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Let F be a local field with characteristic unequal to two, and in which the element 2 is not unitary. Let V be a regular quadratic space over F, L a lattice on V. The group of units of L is the subgroup

$$0(L) = \{\sigma \in 0(V) \mid \sigma L = L\}$$

of the orthogonal group $0(V)$. Two vectors u and v in L are defined to be integrally equivalent if there exists an isometry $\sigma \in 0(L)$ mapping one onto the other. This paper gives necessary and sufficient conditions for integral equivalence of vectors when the underlying lattice L is modular.

A very fundamental theorem in all studies of quadratic forms is the well-known Witt's Theorem. Yet, integral versions of it come scarce. However, there has been some stirring signs of interest and activity of late along this direction. The solution for integral equivalence of vectors would, of course, constitute an one-dimensional integral extension of this classic theorem. Recent works by James [3], Knebusch [4], Rosenzweig [8], Trojan [9], and Wall [10] may be consulted for the few known special cases. Earlier in [2] the author had extended Trojan's unramified modular solution to the special case of the so-called depleted modular lattices over any dyadic local field. This paper removes the restriction to the size of the weight ideal associated with the lattice and thereby completes the solution for arbitrary modular lattices over dyadic local fields.

The technicalities involved when dealing with an arbitrary lattice are substantial and not all of which we have been able to overcome. Here again special cases have been solved and they are included in the author's doctoral dissertation [1].

1. Preliminaries.

We shall freely make use of the results and terminologies of [6]. We do, however, wish to emphasize a few important relevant facts.

The ground field F is a fixed dyadic local field that is a finite (ramified or unramified) extension of the usual 2-adic number field Q_2 (including Q_2). We let $\mathcal{O}$ stand for the ring of integers in F, $\mathcal{U}$ for the group of units, $\mathcal{P}$ for the unique maximal ideal, π for a prime element generating $\mathcal{P}$, ord for the ordinal function, and $|\ \ |$ for the normalized multiplicative valuation in prime spot $\mathcal{P}$. The residue class field is a finite field of characteristic 2 and is therefore perfect. This means, in particular, that every unit $\varepsilon \equiv \mu^2 \bmod \mathscr{P}$ for some unit μ. The *quadratic defect* $\mathscr{D}(\alpha)$ of a field element α is the ideal generated by the element β where $\alpha - \beta$ is a square and $|\beta|$ is minimal. If $\varepsilon \in \mathfrak{U}$, then

(i) $\mathscr{D}(\varepsilon)$ is one of the ideals: $0, 4\mathscr{O}, 4\mathscr{P}^{-1} \cdots, \mathscr{P}^3, \mathscr{P}$;

(ii) $\mathscr{D}(\varepsilon) = 4\mathscr{O}$ if and only if $F(\sqrt{\varepsilon})/F$ is a quadratic unramified extension;

(iii) suppose $\varepsilon = \eta^2 + \alpha$ with $|4| < |\alpha| < 1$ and ord α is odd, we have $\mathscr{D}(\varepsilon) = \alpha\mathscr{O}$.

Hensel's Lemma will frequently be applied, and usually we refer to it as the Local Square Theorem which states: "For any integer $\alpha \in \mathscr{O}$, $1 + 4\pi\alpha$ is a square."

If α, β are nonzero field elements, then $\alpha\beta \sim 1$ (or $\alpha \sim \beta$) means $\operatorname{ord} \alpha \equiv \operatorname{ord} \beta \bmod 2$; otherwise, $\alpha\beta \sim \pi$.

A quadratic space V over a field F is simply a finite dimensional vector space endowed with a symmetric bilinear form B (and its associated quadratic form Q). A *lattice* L on V is a finitely generated $\mathscr{O}$-module in V such that the subspace FL spanned by L equals V. The *coefficient ideal* of a vector x in V with respect to L is

$$\mathfrak{A}_x^L = \{\alpha \in F \mid \alpha x \in L\} .$$

Vector x is called *maximal* (*primitive*) in L if $\mathfrak{A}_x^L = \mathscr{O}$. A sublattice M of L splits if M is an orthogonal direct summand, i.e. $L = M \perp N$ for some N. The $\mathscr{O}$-modules generated by the sets $B(L, L)$ and $Q(L)$ in F are called the *scale* $\mathscr{S}L$ and the *norm ideal* $\mathscr{N}L$ respectively. Let $\mathfrak{A}$ be a fractional ideal, lattice L is said to be $\mathfrak{A}$-modular if and only if $B(x, L) = \mathfrak{A}$ for every primitive vector $x \in L$. The *norm group* $\mathscr{G}L$ of L is the additive subgroup of F generated by $Q(L)$. This object is usually much finer than the norm ideal and it was first introduced by O'Meara to characterize completely isometric modular lattices. Theorem (O'Meara): *Two modular lattices on the same quadratic space are isometric if and only if their scales and norm groups are equal.* Hence, in particular, they are isometric if and only if they represent the same numbers in F. We shall obtain a result very analogous to this. We note here that even if L is modular $Q(L)$ needs not equal $\mathscr{G}L$. O'Meara has shown [6] that if L is modular with $\dim L \geqq 5$, then $Q(L) = \mathscr{G}L$. This was improved by Riehm (see [7], Th. 7.4) to $\dim L \geqq 4$. Notice that if F is unramified (over Q_2) then the concepts of norm groups and norm ideals coincide since the maximal ideal $\mathscr{M}L$ contained in $\mathscr{G}L$ has always the same order parity as $\mathscr{N}L$. This reveals an important point as to why the unramified theory is very much simpler because $\mathscr{N}L$ is a far easier creature to contend with than $\mathscr{G}L$. A *norm generator* of L is an element $a \in \mathscr{G}L$ such that $a\mathscr{O} = \mathscr{N}L$. The object $\mathscr{P}(\mathscr{M}L) + 2\mathscr{S}L$ is called the *weight ideal* $\mathscr{W}L$ of L, and a scalar b is called a *weight generator* if and only if $b\mathscr{O} = \mathscr{W}L$. An element $b \in \mathscr{G}L$ such that $ab \sim \pi$ and $|b|$ is the largest in $\mathscr{G}L$ is called a *base generator* of L (following Riehm). It is well-known that $\mathscr{G}L = a\mathscr{O}^2 + b\mathscr{O}$ where b is either a base or a weight generator and a is a norm generator. A base generator is often also a weight generator (e.g. when $\mathscr{W}L \supset 2\mathscr{S}L$) and we shall use this letter b indiscriminately. L is a *depleted* modular lattice if $\mathscr{W}L = 2\mathscr{S}L$. It was precisely this restriction to the size of $\mathscr{W}L$ that enabled the norm ideal to play a more dominant role and thereby facilitating us in our earlier solution of the integral equivalence problem over such lattices.

