and likewise $\rho \circ \tau \subseteq \tau \circ \rho$. Hence $\rho \circ \tau = \tau \circ \rho$ and R is a sublattice of commuting congruences of $\mathcal{A}(S)$.

Finally, let R/σ be a complete sublattice and let $C' \subseteq R$. Then $\bigvee_{\rho \in C'} \rho/\sigma$ exists and is contained in R/σ , say, $\tau/\sigma = \bigvee_{\rho \in C'} \rho/\sigma$. Then $\rho/\sigma \subseteq \tau/\sigma$, for all $\rho \in C'$ and so $\rho \subseteq \tau$. On the other hand, $\rho \subseteq \tau'$, for all $\rho \in C'$ implies that $\rho/\sigma \subseteq \tau'/\sigma$, for all $\rho \in C'$, and hence that

$$\tau/\sigma = \bigvee_{\rho \in C'} \rho/\sigma \subseteq \tau'/\sigma .$$

Thus $\tau \subseteq \tau'$ and $\bigvee_{\rho \in C'} \rho = \tau \in R$.

The verification that $\bigcap_{\rho \in C'} \rho \in R$ is even simpler. Thus R is a complete sublattice of $\mathcal{A}(S)$.

Note. It is almost immediate that if, in Lemma 3.1, R is a sublattice (sublattice of commuting congruences, complete sublattice) of $\mathcal{A}(S)$ then R/σ is a sublattice (sublattice of commuting congruences, complete sublattice) of $\mathcal{A}(S/\sigma)$.

LEMMA 3.2. ([6] Lemma 2.2) *Let ρ be a congruence on a regular semigroup S . Then each idempotent ρ -class contains an idempotent of S .*

For any semigroup S , let $\Sigma(\mathcal{H}) = \{\rho \in \Lambda(S) : \rho \subseteq \mathcal{H}\}$.
 From [8] Lemmas 1 and 3, we have,

LEMMA 3.3. *Let S be a regular semigroup. Then $\Sigma(\mathcal{H})$ is a sublattice of $\Lambda(S)$ of commuting congruences with a greatest and least element.*

We call a congruence ρ on a semigroup S *idempotent separating* if each ρ -class contains at most one idempotent.

That any congruence ρ on a semigroup such that $\rho \subseteq \mathcal{H}$ is idempotent separating follows from [2], Theorem 2.15, and the fact that every idempotent separating congruence on a regular semigroup is contained in \mathcal{H} follows from Theorem 2.3 of [6]. Thus, for any regular semigroup S , $\Sigma(\mathcal{H})$ is the set of idempotent separating congruences on S .

Now any convex subset, with a largest and smallest member, of a complete lattice is clearly a complete sublattice. Hence, by Lemma 3.3, for any regular semigroup S , since $\Sigma(\mathcal{H})$ is clearly a convex subset of $\Lambda(S)$, it follows that $\Sigma(\mathcal{H})$ (the set of idempotent separating congruences on S) is a complete sublattice of $\Lambda(S)$.

THEOREM 3.4. *Let S be a regular semigroup and let*

$$\begin{aligned} \theta &= \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) : e\rho_1 \cap E_S = e\rho_2 \cap E_S, \text{ for each } e \in E_S\} \\ &= \{(\rho_1, \rho_2) \in \Lambda(S) \times \Lambda(S) : \rho_1|E_S = \rho_2|E_S\}. \end{aligned}$$

Then

- (i) θ is a meet compatible equivalence on $\Lambda(S)$;
- (ii) each θ -class is a complete modular sublattice of $\Lambda(S)$ (with a greatest and least element).

Proof. (i) Clearly θ is an equivalence relation on $\Lambda(S)$. Let $(\rho_1, \rho_2) \in \theta$ and $\rho_3 \in \Lambda(S)$. Then

$$\rho_1 \cap (E_S \times E_S) = \rho_2 \cap (E_S \times E_S)$$

and so

$$\rho_1 \cap \rho_3 \cap (E_S \times E_S) = \rho_2 \cap \rho_3 \cap (E_S \times E_S)$$

that is $(\rho_1 \cap \rho_3, \rho_2 \cap \rho_3) \in \theta$.

- (ii) Let A be a θ -class, let $\sigma = \bigcap_{\tau \in A} \tau$ and let $\rho \in A$. For

$e, f \in E_s$, let $(e, f) \in \rho$. Then $(e, f) \in \tau$, for all $\tau \in A$ and so $(e, f) \in \sigma$. Conversely, as $\sigma \subseteq \rho$, $(e, f) \in \sigma$ implies that $(e, f) \in \rho$. Thus $\rho|_{E_s} = \sigma|_{E_s}$ and $\sigma \in A$. Thus A has a least member.

