Pacific Journal of Mathematics

AUTOMORPHISMS OF POSTLIMINAL C^*-ALGEBRAS

E. Christopher Lance

Vol. 23, No. 3 May 1967

PACIFIC JOURNAL OF MATHEMATICS
Vol. 23, No. 3, 1967

AUTOMORPHISMS OF POSTLIMINAL C^*-ALGEBRAS

E. C. LANCE

Let $\alpha(\mathfrak{A})$ denote the group of automorphisms of a C^*-algebra $\mathfrak{A}$. The object of this paper is to give an intrinsic algebraic characterization of those elements α of $\alpha(\mathfrak{A})$ which are induced by a unitary operator in the weak closure of $\mathfrak{A}$ in every faithful representation, and it is attained for the class of C^*-algebras known as GCR, or more recently postliminal. The relevant condition is that α should map closed two-sided ideals of $\mathfrak{A}$ into themselves, and the main theorem (Theorem 2) may be thought of as an analogue for C^*-algebras of Kaplansky's theorem for von Neumann algebras, namely that an automorphism of a Type I von Neumann algebra is inner if and only if it leaves the centre elementwise fixed. The proof of Theorem 2 requires the—probably unnecessary—assumption that $\mathfrak{A}$ is separable.

By a C^*-algebra we mean a Banach algebra over the complex numbers, with a conjugate-linear anti-automorphic involution $A \rightarrow A^*$ satisfying $\| A^*A \| = \| A^* \| \cdot \| A \|$. The mappings of C^*-algebras which we consider (automorphisms, representations, etc.) will always be assumed to preserve the adjoint operation, and by a homomorphic image of a C^*-algebra $\mathfrak{A}$, we mean the image of a homomorphism from $\mathfrak{A}$ into another C^*-algebra $\mathfrak{B}$ (this is automatically a C^*-subalgebra of $\mathfrak{B}$ [2; 1.8.3]). We shall refer to Dixmier's book [2] for all standard results that we need to quote concerning C^*-algebras. By the theorem of Gelfand-Naĭmark (see, e.g. [2; 2.6.1]), a C^*-algebra has an isometric representation as an algebra of operators on a Hilbert space, and we shall usually think of a given C^*-algebra as being "concretely" represented on some Hilbert space. A *state* of a C^*-algebra $\mathfrak{A}$ is a positive linear functional of norm one. The set $\mathfrak{S}$ of states of $\mathfrak{A}$ is a convex subset of the (Banach) dual space of $\mathfrak{A}$. If $\mathfrak{A}$ has an identity element then $\mathfrak{S}$ is w^*-compact, but in any case $\mathfrak{S}$ contains an abundance of extreme points, which are called *pure states*. The set of pure states of $\mathfrak{A}$ will be denoted by $\mathfrak{P}$.

Given a state ρ of $\mathfrak{A}$, there is a representation ϕ_ρ of $\mathfrak{A}$ on a Hilbert space H_ρ, and a unit vector x_ρ in H_ρ such that $\{\phi_\rho(A)x_\rho : A \in \mathfrak{A}\}$ is dense in H_ρ (i.e. the representation ϕ_ρ is cyclic) and

$$\rho(A) = \langle \phi_\rho(A)x_\rho, x_\rho \rangle$$

for each $A \in \mathfrak{A}$. ϕ_ρ is irreducible if and only if ρ is pure. Given a state ρ of $\mathfrak{A}$, and a representation ϕ of $\mathfrak{A}$ on H, we say that ρ is a *vector state* (in the representation ϕ) if $\rho(A) = \langle \phi(A)x, x \rangle$ for some unit vector x in H; and if ϕ is faithful, we say that ρ is *normal* if the map $A \rightarrow \rho(A)$ is continuous with respect to the topology induced on $\phi(\mathfrak{A})$ by the ultra-weak topology on the algebra $\mathfrak{L}(H)$ of all bounded operators on H. It is clear that a vector state is normal. Let Φ denote the *universal representation* of $\mathfrak{A}$, formed by choosing one element from each unitary equivalence class of cyclic representations of $\mathfrak{A}$ and taking their direct sum; and let Ψ denote the *reduced atomic representation* of $\mathfrak{A}$, formed by choosing one element from each unitary equivalence class of irreducible representations of $\mathfrak{A}$ and taking their direct sum. Both Φ and Ψ are faithful representations, and every state [resp. every pure state] of $\mathfrak{A}$ is a vector state in the representation Φ [resp. Ψ].