The symbol $A(\alpha, \beta)$ denotes a two dimensional unimodular (scale $= \mathscr{O}$) lattice having basis $\{x, y\}$ such that $Q(x) = \alpha$, $Q(y) = \beta$, $B(x, y) = 1$. Similarly, the symbol $\langle\alpha\rangle$ stands for an one dimensional lattice with a basis vector $\{x\}$ whose length is $Q(x) = \alpha$.

The set of all isometries of V leaving L stable is a subgroup $O(L)$ of the orthogonal group $O(V)$. Vectors $u, v \in L$ are *integrally equivalent* (symbolically $u \sim v$) if there exists an isometry $\sigma \in O(L)$ such that $\sigma(u) = v$. Our task is to determine necessary and sufficient conditions for integral equivalence when lattice L is modular. Since the coefficient ideals and the lengths of u and v must clearly be the same for necessity, we shall henceforth take these vectors as being primitive in L with common length δ. By scaling (see [6]), we may assume L is unimodular. Furthermore, since the depleted case has been settled we may assume, whenever necessary, that $\mathscr{W}L \supset 2\mathscr{O}$ which implies, in particular, $\operatorname{ord}(\mathscr{N}L) + \operatorname{ord}(\mathscr{W}L)$ is odd. Also, if $\dim L \geqq 3$, then L represents every weight (base) generator.

Finally, we associate to every maximal vector $x \in L$ its *characteristic set*

$$\mathfrak{M}_x = \{z \in L \mid B(x, z) = 1\} .$$

The numbers represented by this set will be an important invariant needed to classify integrally equivalent vectors.

2. Binary case.

DEFINITION. Let L be binary unimodular and $u, v \in L$. We say u, v are *of the same parity* if and only if for all pairs $(\bar{u}, \bar{v})$ of vectors in L such that $\bar{u} \in \mathfrak{M}_u$, $\bar{v} \in \mathfrak{M}_v$ we have

$$Q(\bar{u}) \equiv Q(\bar{v}) \bmod \omega\mathscr{O}$$

where $\omega = \max\{2, \delta\}$. (Of course, maximum is taken in the sense of their valuations.)

We have proved in [2] the following result.

THEOREM 2.1. *Let L be any binary unimodular lattice. Then, $u \sim v$ if and only if they are of the same parity.*

PROPOSITION 2.2. Suppose L is binary unimodular with $\mathscr{W}L \supset 2\mathscr{O}$, then $u \sim v$ always.

Proof. By ([6], 93:10) we have $L \cong A(a, b)$ where a and b are norm and weight generator respectively. Hence, $O(L) = O(FL)$ by ([7], Lemma 3.5). Now $u \sim v$ by Witt's Theorem.

3. Classification of vectors.

DEFINITIONS. A maximal (primitive) vector x in L is *$\mathscr{N}$-regular* (resp. *$\mathscr{G}$-regular*) if and only if $\mathscr{N}(\langle x\rangle^{\perp}) = \mathscr{N}L$ (resp. $\mathscr{G}(\langle x\rangle^{\perp}) = \mathscr{G}L$), where

$$\langle x\rangle^{\perp} = \{z \in L \mid B(x, z) = 0\} .$$

Otherwise, x is *$\mathscr{N}$-irregular* (resp. *$\mathscr{G}$-irregular*).

Again, putting $\omega = \max\{2, \delta\}$, we call u a vector of *Type* I if $\mathscr{N}(\langle u\rangle^{\perp}) \subseteqq \omega\mathscr{O}$; otherwise, u is of *Type* II.

REMARK. If F is unramified, then the concepts of $\mathscr{N}$-regularity and $\mathscr{G}$-regularity coincide.

DEFINITION. Suppose L is unimodular with $\dim L = 2n, n \geqq 1$. Then, there exists a splitting

$$L = L_1 \perp \cdots \perp L_n$$

where $L_i \cong A(a_i, \gamma_i)$ with $a_i\mathscr{O} = \mathscr{N}L, \gamma_i \in \mathscr{W}L_i \subseteqq \mathscr{W}L$ for $i = 1, \cdots, n$. Such a splitting is called a *quasi-canonical splitting*[1].

It is quite clear that if $Q(u) = \delta \notin \mathscr{U}$ and dim L is odd, then u is $\mathscr{N}$-regular always. Also, u is $\mathscr{N}$-regular whenever δ is a norm generator and $\dim L$ is even. The $\mathscr{N}$-irregular vectors are characterized as follows: (i) Assume $\dim L \geqq 3$. If u is $\mathscr{N}$-irregular, then for every $\bar{u} \in \mathfrak{M}_u, Q(\bar{u})$ is a norm generator. The converse is true provided L is not totally improper (i.e. $\mathscr{N}L \neq 2\mathscr{S}L = 2\mathscr{O}$). (ii) Let $\dim L = 2n, n \geqq 1$. For every quasi-canonical splitting

$$L = \mathop{\perp}_{i=1}^{n} L_i, L_i \cong A(a_i, \gamma_i) = \mathscr{O}x_i + \mathscr{O}y_i, 1 \leqq i \leqq n ,$$

[1] The existence is seen by applying ([6], 93:12 and 93:18) and O'Meara's *op*-transformations (see [5]).

we put

$$u = \sum (\alpha_i x_i + \beta_i y_i),\ \alpha_i, \beta_i \in \mathcal{O}\ .$$

Then, u is $\mathcal{N}$-irregular implies all the β_i's are unitary. Again, the converse holds for L not totally improper.