Now, for any $\rho \in A$, ρ/σ is idempotent separating. For suppose that f_1 and f_2 are idempotents of S/σ such that $(f_1, f_2) \in \rho/\sigma$. By Lemma 3.2, $f_1 = e_1\sigma$ and $f_2 = e_2\sigma$ for some idempotents e_1, e_2 of S . Thus $(e_1\sigma, e_2\sigma) \in \rho/\sigma$ and so $(e_1, e_2) \in \rho$. But $\rho|_{E_s} = \sigma|_{E_s}$ and so

$$f_1 = e_1\sigma = e_2\sigma = f_2.$$

Hence ρ/σ is idempotent separating.

On the other hand, for any congruence τ on S/σ ,

$$\tau' = \{(a, b) \in S \times S : (a\sigma, b\sigma) \in \tau\}$$

is a congruence on S . Suppose that τ is idempotent separating. If, for $e, f \in E_s$, we have $(e, f) \in \tau'$, then $(e\sigma, f\sigma) \in \tau$ and so, as τ is idempotent separating, $e\sigma = f\sigma$. Thus $\tau'|_{E_s} = \sigma|_{E_s}$ and $\tau' \in A$. Now $\tau'/\sigma = \tau$ and so $\{\rho/\sigma : \rho \in A\}$ is just the sublattice of idempotent separating congruences on S/σ . Since this, by Lemma 3.3 and the remarks following it, is a complete sublattice of $\Lambda(S/\sigma)$ of commuting congruences, we conclude from Lemma 3.1 that A is a complete sublattice of $\Lambda(S)$ of commuting congruence and so a complete modular sublattice of $\Lambda(S)$.

Finally, since A is a complete sublattice of $\Lambda(S)$, $\bigvee_{\rho \in A} \rho \in A$ and A has a greatest member.

4. Congruences on inverse semigroups. In this and the following sections we consider inverse semigroups, for which we are able to improve on the results of the previous sections.

DEFINITION 4.1. Let S be an inverse semigroup and

$$P = \{E_\alpha : \alpha \in J\}$$

be a partition of E_s . Then P is a *normal* partition of E_s if

- (i) $\alpha, \beta \in J$ implies that there exists a $\gamma \in J$ such that $E_\alpha E_\beta \subseteq E_\gamma$
- (ii) $\alpha \in J$ and $a \in S$ implies that there exists a $\beta \in J$ such that $aE_\alpha a^{-1} \subseteq E_\beta$.

If, for an inverse semigroup S , $P = \{E_\alpha : \alpha \in J\}$ is a normal partition of E_s , then E_α is convex for each $\alpha \in J$. For if $e, g \in E_\alpha$ with $e \leq f \leq g$ and $f \in E_\beta$, then $ef = e$ implies that $E_\alpha E_\beta \subseteq E_\alpha$ and so $f = gf \in E_\alpha E_\beta \subseteq E_\alpha$. Moreover, we shall denote by π_P the equivalence relation on E_s induced by P and show in the following theorem that there exists a congruence ρ on S such that $\rho|_{E_s} = \pi_P$. In fact, we give characterizations of the largest and smallest such congruences.

THEOREM 4.2. *Let $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of the semilattice of idempotents of an inverse semigroup S . Let $\sigma = \{(a, b) \in S \times S : \text{there exists an } \alpha \in J \text{ with } aa^{-1}, bb^{-1} \in E_\alpha \text{ and, for some } e \in E_\alpha, ea = eb\}$ and $\rho = \{(a, b) \in S \times S : \alpha \in J \text{ implies that, for some } \beta \in J, aE_\alpha a^{-1}, bE_\beta b^{-1} \subseteq E_\beta\}$. Then σ and ρ are, respectively, the smallest and largest congruences on S such that $\sigma|_{E_S} = \rho|_{E_S} = \pi_P$.*