Let $\hat{\mathfrak{A}}$ denote the structure space of $\mathfrak{A}$, i.e. the set of unitary equivalence classes of irreducible representations of $\mathfrak{A}$, with the Jacobson topology [2; § 3.1]. Following Dixmier, we shall call a C^*-algebra *liminal* if every irreducible representation consists of compact operators, *postliminal* if every nonzero homomorphic image has a nonzero closed two-sided liminal ideal, and *antiliminal* if it possesses no nonzero closed two-sided liminal ideals. If $\mathfrak{A}$ is postliminal then $\hat{\mathfrak{A}}$ is a T_0-space [2; 4.3.7 (ii)], and every representation of $\mathfrak{A}$ has a Type I von Neumann algebra as weak closure [2; 5.5.2]. Also, $\mathfrak{A}$ has a composition series $(I_\rho)_{0 \leqq \rho \leqq \delta}$ (i.e. an increasing nest of closed two-sided ideals of $\mathfrak{A}$ indexed by the ordinals less than or equal to some ordinal δ, such that $I_0 = (0), I_\delta = \mathfrak{A}$ and I_ρ is the closure of $\bigcup_{\rho' < \rho} I_{\rho'}$ for every limit ordinal $\rho \leqq \delta$) such that each difference algebra $I_{\rho+1} - I_\rho$ has Hausdorff structure space [2; 4.5.5 and 4.5.3].

Given a C^*-algebra $\mathfrak{A}$, we denote by $\alpha(\mathfrak{A})$ the group of automorphisms of $\mathfrak{A}$. Each element of $\alpha(\mathfrak{A})$ is an isometric isomorphism of $\mathfrak{A}$ onto itself [2; 1.3.7 and 1.8.1]. If ϕ is a faithful representation of $\mathfrak{A}$ on H, an automorphism α of $\mathfrak{A}$ is said to be *extendable* (in the representation ϕ) if there is an automorphism of the weak closure of $\phi(\mathfrak{A})$ which agrees with $\phi \circ \alpha \circ \phi^{-1}$ on $\phi(\mathfrak{A})$; and *weakly-inner* if $\phi(\alpha(A)) = U^* \phi(A) U$ for each A in $\mathfrak{A}$, where U is a unitary operator in the weak closure of $\phi(\mathfrak{A})$. If $\alpha(A) = U^* A U$ for a unitary operator U in $\mathfrak{A}$, then we say that α is *inner*. Following [6], we denote by $\varepsilon_\phi(\mathfrak{A})$ [resp. $\iota_\phi(\mathfrak{A})$] the set of elements of $\alpha(\mathfrak{A})$ which are extendable [resp. weakly-inner] in the representation ϕ, and by $\pi(\mathfrak{A})$ the intersection of all the sets $\iota_\phi(\mathfrak{A})$ as ϕ ranges through the faithful representations of $\mathfrak{A}$ (the elements of $\pi(\mathfrak{A})$ are called *permanently weakly-inner*, or *π-inner* automorphisms). The sets $\varepsilon_\phi(\mathfrak{A}), \iota_\phi(\mathfrak{A})$ and $\pi(\mathfrak{A})$ are all subgroups of $\alpha(\mathfrak{A})$. According to [6; Lemma 3], $\alpha \in \varepsilon_\phi(\mathfrak{A})$ if $\phi \circ \alpha \circ \phi^{-1}$ is ultra-weakly bicontinuous, equivalently if $\rho \circ \alpha$ is a normal state in the representation ϕ if and only if ρ is. It follows that $\varepsilon_\phi(\mathfrak{A}) = \alpha(\mathfrak{A})$

since every state is normal in the universal representation.