CONDITION (D). (Assume $\dim L \geqq 3$, and $\mathscr{W}L \supset 2\mathcal{O}$). An element α in $\mathscr{G}L$ is said to satisfy condition (D) provided the quadratic defect satisfies the inequality

$$\mathscr{D}(a\alpha) \subset \mathcal{N}L\mathscr{W}L$$

for every norm generator $a \in \mathscr{G}L$.

LEMMA 3.1. *Suppose $\delta \in \mathscr{G}L$ and $\delta\mathcal{O} \neq \mathcal{N}L$. Then, if there exists one norm generator a' such that $\mathscr{D}(\delta a') \subset \mathcal{N}L\mathscr{W}L$, we have δ satisfying condition* (D).

Proof. Write $\mathscr{G}L = a'\mathcal{O}^2 + b\mathcal{O}$ for the given a' and an arbitrary base generator b. $\delta\mathcal{O} \neq \mathcal{N}L$ implies $\delta = a't^2 + b\alpha$ has $|t| < 1$. Since $a'b \sim \pi$ we see $|\alpha| < 1$ by the assumption that $\mathscr{D}(\delta a') \subset \mathcal{N}L\mathscr{W}L$. The rest is computational.

PROPOSITION 3.2. Let u be an $\mathcal{N}$-regular vector with length δ satisfying condition (D). Then u is also $\mathscr{G}$-regular if and only if there exists a vector $\bar{u} \in \mathfrak{M}_u$ such that $Q(\bar{u}) \in \mathscr{P}\mathscr{W}L$.

Proof. Since δ satisfies condition (D), we have the implicit assumptions of $\dim L \geqq 3$ and $\mathscr{W}L \supset 2\mathcal{O}$. Therefore, δ is not a norm generator since otherwise $\delta + b$ is a norm generator also (here b is any base generator) and

$$\mathscr{D}(\delta(\delta + b)) = \delta b\mathcal{O} = \mathcal{N}L\mathscr{W}L\ ,$$

implying that δ does not satisfy condition (D). Putting $L = K \perp M$ where

$$K = \mathcal{O}u + \mathcal{O}\bar{u} \cong A(\delta, Q(\bar{u})) \text{ with } Q(\bar{u}) \in \mathscr{P}\mathscr{W}L\ ,$$

it is quite clear that $\mathcal{N}M = \mathcal{N}L$. Write

$$\mathscr{G}M = a_m\mathcal{O}^2 + b_m\mathcal{O} \text{ and } \mathscr{G}K = a_k\mathcal{O}^2 + b_k\mathcal{O}\ .$$

But,

$$\mathscr{W}L = \sum_{\gamma} a_m^{-1}\mathscr{D}(a_m\gamma)$$

where γ runs through the set $\{b_m, a_k, b_k\}$ (see page 31, [7]). Now,

$\mathscr{D}(a_m a_k) \subset \mathscr{N}L\mathscr{W}L$ since δ satisfies condition (D) and $Q(\bar{u}) \in \mathscr{P}\mathscr{W}L$. If $|\delta| \geqq |\mathscr{W}L|$, then

$$b_k\mathscr{O} = \delta^{-1}\mathscr{D}(\delta Q(\bar{u})) + 2\mathscr{O} \subset \mathscr{W}L .$$

On the other hand, $\delta \in \mathscr{P}\mathscr{W}L$ implies $b_k\mathscr{O} \subset \mathscr{W}L$ since L is not depleted by hypothesis. Thus, $\mathscr{W}L = \mathscr{W}M$ proving u is G-regular.

Conversely, assume $\mathscr{G}(\langle u\rangle^{\perp} = \mathscr{G}L$. Suppose the contrary is true, i.e. every $u' \in \mathfrak{M}_u$ is such that $Q(u') \notin \mathscr{P}\mathscr{W}L$. Putting $L = (\mathscr{O}u + \mathscr{O}u') \perp T$, we see that $\mathscr{N}T = \mathscr{N}L$ since u is $\mathscr{N}$-regular with δ not being a norm generator. We also have,

$$T = \begin{cases} \langle a\rangle \perp \cdots = \mathscr{O}x \perp \cdots \\ A(a, \cdots) \perp \cdots = (x\mathscr{O} + \cdots) \perp \cdots \end{cases}$$

Suppose $Q(u')$ is already a norm generator, we apply $op(u') = u' \perp \varepsilon x$ where ε is an unit such that

$$Q(\varepsilon x) \equiv Q(u') \bmod \mathscr{W}L .$$

Now, $Q(op(u'))$ lies in $\mathscr{W}L$. On the other hand, if $|Q(u')| < |\mathscr{N}L|$, by applying $op(u') = u' \perp x$ we have made $op(u')$ a norm generator and furthermore,

$$L = (\mathscr{O}u + \mathscr{O}(op(u'))) \perp T', \mathscr{N}T' = \mathscr{N}L .$$

Therefore, in either case we know that by applying op-transformations, at most twice if necessary, there exists a vector $\bar{u} \in \mathscr{W}_u$ with $\operatorname{ord} Q(\bar{u}) = \operatorname{ord}(\mathscr{W}L)$. Let $Q(\bar{u}) = b$ and write

$$L = K \perp M$$

again as above. Then, $\mathscr{N}M = \mathscr{N}L$ and

$$(*) \qquad \langle u\rangle^{\perp} \cong \langle dK\delta\rangle \perp M .$$

where dK is the discriminant of K. Now, writing $a_m = a$, it is easy to see that $\mathscr{D}(a\delta) \subset ab\mathscr{O}$ implies also that $\mathscr{D}(a\delta dK) \subset ab\mathscr{O}$. Hence, $\mathscr{G}M = \mathscr{G}L$. This means, in particular, that M represents every weight (base) generator of GL whenever $\dim M \geqq 3$. By applying op-transformations, if necessary, and by the perfectness of the residue class field, we can find an $\bar{u} \in \mathfrak{M}_u$ with $Q(\bar{u}) \in \mathscr{P}\mathscr{W}L$ and so we are done except for $\dim M = 1, 2$. But $\dim M = 1$ is not possible since L is not depleted. (Referring to $(*)$ above, one sees immediately that since δ does not satisfy condition (D)—nor does δdK—$\dim M = 1$ would imply u is $\mathscr{G}$-irregular contradicting hypothesis.) Finally, suppose $\dim M = 2$. We express

$$M \cong A(a, -\alpha a^{-1})$$

where $|-\alpha a^{-1}| = |\mathscr{W}L|$, (see [6], 93:10 & 93:17). Again, it is not difficult to see that there is a suitable op-transformation such that $op(\bar{u}) \in \mathfrak{M}_u$ *with length contained in* $\mathscr{P}\mathscr{W}L$, *contradicting the initial* assumption.