Proof. Clearly σ is an equivalence relation. So let $(a, b) \in \sigma$ and $c \in S$, where $aa^{-1}, bb^{-1} \in E_\alpha$ and $ea = eb$, for some $e \in E_\alpha$. Now suppose that $(ac)(ac)^{-1} = acc^{-1}a^{-1} \in E_\gamma$ while $(bc)(bc)^{-1} = bcc^{-1}b^{-1} \in E_\delta$. Then, since $(aa^{-1})(acc^{-1}a^{-1}) = acc^{-1}a^{-1}$ and $(aa^{-1})(acc^{-1}a^{-1}) \in E_\alpha E_\gamma$, it follows that $E_\alpha E_\gamma \subseteq E_\gamma$. Likewise $E_\alpha E_\delta \subseteq E_\delta$. Now, $eacc^{-1}a^{-1} = eacc^{-1}a^{-1}e = ebcc^{-1}b^{-1}e = ebcc^{-1}b^{-1}$ where $eacc^{-1}a^{-1} \in E_\alpha E_\gamma \subseteq E_\gamma$ and $ebcc^{-1}b^{-1} \in E_\alpha E_\delta \subseteq E_\delta$. Hence $E_\gamma = E_\delta$. Now, for any $f \in E_\gamma, fe \in E_\alpha E_\gamma \subseteq E_\gamma$ and

$$(fe)ac = f(ea)c = f(eb)c = (fe)bc .$$

Thus $(ac, bc) \in \sigma$.

For some $\gamma \in J, cE_\alpha c^{-1} \subseteq E_\gamma$ and so $(ca)(ca)^{-1} = caa^{-1}c^{-1} \in cE_\alpha c^{-1} \subseteq E_\gamma$ and $(cb)(cb)^{-1} = cbb^{-1}c^{-1} \in cE_\alpha c^{-1} \subseteq E_\gamma$. Also, if $f = caa^{-1}ec^{-1}$ then $f \in cE_\alpha c^{-1} \subseteq E_\gamma$ and

$$\begin{aligned} fca &= caa^{-1}ec^{-1}ca = caa^{-1}c^{-1}cea \\ &= caa^{-1}c^{-1}ceb = caa^{-1}ec^{-1}cb = fcb . \end{aligned}$$

Thus $(ca, cb) \in \sigma$ and σ is a congruence on S . Moreover, it is evident that $\sigma|_{E_S} = \pi_P$.

Now suppose that τ is any congruence on S such that

$$\tau|_{E_S} = \sigma|_{E_S} = \pi_P$$

and let a, b be as above. Then $aa^{-1}\tau = bb^{-1}\tau = e\tau$ and so

$$\begin{aligned} a\tau &= (aa^{-1}a)\tau = (aa^{-1})\tau a\tau = e\tau a\tau = (ea)\tau = (eb)\tau \\ &= e\tau b\tau = bb^{-1}\tau b\tau = b\tau . \end{aligned}$$

Thus $\sigma \subseteq \tau$ and σ is the finest congruence on S such that $\sigma|_{E_S} = \pi_P$.

The verification that ρ is the largest such congruence is similar but simpler and so we omit it.

We devote the remainder of this section to obtaining an alternative characterization of the congruences σ, ρ of Theorem 4.2 in terms of kernel normal systems.

DEFINITION 4.3 [9]. Let S be an inverse semigroup. We call \mathcal{N} a *kernel normal system* of S if \mathcal{N} is a collection of inverse subsemigroups of $S, \mathcal{N} = \{N_\alpha : \alpha \in J\}$ such that, if $E_\alpha = E_{N_\alpha}$ then

- (1) $\{E_\alpha : \alpha \in J\}$ is a normal partition of E_S ;
- (2) $aa^{-1}, bb^{-1} \in E_\alpha$ and $a, ab^{-1} \in N_\alpha$ implies that $b \in N_\alpha$;
- (3) $aa^{-1}, bb^{-1} \in E_\alpha, ab^{-1} \in N_\alpha$ and $aE_\beta a^{-1} \subseteq E_\gamma$ implies that $aN_\beta b^{-1} \subseteq N_\gamma$.

Then we have

THEOREM 4.4. (*Preston [9], Th. 1*). *Let S be an inverse semigroup and let $\mathcal{N} = \{N_\alpha : \alpha \in J\}$ be a kernel normal system of S . Let $\rho_{\mathcal{N}} = \{(a, b) \in S \times S : aa^{-1}, bb^{-1} \in E_\alpha \text{ and } ab^{-1} \in N_\alpha \text{ for some } \alpha \in J\}$. Then $\rho_{\mathcal{N}}$ is a congruence on S and $\{N_\alpha : \alpha \in J\}$ is the set of idempotents in $S/\rho_{\mathcal{N}}$.*

Conversely, let ρ be a congruence on S . Then $\mathcal{N} = \{e\rho : e \in E_S\}$ is a kernel normal system of S and $\rho = \rho_{\mathcal{N}}$.