If $\alpha \in \alpha(\mathfrak{A})$, we shall say that α *preserves ideals* if $\alpha(I) \subseteqq I$ for every closed two-sided ideal I of $\mathfrak{A}$, and that α preserves ideals *carefully* if $\alpha(I) = I$ for each such ideal I. We shall denote by $\tau(\mathfrak{A})$ [resp. $\tau_0(\mathfrak{A})$] the set of elements of $\alpha(\mathfrak{A})$ which preserve ideals [resp. preserve ideals carefully]. It is clear that $\tau_0(\mathfrak{A})$ is a subgroup of $\alpha(\mathfrak{A})$, and that $\tau(\mathfrak{A})$ is a subsemigroup of $\alpha(\mathfrak{A})$, but it is not clear whether $\tau(\mathfrak{A})$ can contain elements not in $\tau_0(\mathfrak{A})$ (*cf.* Corollary 1 of Theorem 1). Since an automorphism preserves the property of being a maximal ideal, an element of $\tau(\mathfrak{A})$ must preserve maximal two-sided ideals carefully, so that $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A})$ if every closed two-sided ideal of $\mathfrak{A}$ is an intersection of maximal ones.

LEMMA 1. *For any C^*-algebra $\mathfrak{A}$, $\varepsilon_\Psi(\mathfrak{A}) = \alpha(\mathfrak{A})$.*

Proof. To save writing Ψ constantly, we shall suppose that $\mathfrak{A}$ is given in its reduced atomic representation. Let $\mathfrak{N}$ denote the closure in the norm topology on $\mathfrak{S}$ of the convex hull of $\mathfrak{P}$. Let $\alpha \in \alpha(\mathfrak{A})$, then it is easy to see that α preserves pure states, i.e. $\rho \in \mathfrak{P} \Leftrightarrow \rho \circ \alpha \in \mathfrak{P}$. Also, for any bounded linear functional f on $\mathfrak{A}$, $\| f \circ \alpha \| = \| f \|$. It follows that $\sigma \in \mathfrak{N} \Leftrightarrow \sigma \circ \alpha \in \mathfrak{N}$.

Let $\mathfrak{N}_0$ denote the set of normal states of $\mathfrak{A}$. We shall show that $\mathfrak{N}_0 = \mathfrak{N}$ from which it follows that α and α^{-1} preserve normal states and by [6; Lemma 3] the lemma will be proved. Now $\mathfrak{N}_0$ is norm-closed and convex, and contains $\mathfrak{P}$ since every pure state is a vector state in the given representation, hence $\mathfrak{N}_0 \supseteqq \mathfrak{N}$. Conversely, if $\rho \in \mathfrak{N}_0$, then ρ is a norm limit of convex combinations of vector states [1; Chap. I § 4 Théorème 1] so it will suffice to show that each vector state is in $\mathfrak{N}$.

Denote by ω_x the state $A \to \langle Ax, x \rangle$ where x is a unit vector in the space H on which $\mathfrak{A}$ acts. Since $\mathfrak{A}$ is given in the reduced atomic representation we can write $H = \bigoplus_{\gamma \in \Gamma} H_\gamma$ where each H_γ is a subspace of H invariant under $\mathfrak{A}$, and the restriction $\mathfrak{A}|_{H_\gamma}$ is irreducible. Write $x = \sum_{\gamma \in \Gamma} x_\gamma$, with $x_\gamma \in H_\gamma$. Then

$$\begin{aligned} A \in \mathfrak{A} &\Longrightarrow Ax_\gamma \in H_\gamma \quad \text{for each } \gamma \in \Gamma \\ &\Longrightarrow \langle Ax, x \rangle = \sum_{\gamma \in \Gamma} \langle Ax_\gamma, x_\gamma \rangle \,, \end{aligned}$$

so that

$$\omega_x = \sum_{\gamma \in \Gamma} \omega_{x_\gamma} \,, \quad \text{where} \quad \sum_{\gamma \in \Gamma} \| x_\gamma \|^2 = 1 \,. \tag{1}$$

But ω_{x_γ} is either zero (if $x_\gamma = 0$) or a multiple $\| x_\gamma \|^{-2}$ of a vector state of an irreducible representation, which is pure. It follows from

(1) that $\omega_x \in \mathfrak{N}$, showing that $\mathfrak{N}_0 \subseteqq \mathfrak{N}$.