COROLLARY 3.3. *If u is a $\mathscr{G}$-regular vector with length $Q(u) = \delta$ satisfying condition* (D), *then* (i) $\dim L \geqq 4$, *and* (ii) *for every* $\bar{u} \in \mathfrak{M}_u$, $\mathscr{G}(\mathscr{O}u + \mathscr{O}\bar{u})^{\perp} = \mathscr{G}L$.

PROPOSITION 3.4. Let $\dim L \geqq 3$ and δ not satisfying condition (D). Then, $u \sim v$ if and only if $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$.

Proof. We may assume δ is not a norm generator by Proposition 2 in [2]. Therefore, $\mathscr{D}(\delta a') = \mathscr{N}L\mathscr{W}L$ for each norm generator $a' \in \mathscr{G}L$. Putting $\mathscr{G}L = a'\mathscr{O}^2 + b\mathscr{O}$ for some base generator b, we have

$$\delta = a't^2 + b\varepsilon, |\varepsilon| = 1, |t| < 1 .$$

Case I. Suppose both u and v are $\mathscr{N}$-regular, then there is $\bar{u} \in \mathfrak{M}_u$ with $Q(\bar{u}) \in b\mathscr{O}$. Let $K_u = \mathscr{O}u + \mathscr{O}\bar{u}$, $M_u = K_u^{\perp}$. Then, $\mathscr{N}M_u = \mathscr{N}L$. Since $a'b \sim \pi$, $\operatorname{ord}\delta \leqq \operatorname{ord} b$. Let $\sigma\colon \langle u\rangle^{\perp} \to \langle v\rangle^{\perp}$ be the given isometry and $\sigma(M_u) = M_v$. Then M_v splits.

$$L = K_v \perp M_v \text{ with } K_v = \mathscr{O}v + \mathscr{O}\bar{v} \cong A(\delta, Q(\bar{v}))$$

for some $\bar{v} \in M_v$. We claim

$$Q(\bar{v}) \in \delta\mathscr{O} .$$

Suppose not. Put

$$Q(\bar{v}) = a's^2 + br \text{ with } |a's^2| > |b| .$$

Then,

$$\delta Q(\bar{v}) = (a'ts)^2 + a's^2b\varepsilon + b^2\varepsilon r + at^2br .$$

Clearly, we may assume that $\operatorname{ord} s < \operatorname{ord} t$. But,

$$dK_u \cong dK_v \Rightarrow \mathscr{D}(-dK_u) = \mathscr{D}(-dK_v) .$$

Now, $-dK_u$ has quadratic defect contained in $\delta b\mathscr{O}$. On the other hand, by direct computations, we see

$$\mathscr{D}(-dK_v) = Q(\bar{v})b\mathscr{O} \supset \mathscr{D}(-dK_u) .$$

This is a contradiction so the claim is true. Hence, $\mathscr{N}K_u = \mathscr{N}K_v = \delta\mathscr{O}$ and we see readily that $\mathscr{G}K_u = \mathscr{G}K_v$. By Witt's Theorem and

O'Meara's Theorem on isometry of modular lattices, $K_u \cong K_v$. Finally, $u \sim v$ follows from another application of Proposition 2, [2].

Case II. Suppose both are $\mathscr{N}$-irregular. Let $\bar{u} \in \mathfrak{M}_u$ be arbitrary and $K_u, M_u, M_v, K_v = \mathscr{O}v + \mathscr{O}\bar{v}$ for some $\bar{v} \in \mathfrak{M}_v$ as before. But now, $Q(\bar{u})$ and $Q(\bar{v})$ are norm generators for $\mathscr{G}L$. FK_u is isometric to FK_v by Witt. It is an easy computation to check that the sublattices K_u and K_v are not depleted; indeed, $\mathscr{G}K_u = \mathscr{G}K_v = \mathscr{G}L$. Hence, $u \sim v$ by Proposition 2.2.

REMARK. It can be shown that when $\dim L = 3$ and if δ does not satisfy condition (D), then $u \sim v$ always provided $\delta \notin \mathscr{U}$. In proving this fact, we show that $\langle u \rangle^{\perp} \cong \langle v \rangle^{\perp}$ by using O'Meara's Theorem 93:28, [6].

PROPOSITION 3.5. Suppose $\dim L \geqq 3$ and both u and v are Type I vectors. Then, $u \sim v$ if and only if (i) u, v are of the same parity, and (ii) $\langle u \rangle^{\perp} \cong \langle v \rangle^{\perp}$.

Proof. The case of δ being an unit is obvious. So, let $\delta \notin \mathscr{U}$. Hence, choose any $\bar{u} \in \mathfrak{M}_u$ and put

$$K_u = \mathscr{O}u + \mathscr{O}\bar{u},\ T_u = K_u^{\perp} .$$

Suppose $\sigma: \langle u \rangle^{\perp} \longrightarrow \langle v \rangle^{\perp}$ is the given isometry. Then, $\sigma(T_u) = T_v$ splits and we have

$$L = K_v \perp T_v$$

where $K_v = \mathscr{O}v + \mathscr{O}\bar{v}$ for some $\bar{v} \in \mathfrak{M}_v$. If $|\delta| \leqq |2|$, then (i) implies $\mathscr{G}K_v = \mathscr{G}K_u$ so that $K_u \cong K_v$ by Witt and O'Meara and therefore $u \sim v$ follows from Theorem 2.1. Otherwise, define the mapping $\phi: FK_u \longrightarrow FK_v$ *by*: $\phi(u) = v$, $\phi(u - \delta\bar{u}) = \mu(v - \delta\bar{v})$ where

$$\mu^2 = \frac{1 - \delta Q(\bar{u})}{1 - \delta Q(\bar{v})} .$$

Now, again condition (i) implies that

$$\mu^2 \equiv 1 \bmod \delta^2\mathscr{O} .$$

It is easily checked that ϕ, in fact, maps K_u onto K_v and we are done.

