

# Pacific Journal of Mathematics

**POLYNOMIALS IN CENTRAL ENDOMORPHISMS**

FRANKLIN HAIMO

# POLYNOMIALS IN CENTRAL ENDOMORPHISMS

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Let  $\lambda$  be a central endomorphism of a group  $G$  in the sense that  $\lambda$  induces the identity map on the inner automorphism group of  $G$ . Despite the nearness of the situation to commutativity, it is not necessarily true that the central endomorphisms of  $G$  form a ring or even that the subset generated by  $\lambda$  be a ring. The displacement map  $\tau$ , given by  $\tau(g) = g^{-1}\lambda(g)$  for each  $g \in G$ , is an endomorphism with central values. We shall show (Theorem 1) that if  $\tau$  satisfies a certain pair of simultaneous equations then  $\lambda$  or  $\lambda^2$  is idempotent. Let  $P$  be a formal polynomial with integral coefficients, and let  $t$  be the sum of these coefficients. Then (Theorem 2)  $P(\lambda)$  is an endomorphism if and only if  $t$  induces an integral endomorphism on  $G$ . If  $G$  is nilpotent of class 2 then (Theorem 3)  $P(\lambda)$  is an endomorphism if and only if  $t(t-1)/2$  is an exponent for the commutator subgroup  $Q$  of  $G$ .

Theorem 3 gives us an alternate proof of an older (essentially equivalent) result [2, Th. 7, Corollary]. If  $\alpha$  and  $\beta$  are two maps in  $G^G$ , then  $\gamma = \alpha + \beta$  is to mean the map given by  $\gamma(g) = \alpha(g)\beta(g)$  for all  $g \in G$ . The symbol  $\iota$  will be reserved for the identity map on  $G$ . By  $\text{diag}_m x$  we mean the  $m$ -by- $m$  matrix with  $x$  repeated down the main diagonal and with zeros elsewhere. If  $1_G$  is the unity of the group  $G$ , we say that an integer  $m$  is an exponent of  $G$  if  $g^m = 1_G$  for each  $g \in G$ . An integer  $m$  is said to induce an integral endomorphism on a group  $G$  if  $(xy)^m = x^m y^m$  for all  $x, y \in G$ .

1. Preliminaries. Let  $\tau$  be a center-endomorphism of a group  $G$ . That is,  $\tau$  is an endomorphism of  $G$ , and  $\text{Im } \tau \leq Z$ , the center of  $G$ . The map  $\lambda \in G^G$  given by  $\lambda(x) = x\tau(x)$  for each  $x \in G$  is a normal endomorphism of  $G$  in that it commutes with each inner automorphism of  $G$ . It is a central endomorphism in that  $\lambda = \iota + \tau$  where  $\tau$  is a center-endomorphism. See [3]. Each center-endomorphism of  $G$  is likewise a normal endomorphism; but if  $G$  is nonabelian, no such endomorphism is a central endomorphism. The central endomorphism  $\lambda = \iota + \tau$  is said to be related to the center-endomorphism  $\tau$ . The set of all center-endomorphisms of a group  $G$  is a ring  $C(G)$  under endomorphism addition and composition.

If  $\tau$  is a center-endomorphism of  $G$  with related central endomorphism  $\lambda$ , then, with multiplication proceeding from left to right with increasing  $i$  and with  $C(n, i)$  as the usual binomial coefficient, we have

$$(A_n) \quad \lambda^n(x) = x \prod_{i=1}^n \tau^i(x^{C(n,i)})$$

and

$$(B_n) \quad \tau^n(x) = \left[ x \prod_{i=1}^n \lambda^i(x^{(-1)^i C(n,i)}) \right]^{(-1)^n}$$

for each  $x \in G$  and for each positive integer  $n$ . From  $(A_n)$ , each  $\lambda^n$  is a central endomorphism related to  $\sum_{i=1}^n C(n,i)\tau^i \in C(G)$  where  $\lambda$  is related to  $\tau$ . One readily sees that  $\lambda$  is idempotent if and only if  $-\tau$  is idempotent. Under this assumption,  $\tau^{2j+1} = \tau = -\tau^{2j}$  for each positive integer  $j$ .

Observe that the  $2^n$  factors on the right of  $(B_n)$  can be rearranged at will. In fact, if one considers the mapping  $P(\lambda) = \sum_{i=0}^n a_i \lambda^i$  where the  $a_i$  are integers with  $a_n \neq 0$ , where  $\lambda^0 = \iota$ , and where  $P(\lambda)x = \prod_{i=0}^n \lambda^i(x^{a_i})$  for each  $x \in G$ , then the terms of  $P(\lambda)$  can be rearranged in any way. Nevertheless,  $P(\lambda)$  need not be an endomorphism. If, however, it is an endomorphism, then it is normal. Call  $n$  the degree of  $P$ .