Thus a congruence on an inverse semigroup is uniquely determined by the congruence classes which contain the idempotents.

THEOREM 4.5. *Let S be an inverse semigroup and $P = \{E_\alpha : \alpha \in J\}$ be a normal partition of E_S . For each $\alpha \in J$, let T_α be the largest inverse subsemigroup of S such that $E_{T_\alpha} = E_\alpha$, let $M_\alpha = \{x \in T_\alpha : ex = e \text{ for some } e \in E_\alpha\}$ and let $N_\alpha = \{x \in T_\alpha : E_\alpha E_\beta \subseteq E_\gamma \text{ implies that } xE_\beta x^{-1} \subseteq E_\gamma\}$. Then $\mathcal{M} = \{M_\alpha : \alpha \in J\}$ and $\mathcal{N} = \{N_\alpha : \alpha \in J\}$ are kernel normal systems of S , $\rho_{\mathcal{M}} = \sigma$ and $\rho_{\mathcal{N}} = \rho$, where σ and ρ are defined as in Theorem 4.2.*

Proof. For each $\alpha \in J$, let U_α, V_α be the σ and ρ -classes, respectively, of S containing E_α .

Clearly $M_\alpha \subseteq U_\alpha$. Also, since $E_{U_\alpha} = E_\alpha$, it follows that $U_\alpha \subseteq T_\alpha$. Hence for $x \in U_\alpha$, we have first that $x \in T_\alpha$. Moreover, from the definition of σ , since $x, xx^{-1} \in U_\alpha$, that is, $(x, xx^{-1}) \in \sigma$, we have that, for some idempotent $e \in E_\alpha$, $ex = exx^{-1}$ and so $exe = exx^{-1}e = exx^{-1} \in E_\alpha$. Hence $(exe)x = exex = ex = exe$ and so $x \in M_\alpha$, $M_\alpha = U_\alpha$ and $\rho_{\mathcal{M}} = \sigma$.

For $x \in N_\alpha$, we have $xx^{-1} \in E_\alpha$. Now, for any $\beta \in J$, $xx^{-1}E_\beta xx^{-1} = xx^{-1}E_\beta \subseteq E_\alpha E_\beta \subseteq E_\gamma$, say. Then $xE_\beta x^{-1} \subseteq E_\gamma$, by the definition of N_α , and so $(x, xx^{-1}) \in \rho$. Consequently, $x \in V_\alpha$. On the other hand, since $E_{V_\alpha} = E_\alpha$, $V_\alpha \subseteq T_\alpha$ and so $x \in V_\alpha$ implies that $x, x^{-1} \in T_\alpha$. For $\beta \in J$, suppose that $E_\alpha E_\beta \subseteq E_\gamma$ and let $e \in E_\alpha, f \in E_\beta$ and $g \in E_\gamma$. Then $efe \in E_\alpha E_\beta E_\alpha \subseteq E_\gamma$ and so $(e\rho)(f\rho)(e\rho) = (g\rho)$. Hence, since $e\rho = x\rho = x^{-1}\rho$, $(xfx^{-1})\rho = (x\rho)(f\rho)(x^{-1}\rho) = g\rho$; that is, $xfx^{-1} \in E_\gamma$. Since P is a normal partition of E_S , it follows that $xE_\beta x^{-1} \subseteq E_\gamma$ and that $x \in N_\alpha$. Thus $V_\alpha = N_\alpha$ and $\rho_{\mathcal{N}} = \rho$.

5. The lattice of congruences on an inverse semigroup.

THEOREM 5.1. *Let S be an inverse semigroup and let*

$$\theta = \{(\rho_1, \rho_2) \in \mathcal{A}(S) \times \mathcal{A}(S) : \rho_1|_{E_S} = \rho_2|_{E_S}\}.$$

Then

- (i) θ is a congruence on $\mathcal{A}(S)$;
- (ii) each θ -class is a complete modular sublattices of $\mathcal{A}(S)$ (with a greatest and least element);
- (iii) the quotient lattice $\mathcal{A}(S)/\theta$ is complete and the natural homomorphism θ^\natural of $\mathcal{A}(S)$ onto $\mathcal{A}(S)/\theta$ is a complete lattice homomorphism.