4. **Main results.** We recall that to every maximal vector x in L, there is associated with it a characteristic subset $\mathfrak{M}_x$ of the lattice

$$\mathfrak{M}_x = \{z \in L \mid B(x, z) = 1\} .$$

A central result given below states that u is integrally equivalent to v if and only if $\mathfrak{M}_u$ and $\mathfrak{M}_v$ represent the same field elements when the dimension of the given lattice is sufficiently large. This theorem may be viewed (for $\dim L$ large enough) as an analogue to the well-known theorem on the integral classification of modular quadratic forms over local fields.

THEOREM 4.1. *Let L be an unimodular lattice over a dyadic local field of characteristic zero, and that $\dim L \neq 4, 5, 6$. Then, two maximal vectors u and v in L are integrally equivalent if and only if $Q(u) = \delta = Q(v)$ and $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$.*

Proof. Necessity is obvious. As for sufficiency we proceed in several steps.

1. Pick any $\bar{u} \in \mathfrak{M}_u$, $\bar{v} \in \mathfrak{M}_v$ such that $Q(\bar{u}) = Q(\bar{v})$. Then,

$$\mathfrak{M}_u = \bar{u} + \mathcal{O}(u - \delta\bar{u}) \perp T_u = \bar{u} + \langle u \rangle^{\perp}$$
$$\mathfrak{M}_v = \bar{v} + \mathcal{O}(v - \delta\bar{v}) \perp T_v = \bar{v} + \langle v \rangle^{\perp} .$$

Hence, $Q(\langle u \rangle^{\perp}) = Q(\langle v \rangle^{\perp}) \bmod 2\mathcal{O}$. (i.e. for every $z \in \langle u \rangle^{\perp}$, $\exists w \in \langle v \rangle^{\perp}$ such that $Q(z) \equiv Q(w) \bmod 2\mathcal{O}$) Therefore, the norm groups are equal

$$\mathscr{G} = \mathscr{G}(\langle u \rangle^{\perp}) = \mathscr{G}(\langle v \rangle^{\perp}) .$$

It is also clear that $F\langle u \rangle^{\perp} \cong F\langle v \rangle^{\perp}$ and $FT_u \cong FT_v$.

2. Suppose $\dim L \geqq 9$ so that $\dim T_u = \dim T_v \geqq 7$. Then, it is well-known (see [6], 93:18) that

$$T_u \cong A(0, 0) \perp A(0, 0) \perp \cdots .$$

Take a norm generator a' and a base (weight) generator b for $\mathscr{G}(\langle u \rangle^{\perp})$. So, $\mathscr{G}(\langle u \rangle^{\perp}) = a'\mathcal{O}^2 + b\mathcal{O}$. Therefore,

$$\langle u \rangle^{\perp} \cong A(0, 0) \perp A(0, 0) \perp K_u$$

for some K_u. But now,

$$\mathscr{G}(K_u^{\mathcal{O}}) = 2\mathcal{O} + \mathscr{G}(K_u^{\mathcal{O}}) = \mathscr{G}(\langle u \rangle^{\perp})$$

so that a', b lie in $\mathscr{G}(K_u^{\mathcal{O}})$. (Here $K_u^{\mathcal{O}} = \{x \in K_u \mid B(x, K_u) \subseteqq \mathcal{O}\}$.) Hence, by ([6], 93:13), we have

$$\langle u \rangle^{\perp} \cong A(a', 0) \perp A(b, 0) \perp K_u .$$

This means there exists a Jordan decomposition

$$\langle u \rangle^{\perp} = W_1 \perp W_2$$

where $\mathscr{G} W_1 = \mathscr{G}(\langle u \rangle^{\perp})$ and $W_2 \cong \mathcal{O}(u - \delta\bar{u})$.

3. Let $\dim L \geqq 7$. Just adjoin an hyperbolic plane H to L, $H \cong A(0, 0)$. Now, apply step (2) and we have

$$\langle u\rangle^{\perp} \perp H = W_1^* \perp W_2^*$$

with $\mathscr{G} W_1^* = \mathscr{G}(\langle u\rangle^{\perp})$ and $W_2^* \cong \mathscr{Q}(u - \delta\bar{u})$. But, $\dim W_1^* \geqq 7$ here so that W_1^* admits a splitting

$$W_1^* \cong A(0, 0) \perp A(0, 0) \perp W_1' .$$

Clearly, $\mathscr{G} W_1' = \mathscr{G} W_1^* = \mathscr{G}(\langle u\rangle^{\perp})$. Upon cancelling the hyperbolic plane, we obtain

$$\langle u\rangle^{\perp} \cong W_1' \perp W_2^* .$$

Similarly for $\langle v\rangle^{\perp}$. Hence, $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$.

4. By (2) and (3), we put $L_u \perp P_u = L = L_v \perp P_v$, where $P_u \cong P_v$ have norm groups equal to $\mathscr{G}$, and $L_u = \mathscr{O}u + \mathscr{O}u'$, $L_v = \mathscr{O}v + \mathscr{O}v'$. Let $v^* \in \mathfrak{M}_v$ such that $Q(v^*) = Q(u')$. Hence, we have

$$v' = v^* + w,\ w \in \langle v\rangle^{\perp} .$$

Therefore,

$$Q(u') = Q(v') + \alpha , \qquad \text{for some } \alpha \text{ in } \mathscr{G}.$$

On the other hand, we have

$$P_v \cong A(0, 0) \perp R_v \qquad \text{where } \mathscr{G} R_v = \mathscr{G} \text{ still!}$$

By ([6], 93:13),

$$P_v \cong A(\alpha, 0) \perp R_v = (\mathscr{O}y + \cdots) \perp R_v .$$

Applying the op-transformation: $v' \to op(v') = v' \perp y$, we see that

$$L_v' = \mathscr{O}v + \mathscr{O}(op(v')) \text{ splits } L$$

with P_v' as its orthogonal complement and furthermore, $\mathscr{G} P_v'$ still equals $\mathscr{G}$. Now, $u \sim v$ is clear.