LEMMA 2. *For any C^*-algebra $\mathfrak{A}$, $\iota_\Psi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A})$.*

Proof. We shall again suppose that $\mathfrak{A}$ is given in its reduced atomic representation with weak closure $\mathfrak{A}^-$. Writing $H = \bigoplus_{\gamma \in \Gamma} H_\gamma$ as in Lemma 1, we have ([3]) $\mathfrak{A}^- = \bigoplus_{\gamma \in \Gamma} \mathfrak{L}(H_\gamma)$. If $\alpha \in \iota_\Psi(\mathfrak{A})$, let $U = \sum U_\gamma$ be a unitary in $\mathfrak{A}^-$ which induces α, where U_γ is a unitary operator on $H_\gamma (\gamma \in \Gamma)$. Let π_γ be the irreducible representation of $\mathfrak{A}$ on H_γ defined by $A \rightarrow A|_{H_\gamma}$ (for some $\gamma \in \Gamma$), and suppose $\pi_\gamma(A) = 0$. Then

$$\begin{aligned} \pi_\gamma(\alpha(A)) &= U^* A U|_{H_\gamma} \\ &= U_\gamma^* A U_\gamma \\ &= 0 . \end{aligned}$$

Thus a preserves the primitive ideal $\pi_\gamma^{-1}(0)$. But every primitive ideal is of this form, and every closed two-sided ideal in $\mathfrak{A}$ is an intersection of primitive ideals, hence α preserves ideals.

Since $\iota_\Psi(\mathfrak{A})$ is a group, α^{-1} also preserves ideals, and so α preserves ideals carefully.

As an immediate corollary to the above lemma, we have $\pi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A})$ for any C^*-algebra $\mathfrak{A}$, a fact which has previously been noted by R. V. Kadison (private communication).

THEOREM 1. *If $\mathfrak{A}$ is a postliminal C^*-algebra, then $\tau(\mathfrak{A}) = \iota_\Psi(\mathfrak{A})$.*

Proof. We continue to assume that $\mathfrak{A}$ is given in the reduced atomic representation, and we shall use the notation established in Lemma 2. By that lemma, we have only to prove that $\tau(\mathfrak{A}) \subseteqq \iota_\Psi(\mathfrak{A})$.

For each closed two-sided ideal I of $\mathfrak{A}$, define subsets $\mathfrak{U}(I)$ and $\mathfrak{B}(I)$ of the structure space $\hat{\mathfrak{A}}$ by

$$\begin{aligned} \mathfrak{U}(I) &= \{\pi \in \hat{\mathfrak{A}} : \pi(I) = (0)\} , \\ \mathfrak{B}(I) &= \{\pi \in \hat{\mathfrak{A}} : \pi(I) \neq (0)\} . \end{aligned}$$

These sets are, respectively, closed and open in $\hat{\mathfrak{A}}$ [2; 3.2.1].

Suppose that $\alpha \in \tau(\mathfrak{A})$. By Lemma 1, α has an extension to an automorphism $\bar{\alpha}$ of $\mathfrak{A}^- = \bigoplus_{\gamma \in \Gamma} \mathfrak{L}(H_\gamma)$. Given $\pi \in \hat{\mathfrak{A}}$ there is a unique subspace H_γ of H such that π is unitarily equivalent to π_γ. Let $E_\pi \in \mathfrak{A}^-$ denote the projection from H onto H_γ. The elements $\{E_\pi : \pi \in \hat{\mathfrak{A}}\}$ are precisely the minimal central projections of $\mathfrak{A}^-$, and they generate the centre of $\mathfrak{A}^-$ (as a von Neumann algebra). An automorphism preserves the property of being a minimal central projection, so $\bar{\alpha}$ permutes the E_π.