Proof. We already know from Theorem 3.4 that θ is a meet compatible equivalence on $\mathcal{A}(S)$. To establish that θ is a congruence it only remains to be shown that for $(\rho_1, \rho_2) \in \theta$, $\rho_3 \in \mathcal{A}(S)$ we have $(\rho_1 \vee \rho_3, \rho_2 \vee \rho_3) \in \theta$. Let $e \in E_S$ and $f \in e(\rho_1 \vee \rho_3) \cap E_S$. Then $f \in E_S$ and $(e, f) \in \rho_1 \vee \rho_3$. Hence there exist $x_1, x_2, \dots, x_k \in S$ such that $(e, x_1) \in \rho_1$, $(x_1, x_2) \in \rho_3, \dots, (x_k, f) \in \rho_3$. Thus $(e, x_1x_1^{-1}) \in \rho_1$, $(x_1x_1^{-1}, x_2x_2^{-1}) \in \rho_3, \dots, (x_kx_k^{-1}, f) \in \rho_3$. But $(\rho_1, \rho_2) \in \theta$ and so $(e, x_1x_1^{-1}) \in \rho_2$, $(x_1x_1^{-1}, x_2x_2^{-1}) \in \rho_3, \dots, (x_kx_k^{-1}, f) \in \rho_3$. Consequently, $(e, f) \in \rho_2 \vee \rho_3$ and $f \in e(\rho_2 \vee \rho_3) \cap E_S$. Similarly $e(\rho_2 \vee \rho_3) \cap E_S \subseteq e(\rho_1 \vee \rho_3) \cap E_S$. Hence $\rho_1 \vee \rho_3|_{E_S} = \rho_2 \vee \rho_3|_{E_S}$ and $(\rho_1 \vee \rho_3, \rho_2 \vee \rho_3) \in \theta$.

Part (ii) follows immediately from Theorem 3.4.

(iii) To show that $\mathcal{A}(S)/\theta$ is complete and the natural homomorphism θ^\natural of $\mathcal{A}(S)$ onto $\mathcal{A}(S)/\theta$ is complete, (i.e., θ^\natural preserves arbitrary joins and intersections as well as pairwise joins and intersections) it is sufficient to show that θ is a complete congruence in the following sense: if, for some index set I , $\rho_i, \rho'_i \in \mathcal{A}(S)$ for all $i \in I$, and $(\rho_i, \rho'_i) \in \theta$, for all $i \in I$, then

(a)
$$\left(\bigcap_{i \in I} \rho_i, \bigcap_{i \in I} \rho'_i\right) \in \theta$$

and

(b)
$$\left(\bigvee_{i \in I} \rho_i, \bigvee_{i \in I} \rho'_i\right) \in \theta.$$

However, quite minor alterations to the proofs of (i) in Theorem 3.4 and (i) in Theorem 5.1 will establish (a) and (b), respectively. Hence we have (iii).

6. Kernel normal systems. Let S be an inverse semigroup and define θ on $\mathcal{A}(S)$ as in Theorem 5.1.

DEFINITION 6.1 [3]. Let T be a semigroup, ρ a congruence on T , and $B \subseteq T$. Then $[B]\rho = \{x \mid (b, x) \in \rho \text{ for some } b \in B\}$.

The proofs of the following two lemmas are based on the methods of Goldie [3] and Preston [9].

LEMMA 6.2. *Let T be a semigroup, ρ_1, ρ_2 congruences on T , and $B \subseteq T$. Then $[[B]\rho_1]\rho_2 = [[B]\rho_2]\rho_1$ implies that $[[B]\rho_1]\rho_2 = B(\rho_1 \vee \rho_2)$.*

Proof. (i) It is immediate from the definition that $[[B]\rho_1]\rho_2 \subseteq [B](\rho_1 \vee \rho_2)$.

(ii) Let $x \in [B](\rho_1 \vee \rho_2)$. Then $(b, x) \in \rho_1 \vee \rho_2$ for some $b \in B$ so that there exist $x_1, x_2, \dots, x_k \in T$ such that $(b, x_1) \in \rho_1, (x_1, x_2) \in \rho_2, \dots, (x_k, x) \in \rho_2$. Hence $x_1 \in [B]\rho_1$ and so $x_2 \in [[B]\rho_1]\rho_2 = [[B]\rho_2]\rho_1$. Thus $x_3 \in [[[B]\rho_2]\rho_1]\rho_1 = [[B]\rho_2]\rho_1 = [[B]\rho_1]\rho_2$. Proceeding by induction, it is easy to see that $x \in [[B]\rho_1]\rho_2$.