**THEOREM 1.** *Let  $\tau$  be a center-endomorphism with related central endomorphism  $\lambda$  on a group  $G$ .*

(a) *Suppose that there exist integers  $m > 0$  and  $k \geq 0$  such that  $\tau^{2m+k} + \tau^m = 0$ . Then there exists a formal polynomial  $P$  with integral coefficients and of degree  $2m + 2k$  for which  $\lambda$  is a zero.*

(b) *If there exists an integer  $n \geq 3$  such that  $\tau + \tau^{n-1} = 0 = \tau^2 + \tau^{n-2}$ , then  $\lambda$  is idempotent if  $n$  is odd; while if  $n$  is even,  $\text{Im } \tau$  is elementary 2-abelian,  $\lambda^3 = \lambda^2$ , and  $\lambda^2$  is idempotent.*

*Proof.* (a) From  $\tau^{2m+2k} + \tau^{m+k} = 0$  and the above remark on idempotents, the central endomorphism  $\sigma$  related to  $\tau^{m+k}$  must be idempotent. From  $(B_{m+k})$ ,  $\sigma$  must be of degree  $m+k$  as a polynomial in  $\lambda$ . Let  $T$  be the formal polynomial corresponding to  $\sigma$ . Let  $P = T^2 - T$ .

(b)  $\tau = \tau^3$  so that  $\tau^3 = \tau^4$ , all odd powers reducing to  $\tau$ , even to  $\tau^2$ . If  $n$  is odd, then  $\tau^{n-1} = \tau^2$  while  $\tau^{n-2} = \tau$ , from which  $\tau^2 = -\tau$  and  $\lambda^2 = \lambda$ . If  $n$  is even,  $\tau^{n-1} = \tau$  whence  $\tau^{n-1} + \tau = 0$  yields  $\tau(x^2) = 1_G$  for every  $x \in G$ . At once,  $\text{Im } \tau$  is elementary 2-abelian. Now,  $(A_2)$  leads to  $\lambda^2(x) = x\tau^2(x)$  in this case. Applying  $\lambda$ ,  $\lambda^3(x) = x\tau(x^2)\tau^2(x) = \lambda^2(x)$ . Thus,  $\lambda^3 = \lambda^2$ , all higher powers reducing to  $\lambda^2$ . In particular,  $\lambda^2$  is idempotent.

As an example of (b), take  $G$  to be the group of  $m$ -by- $m$  non-singular real matrices, and, for each matrix  $A$  therein, let  $\tau(A) = \text{diag}_m(|\det A|^{-1/m})$ . It is clear that  $\tau$  is a center-endomorphism of  $G$  and that  $\tau^2 + \tau = 0$ . If we take  $n = 3$ , we have the situation in (b).

2. **The sum of the coefficients.** If  $P$  is a polynomial with integral coefficients, let  $t(P)$  denote the sum of these coefficients.

**LEMMA.** *Let  $\alpha$  be a center-endomorphism of a group  $G$ , and let  $\beta$  be a member of  $G^G$ . Then  $\alpha + \beta$  is an endomorphism of  $G$  if and only if  $\beta$  is an endomorphism.*

*Proof.*  $(\alpha + \beta)(xy) = \alpha(x)\alpha(y)\beta(xy)$  while  $(\alpha + \beta)(x)(\alpha + \beta)(y) = \alpha(x)\beta(x)\alpha(y)\beta(y)$ . Since  $\alpha(y)$  is in the center, the result is clear.

If  $k$  is an integer, let  $[k]$  be that member of  $G^G$  which is given by  $[k]x = x^k$  for each  $x \in G$ . Observe that if  $\tau$  is a center-endomorphism of  $G$ , then  $\tau$  generates a subring  $\{\tau\}$  of  $C(G)$ .

**THEOREM 2.** *Let  $\tau$  be a center-endomorphism of a group  $G$ , and let  $\lambda$  be its related central endomorphism. Let  $P$  be a polynomial with integral coefficients.*

(a) *If  $t(P) = 0$ , then  $P(\lambda)$  is a center-endomorphism, a member of  $\{\tau\}$ .*

(b) *If  $t(P) = 1$ , then  $P(\lambda)$  is a central endomorphism related to some member of  $\{\tau\}$ .*

(c) *If  $G$  is noncommutative and if  $t(P) = 2$ , then  $P(\lambda)$  is no endomorphism.*

(d) *If  $t(P) \neq 0, 1, 2$ , then  $P(\lambda)$  is: (1) an endomorphism if and only, if  $[t(P)]$  is an endomorphism on  $G$ ; (2) a center-endomorphism if and only if  $[t(P)]$  is a center-endomorphism on  $G$ ; (3) a central endomorphism if and only if  $[t(P) - 1]$  is a center-endomorphism on  $G$ .*