5. When $\dim L$ is less than 4, the proof of the theorem is entirely trivial.

COROLLARY 4.2. *Let L be an unimodular lattice with arbitrary dimension, u and v be to two maximal vectors in L having the same length. If $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$, then, $\langle u\rangle^{\perp} \cong \langle v\rangle^{\perp}$.*

Proof. By adjoining a suitable number of $A(0, 0)$'s and calling the enlarged lattice L', we have u integrally equivalent to v over L'. Hence,

$$\langle u\rangle^{\perp}(\text{in } L') \cong \langle v\rangle^{\perp}(\text{in } L') .$$

But,

$$\langle u\rangle^{\perp}(\text{in } L') = \langle u\rangle^{\perp}(\text{in } L) \perp A(0, 0) \perp \cdots \perp A(0, 0)$$

and similarly for $\langle v\rangle^{\perp}$. Now, cancel out the $A(0, 0)$'s .

COROLLARY 4.3. *Let L be unimodular having arbitrary dimension, $\delta \in 2\mathcal{O}$, and $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$. Then, $u \sim v$ always.*

Proof. Choose any $\bar{u} \in \mathfrak{M}_u$ and $\bar{v} \in \mathfrak{M}_v$ such that $Q(\bar{u}) = Q(\bar{v})$, and put

$$L_u = \mathcal{O}u + \mathcal{O}\bar{u},\ P_u = L_u^{\perp},\ L_v = \mathcal{O}v + \mathcal{O}\bar{v},\ P_v = L_v^{\perp} .$$

Then, $FP_u \cong FP_v$. Since δ lies in $2\mathcal{O}$, it is clear that $\mathcal{G}P_u = \mathcal{G}P_v$. The rest is obvious.

THEOREM 4.4. *Let $\dim L = 4, 5$ and $Q(\mathfrak{M}_u) = Q(\mathfrak{M}_v)$. If there exists a vector $\bar{u} \in \mathfrak{M}_u$ such that $(\mathcal{O}u + \mathcal{O}\bar{u})^{\perp}$ is isotropic, then $u \sim v$.*

We shall first prove a lemma.

LEMMA 4.5. *Under the same hypothesis as in the theorem except $\dim L$ may be 6, there exist then vectors $x_u \in \mathfrak{M}_u$, $x_v \in \mathfrak{M}_v$ such that $Q(x_u) = Q(x_v)$ and moreover, by denoting $L_u = \mathcal{O}u + \mathcal{O}x_u$ and $L_v = \mathcal{O}v + \mathcal{O}x_v$, we will have $FL_u^{\perp} \cong FL_v^{\perp}$ are isotropic spaces, and $\mathcal{N}(L_u^{\perp}) = \mathcal{N}(\langle u\rangle^{\perp}) = \mathcal{N}(L_v^{\perp})$.*

Proof. The case of $\delta \in \mathcal{U}$ is quite obvious. Let $\bar{u}$ be the given vector, we put $K_u = \mathcal{O}u + \mathcal{O}\bar{u}$. If u is a Type II vector (hence so is v), then $\mathcal{N}(K_u^{\perp})$ already equals $\mathcal{N}(\langle u\rangle^{\perp})$ and everything is clear. So let both be Type I vectors. By a suitable op-transformation, we may assume $\mathcal{N}(K_u^{\perp}) = \mathcal{N}(\langle u\rangle^{\perp})$. Pick $\bar{v}$ from $\mathfrak{M}_v$ with $Q(\bar{v}) = Q(\bar{u})$, and denote $K_v = \mathcal{O}v + \mathcal{O}\bar{v}$. By Witt's Theorem, $FK_v^{\perp}$ is isotropic. Hence, $K_v^{\perp}$ has a splitting of the form

$$K_v^{\perp} = A(s, 0) \perp \cdots = (\mathcal{O}x + \mathcal{O}z) \perp \cdots .$$

We know that

$$\langle v\rangle^{\perp} = K_v^{\perp} \perp \mathcal{O}(v - \delta\bar{v}) .$$

Apply $op(x) = x \perp (v - \delta\bar{v})$, then $K_v^{\perp}$ becomes $(\mathcal{O}(op(x)) + \mathcal{O}z) \perp \cdots$ and call this $T_v^{\perp}$ where $T_v = \mathcal{O}v + \mathcal{O}w$ for some $w \in \mathfrak{M}_v$. In fact,

$$w = \bar{v} + \alpha(v - \delta\bar{v}) + \beta x + \gamma z,\ \alpha, \beta, \gamma \in \mathcal{O}.$$

Using the fact that w is orthogonal to both z and $op(x)$, one deduces $\beta = 0$ and $\gamma = (1 - \alpha\delta)D$, where D is the discriminant of K_v. Consequently,

$$Q(w) = Q(\bar{v}) + \alpha^2\delta D - 2\alpha D .$$

Here, α can be quite arbitrary so that by choosing order of α to be sufficiently large, we see that $T_v \cong K_v$. But now, $T_v^\perp$ is easily seen to have its norm ideal equals to $\mathscr{N}(\langle v\rangle^\perp) = \mathscr{N}(\langle u\rangle^\perp)$, because if $|\delta| < |\mathscr{N}(\langle v\rangle^\perp)|$ then $K_v^\perp$ already has norm equal to $\mathscr{N}(\langle v\rangle^\perp)$. The existence of such vectors x_u and x_v is now clear.

Proof of the theorem. Let $L_u = \mathscr{O}u + \mathscr{O}x_u$ and $L_v = \mathscr{O}v + \mathscr{O}x_v$ enjoy the properties as stated in the lemma. We put $D = dL_u = dL_v$. Suppose, for the moment, that L is quarternary. Then, we write

$$L_u^\perp = A(a_u, 0) , \qquad L_v^\perp = \mathscr{O}x + \mathscr{O}z = A(a_v, 0)$$

where a_u and a_v are norm generators for $\mathscr{G}(\langle u\rangle^\perp)$. If $\mathscr{W}(\langle u\rangle^\perp) = 2\mathscr{O}$, then it is easy to see that $L_u^\perp \cong L_v^\perp$ and so $u \sim v$. Therefore, $\mathscr{D}(a_u\delta D)$ must equal $a_u\mathscr{O}\ \mathscr{W}(\langle u\rangle^\perp)$! As in the lemma, since $a_u \in Q(\langle v\rangle^\perp)$ we have

$$a_u = A^2a_v + B^2\delta D + 2A' , \qquad |A| = 1, B, A' \in \mathscr{O} .$$

Applying the op-transform, $op(x) = x \perp BA^{-1}(v - \delta x_v)$ one sees by direct computations that

$$L = (\mathscr{O}v + \mathscr{O}(x_v \perp DBA^{-1}z)) \perp (\mathscr{O}(op(x)) + \mathscr{O}z) .$$

(The choice of $x_v \perp DBA^{-1}z$ corresponds to the choice of α equals zero in the proof of the lemma.) Now, observe that the first term on the right-hand-side is isometric to L_u and the second term is isometric to $L_u^\perp$ because

$$Q(op(x)) \equiv a_uA^2 \bmod 2\mathscr{O}, \text{ with } A \in \mathscr{U} .$$

Hence, $u \sim v$.