LEMMA 6.3. *Let $\rho, \sigma \in \mathcal{A}(S)$ be such that $(\rho, \sigma) \in \theta$ and let $\{N_\alpha \mid \alpha \in J\}, \{M_\alpha \mid \alpha \in J\}$ be the kernel normal systems of ρ and σ , respectively. Define $(N \vee M)_\alpha = \{k \mid kk^{-1} \in E_\alpha \text{ and } kn = m \text{ for some } n \in N_\alpha \text{ and } m \in M_\alpha\}$. Then $(N \vee M)_\alpha = [N_\alpha]\sigma = [M_\alpha]\rho$.*

Proof. (i) $(N \vee M)_\alpha \subseteq [N_\alpha]\sigma \cap [M_\alpha]\rho$. Let $k \in (N \vee M)_\alpha$. Then $kk^{-1} \in E_\alpha$ and $kn = m$ for some $n \in N_\alpha$ and $m \in M_\alpha$. Thus $kk^{-1}, (n^{-1})(n^{-1})^{-1} \in E_\alpha$ and $k(n^{-1})^{-1} \in M_\alpha$ so that $(k, n^{-1}) \in \sigma$ and hence $k \in [N_\alpha]\sigma$.

Now, $k \in [N_\alpha]\sigma$ implies that $(k, a) \in \sigma$, for some $a \in N_\alpha$ and so $(k^{-1}k, a^{-1}a) \in \sigma$, where $a^{-1}a \in E_\alpha$. Hence $k^{-1}k \in E_\alpha$ and $(k^{-1}k, n) \in \rho$. Then $(k, m) = (kk^{-1}k, kn) \in \rho$ or $k \in [M_\alpha]\rho$.

(ii) $[N_\alpha]\sigma \subseteq (N \vee M)_\alpha$. Let $k \in [N_\alpha]\sigma$. Then $(k, n) \in \sigma$ for some n in N_α . Thus $kk^{-1}, nn^{-1} \in E_\beta$ and $kn^{-1} \in M_\beta$ for some $\beta \in J$. But $n \in N_\alpha$ and so $nn^{-1} \in E_\alpha$ so that $E_\alpha = E_\beta$. Now $kn^{-1} \in M_\alpha$ implies that $k \in (N \vee M)_\alpha$.

(iii) $[M_\alpha]\rho \subseteq [N \vee M]_\alpha$. Let $k \in [M_\alpha]\rho$, say $(k, m) \in \rho$ where $m \in M_\alpha$. Then, as $mm^{-1} \in E_\alpha$, we have $kk^{-1} \in E_\alpha$. Also, $(k^{-1}, m^{-1}) \in \rho$ and $m^{-1}m \in E_\alpha$ imply that $k^{-1}m \in N_\alpha$, say $k^{-1}m = n \in N_\alpha$. Then $kn = kk^{-1}m \in E_\alpha M_\alpha \subseteq M_\alpha$ or $k \in (N \vee M)_\alpha$.

THEOREM 6.4. *Let $\{N_\alpha \mid \alpha \in J\}$ and $\{M_\alpha \mid \alpha \in J\}$ denote the kernel normal systems of ρ and σ , respectively, where $(\rho, \sigma) \in \theta$. Let $(N \vee M)_\alpha = \{k \mid kk^{-1} \in E_\alpha \text{ and } kn = m \text{ for some } n \in N_\alpha \text{ and } m \in M_\alpha\}$ and $(N \wedge M)_\alpha = N_\alpha \cap M_\alpha$. Then $\{(N \vee M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \vee \sigma$ and $\{(N \wedge M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \cap \sigma$.*

Proof. It is immediate that $\{(N \wedge M)_\alpha \mid \alpha \in J\}$ is the kernel normal system of $\rho \cap \sigma$ since for each $e \in E_\alpha$, $e(\rho \cap \sigma) = e\rho \cap e\sigma = N_\alpha \cap M_\alpha$.

If $e \in E_\alpha$, then $e\rho = [E_\alpha]\rho = N_\alpha$ and $e\sigma = [E_\alpha]\sigma = M_\alpha$. Since the θ -classes are sublattices of $\mathcal{A}(S)$, $e \in E_\alpha$ implies that $e(\rho \vee \sigma) = [E_\alpha](\rho \vee \sigma)$. Lemma 6.3 shows that $[[E_\alpha]\rho]\sigma = [[E_\alpha]\sigma]\rho$ and hence Lemma 6.2 implies that $[E_\alpha](\rho \vee \sigma) = [[E_\alpha]\rho]\sigma = [N_\alpha]\sigma = (N \vee M)_\alpha$.

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