*Proof.* Suppose that  $P(\lambda) = \sum_{i=0}^n a_i \lambda^i$  for integers  $a_i$ . Note that  $\lambda^0 = \iota$  and that, from  $(A_i)$ ,  $\lambda^i = \iota + \sum_{j=1}^i C(i, j)\tau^j$  if  $i > 0$ . Upon substitution,  $P(\lambda) = \sum_{i=0}^n a_i(\iota + \sum_{j=1}^i C(i, j)\tau^j) = t(P)\iota + \sum_{i=1}^n q_i \tau^i$  for suitable integers  $q_i$ . (a) and (b) are now immediate. If  $t(P) = 2$ , the lemma says that  $2\iota = [2]$  is an endomorphism of  $G$  if and only if  $P(\lambda)$  is an endomorphism. But  $[2]$  is an endomorphism if and only if  $G$  is abelian, establishing (c). For  $t(P) \neq 0, 1, 2$ , the lemma gives (d), (1) and (2), directly. Now  $P(\lambda)$  is central and related to a center-endomorphism if and only if  $P(\lambda) = \iota + \sigma$  for some center-endomorphism  $\sigma$ . Equivalently,  $(t(P) - 1)\iota + \sum_{i=1}^n q_i \tau^i - \sigma = 0$ ; that is,  $(t(P) - 1)\iota = [t(P) - 1]$  is a center-endomorphism on  $G$ , establishing (d), (3).

By (a) above, each  $\lambda^n - \lambda$  is a center-endomorphism,  $n = 1, 2, \dots$ . By (c), if  $G$  is noncommutative, no  $\lambda^n + \lambda$  is an endomorphism,  $n = 1, 2, \dots$ .

Recall that a group is (nilpotent) of class 2 if its inner automorphism group is abelian.

**THEOREM 3.** *Let  $G$  be a class 2 group,  $\lambda$  a central endomorphism of  $G$ , and  $P$  a polynomial with integral coefficients. Then  $P(\lambda)$  is a normal endomorphism of  $G$  if and only if  $(t(P) - 1)t(P)/2$  is an exponent of  $Q$ .*

*Proof.* Note that  $P(\lambda) = \sum_{i=0}^n a_i \lambda^i$  is a normal endomorphism if and only if it is an endomorphism. Each  $\lambda^i$  is central (by  $A_i$ ). For  $x, y \in G$ , let  $w$  denote  $[y^{-1}, x^{-1}] = y^{-1}x^{-1}yx$ . For a class 2 group, recall that  $y^b x^a = x^a y^b w^{ab}$  and that  $(xy)^a = x^a y^a w^{a(a-1)/2}$  for all integers  $a$  and  $b$ . By the centrality of the powers of  $\lambda$ ,  $\lambda^i(y^b)\lambda^j(x^a) = \lambda^j(x^a)\lambda^i(y^b)w^{ab}$  for all  $x, y \in G$ , all nonnegative integers  $i$  and  $j$ , and all integers  $a$  and  $b$ . It is now easy to show that  $P(\lambda)(xy) = P(\lambda)(x)P(\lambda)(y)w^E$  where the integer  $E = \sum_{i=0}^n a_i(a_i - 1)/2 + \sum_{i < j} a_i a_j$ . From a routine observation one sees that  $E = (t(P) - 1)t(P)/2$ .

**COROLLARY.** [2, Th. 7, Corollary] *Let  $s$  be an integer  $\neq 0, 1, 2$ . Let  $G$  be a class 2 group for which  $s(s - 1)/2$  is an exponent for  $Q$ . Then  $[s]$  is an integral endomorphism for  $Q$ .*

*Proof.* By the theorem, any polynomial  $P$  with integral coefficients and with coefficient-sum  $s$  has  $P(\lambda)$  an endomorphism for each central endomorphism  $\lambda$ , and the set of all such  $\lambda$  is nonempty. By Theorem 2, (d),  $[s]$  is an endomorphism on  $G$ .

As an example of this corollary, let  $F$  be a commutative ring of finite characteristic and with a unity. Suppose that the characteristic  $k = s(s - 1)/2$  for some integer  $s > 2$ . Let  $G$  be the set of all ordered triples  $\langle a, b, c \rangle$  over  $F$  with multiplication given by  $\langle a, b, c \rangle \langle a', b', c' \rangle = \langle a + a', b + b', c + c' + ba' \rangle$ . We have the well known class 2 group  $G$  of triangular matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix}$$

where  $Z = Q$  is the set of all  $(0, 0, x)$ . Since  $(0, 0, x)^n = (0, 0, nx)$ , the characteristic  $k$  is an exponent for  $Q$ . In general,  $\langle a, b, c \rangle^n = \langle na, nb, nc + (n(n - 1)/2)bc \rangle$  for each integer  $n$ . An easy calculation now shows that  $\langle \langle a, b, c \rangle \langle a', b', c' \rangle \rangle^s - \langle a, b, c \rangle^s \langle a', b', c' \rangle^s = (0, 0, (s - s^2)ba')$ . But  $(s - s^2)ba' = -2kba' = 0$ , so that  $[s]$  is indeed an integral endomorphism of  $G$ .

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