Now, let $\dim L = 5$. By proving a result similar to Lemma 4.5, we may assume $(\mathscr{O}u + \mathscr{O}x_u)^\perp$ has norm group equal to $\mathscr{G}(\langle u\rangle^\perp)$. But, $(\mathscr{O}v + \mathscr{O}x_v)^\perp$ will not, in general, *simultaneously* enjoy this same property.

Let us call $P_u = L_u^\perp, P_v = L_v^\perp, \mathscr{G}(\langle u\rangle^\perp) = \mathscr{G} = \bar{a}\mathscr{O}^2 + \bar{b}\mathscr{O}$; the discriminant of P_u we denote by d and therefore by ([6], 93 : 18)

$$P_u \cong \langle -d\rangle \perp A(\bar{b}, 0)$$
$$P_v \cong \langle -d\rangle \perp A(b', 0) ,$$

where b' is a weight generator for $\mathscr{G}P_v$, which we may assume to be of having larger order than that of $\bar{b}$ because otherwise, $P_u \cong P_v$ already.

Suppose the component $A(b', 0)$ is adapted to a basis $\mathscr{O}x + \mathscr{O}z$ again. Take $op(x) = x \perp (v - \delta x_v)$. If D still denotes the discriminant of $L_v = \mathscr{O}v + \mathscr{O}x_v$, it is readily seen that

$$L = (\mathscr{O}v + \mathscr{O}(x_v \perp Dz)) \perp \langle -d \rangle \perp (\mathscr{O}(op(x)) + \mathscr{O}z) ,$$

where the first term on the right side again is isometric to L_u. Because of the assumption that $|b'| < |\bar{b}|$, we must have $\mathscr{D}(-d(b' + \delta D))$ equal to $\mathscr{W}(\langle u \rangle^{\perp})$. Hence,

$$P_u \cong \langle -d \rangle \perp (\mathscr{O}(op(x)) + \mathscr{O}z) .$$

The rest is obvious.

REMARKS. (i) It follows from the proofs of the theorem and the lemma that if either $\delta \in \mathscr{W}(\langle u \rangle^{\perp})\mathscr{P}$ or $\mathscr{W}(\langle u \rangle^{\perp}) \neq 2\mathscr{O}$, then $u \sim v$ regardless of the dimension and the existence of the vector $\bar{u}$ with the stated property.

(ii) A 4-dimensional unimodular lattice with given discriminant assumes two possible forms (either J or K in 93 : 18 of [6]). By a result of Riehm (Theorem 7.4, [7]) it is known that such a lattice represents every element of its norm group. Now, employing the same notations as in the proof of the last theorem, it is readily seen that we may assume when $\dim L = 6$ (as in the $\dim L = 5$ case) that $(\mathscr{O}u + \mathscr{O}x_u)^{\perp}$ has norm group equal to $\mathscr{G}$ already. Hence, the proof of Theorem 4.4 still goes through if P_u takes the "J-form". (It is easily seen that P_u takes the "J-form" if and only if P_v does so since the spaces on which they sit are isometric.)

(iii) Theorem 4.4 also goes through when $\dim L = 6$ and when both u and v are $\mathscr{N}$-irregular vectors.

(iv) Finally, we remark that Theorem 4.1 remains valid if and only if the characteristic sets satisfy the somewhat weaker property: For each $\bar{u} \in \mathfrak{M}_u$, there is a vector $\bar{v} \in \mathscr{M}_v$ with

$$Q(\bar{u}) \equiv Q(\bar{v}) \bmod 2\mathscr{O} .$$

DEFINITION. Let $\mathscr{G}$ be an additive subgroup of F. We say u and v are *of the same parity* $\bmod \mathscr{G}$ if $Q(\bar{u}) \equiv Q(\bar{v}) \bmod \mathscr{G}$ for all $\bar{u} \in \mathfrak{M}_u$, $\bar{v} \in \mathfrak{M}_v$.

We have then the immediate consequence which we mention here only because it is generally slightly easier to apply than Theorem 4.1 itself.

PROPOSITION 4.6. Let $\dim L \geqq 7$. Then, two maximal vectors u and v having the same lengths are integrally equivalent if and only if (i) $\mathscr{G}(\langle u\rangle^\perp) = \mathscr{G} = \mathscr{G}(\langle v\rangle^\perp)$, and (ii) u and v are of the same parity mod G.

We wish to make the conjecture here that both Theorem 4.1 and Proposition 4.6 hold for $\dim L = 4, 5, 6$ in the general situation as well.

5. Ternary case.

PROPOSITION 5.1. Let L be ternary unimodular. Then, $u \sim v$ if and only if. (i) there exist vectors $x_u \in \mathfrak{M}_u$, $x_v \in \mathfrak{M}_v$ such that $Q(x_u) \equiv Q(x_v) \bmod 2\mathscr{O}$ when $\delta \in 2\mathscr{P}$; $Q(x_u) \equiv Q(x_v) \bmod 4\delta^{-1}\mathscr{P}$ when $|2| \leqq |\delta| < 1$; (ii) otherwise, $\mathscr{G}(\langle u\rangle^\perp) = \mathscr{G}(\langle v\rangle^\perp)$.

Proof. Necessity is obvious. As for sufficiency, we put $L = L_u \perp \mathscr{O}w_u = L_v \perp \mathscr{O}w_v$, where $L_u = \mathscr{O}u + \mathscr{O}x_u$ & $L_v = \mathscr{O}v + \mathscr{O}x_v$. But, $dL_u \cong dL_v$ by the Local Square Theorem. Therefore, by Witt $Q(w_u) \cong Q(w_v)$ and $FL_u \cong FL_v$.

It is not difficult to see that $L_u \cong L_v$ so that $u \sim v$ by Theorem 2.1 since u, v are of the same parity over isometric binary components. (These statements are true provided $\delta \notin \mathscr{U}$. But, then if $\delta \in \mathscr{U}$, condition (ii) finishes the proof immediately.)

6. $\dim L = 4, 5, 6$. Let us put

$$\mathfrak{M}_x^{(s)} = \{w \in \mathfrak{M}_x \mid \mathscr{N}(\mathscr{O}x + \mathscr{O}w)^\perp = 2\mathscr{P}^{-s}\}$$
$$\mathfrak{M}_x^{(s,t)} = \{w \in \mathfrak{M}_x^{(s)} \mid \mathscr{W}(\mathscr{O}x + \mathscr{O}w)^\perp = 2\mathscr{P}^{-t}\}\,.$$

We shall write $\mathscr{N}(L) \cong \mathscr{N}(K) \bmod \mathscr{A}$ (here $\mathscr{A}$ denotes any fractional ideal in F) to mean that there exist respective norm generators $a_L \in Q(L)$, $a_K \in Q(K)$ such that $a_L \equiv a_K\xi^2 \bmod \mathscr{A}$ for some unit ξ.

PROPOSITION 6.1. Let L be quarternary unimodular. Then, $u \sim v$ if and only if conditions (i) and (ii) in Proposition 5.1 hold, and also $\mathscr{N}(\mathscr{O}u + \mathscr{O}x_u)^\perp \cong \mathscr{N}(\mathscr{O}v + \mathscr{O}x_v)^\perp \bmod 2\mathscr{O}$.

Proof. Sufficiency follows closely to the proof of last proposition. Denote L_u and L_v as before. Again, $L_u \cong L_v$ by direct computation of the norm groups. Condition

$$\mathscr{N}(\mathscr{O}u + \mathscr{O}x_u)^\perp \cong \mathscr{N}(\mathscr{O}v + \mathscr{O}x_v)^\perp \bmod 2\mathscr{O}$$

together with $93:17$ of [6] give us

$$L_u^\perp \cong A(a_u, -\alpha a_u^{-1})\,, \qquad L_v^\perp \cong A(a_v, -\alpha a_v^{-1})$$

for the suitable norm generators a_u, a_v of $L_u^\perp, L_v^\perp$ respectively. Now, it is easy to see that $\mathscr{W}(L_u^\perp) = \mathscr{W}(L_v^\perp) = (\text{say } \mathscr{W})$. Thus, $a_u \cong a_v \bmod \mathscr{W}$, so that $\mathscr{G}(L_u^\perp) = \mathscr{G}(L_v^\perp)$ and $L_u^\perp \cong L_v^\perp$ by Witt and O'Meara.

Direct computations again shows that u, v are of the same parity over isometric binary components and so apply Theorem 2.1. Again, the case of δ being unitary is trivial.

PROPOSITION 6.2. Suppose $\dim L = 5$. Then, $u \sim v$ if and only if: (i) there are vectors $x_u \in \mathfrak{M}_u^{(e,t)}$, $x_v \in \mathfrak{M}_v^{(e,t)}$ for some $t \geqq 0$ such that $Q(x_u) \equiv Q(x_v) \bmod 4\delta^{-1}\mathscr{P}$ when $\delta \notin 2\mathscr{P}$; and $Q(x_u) \equiv Q(x_v) \bmod 2\mathscr{O}$ when $\delta \in 2\mathscr{P}$ (ii) $\mathscr{G}(\langle u\rangle^\perp) = \mathscr{G}(\langle v\rangle^\perp)$ if $\delta \in \mathscr{U}$. (Here, e denotes ord 2.)

Proof. Using the same L_u and L_v, one proves that they are again isometric. So, $FL_u \cong FL_v$ by Witt. But now, $\dim L_u^\perp = \dim L_v^\perp = 3$ so that $L_u^\perp$ represents every weight (base) generator; similarly for $L_v^\perp$. Put $\mathscr{W}(L_u^\perp) = \mathscr{W}(L_v^\perp) = b\mathscr{O}$.

If $\mathscr{N}(L_u^\perp)\mathscr{W}(L_u^\perp) \sim 1$, (i.e. if t is even) then

$$L_u^\perp \cong A(0, 0) \perp \langle -d\rangle \cong L_v^\perp\,,$$

where $d \cong dL_u$. Thus, $u \sim v$ by Theorem 2.1.

If t is odd, then 93:18 of [6] shows that $L_u^\perp \cong A(b, 0) \perp \langle -d\rangle$ if FL_u is isotropic, and $L_u^\perp \cong A(b, 4\rho b^{-1}) \perp \langle -d(1-4\rho)\rangle$ if FL_u anistropic. Similarly, we write out for $L_v^\perp$. Thus, $L_u^\perp \cong L_v^\perp$ always. Apply Theorem 2.1.

PROPOSITION 6.3. Suppose $\dim L = 6$. Then, $u \sim v$ if and only if: (i) there are vectors $x_u \in \mathfrak{M}_u^{(s,t)}$, $x_v \in \mathfrak{M}_v^{(s,t)}$ for some $s, t \geqq 0$ such that $Q(x_u) \equiv Q(x_v) \bmod 2\mathscr{O}$ for $\delta \in 2\mathscr{P}$; and congruence modulo $4\delta^{-1}\mathscr{P}$ if $\delta \notin 2\mathscr{P}$; (ii) $\mathscr{N}(\mathscr{O}u + \mathscr{O}x_u)^\perp \cong \mathscr{N}(\mathscr{O}v + \mathscr{O}x_v)^\perp \bmod 2\mathscr{P}^{-t}$; and (iii) $\mathscr{G}(\langle u\rangle^\perp)$ equals $\mathscr{G}(\langle v\rangle^\perp)$ if $\delta \in \mathscr{U}$.

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