Let $(I_\rho)_{0 \leqq \rho \leqq \delta}$ be a composition series for $\mathfrak{A}$ such that each difference algebra $I_{\rho+1} - I_\rho$ has Hausdorff structure space. Suppose that σ is an ordinal $(0 < \sigma \leqq \delta)$ and that for $\rho < \sigma$ we have shown that

$$\bar{\alpha}(E_\pi) = E_\pi \quad \text{for all} \quad \pi \in \mathfrak{V}(I_\rho) . \tag{2}$$

Clearly (2) is (vacuously) satisfied for $\sigma = 1$. If σ is a limit ordinal then $\mathfrak{V}(I_\sigma) = \bigcup_{\rho<\sigma} \mathfrak{V}(I_\rho)$ so that (2) holds with $\rho = \sigma$. Suppose that σ is not a limit ordinal, and let $\theta \in \mathfrak{V}(I_\sigma)$. Let $\bar{\alpha}(E_\theta) = E_\phi$. We shall suppose $\phi \neq \theta$ and obtain a contradiction.

Let $\{\phi\}^-$ denote the closure of $\{\phi\}$ in the Jacobson topology. We shall first show that $\theta \notin \{\phi\}^-$. To see this, note that

$$\hat{\mathfrak{A}} = \mathfrak{V}(I_{\sigma-1}) \cup (\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})) \cup \mathfrak{U}(I_\sigma) ,$$

so that ϕ must belong to one of these three sets.

(i) for $\pi \in \mathfrak{V}(I_{\sigma-1})$ we have by (2), $\bar{\alpha}(E_\pi) = E_\pi$, so that all the elements $E_\pi (\pi \in \mathfrak{V}(I_{\sigma-1}))$ are already bespoken as values for the (injective) mapping $\bar{\alpha}$, hence it is not possible that $\phi \in \mathfrak{V}(I_{\sigma-1})$ unless $\theta = \phi$. Thus $\phi \notin \mathfrak{V}(I_{\sigma-1})$ and also $\theta \notin \mathfrak{V}(I_{\sigma-1})$.

(ii) $\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$ is homeomorphic with the structure space of $I_\sigma - I_{\sigma-1}$ [2; 3.2.1], and this is Hausdorff (and hence a T_1-space) so that if $\phi \in \mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$, $\theta \notin \{\phi\}^-$ since by (i) θ is also in $\mathfrak{V}(I_\sigma) \cap \mathfrak{U}(I_{\sigma-1})$.

(iii) $\mathfrak{U}(I_\sigma)$ is closed, and $\theta \notin \mathfrak{U}(I_\sigma)$. Thus if $\phi \in \mathfrak{U}(I_\sigma)$, it follows that $\{\phi\}^- \subseteqq \mathfrak{U}(I_\sigma)$ and $\theta \notin \{\phi\}^-$.

Thus in any case $\theta \notin \{\phi\}^-$, i.e. $\operatorname{Ker}(\phi) \nsubseteqq \operatorname{Ker}(\theta)$. Choose $A \in \mathfrak{A}$ such that $\phi(A) = 0$, $\theta(A) \neq 0$. Then

$$\begin{aligned} \theta(A) \neq 0 &\Longrightarrow AE_\theta \neq 0 \\ &\Longrightarrow \bar{\alpha}(AE_\theta) \neq 0 \\ &\Longrightarrow \bar{\alpha}(A)\cdot\bar{\alpha}(E_\theta) \neq 0 \\ &\Longrightarrow \alpha(A)\cdot E_\phi \neq 0 . \end{aligned}$$

On the other hand, $\alpha \in \tau(\mathfrak{A})$ so α preserves $\operatorname{Ker}(\phi)$, hence

$$\begin{aligned} \phi(A) = 0 &\Longrightarrow A \in \operatorname{Ker}(\phi) \\ &\Longrightarrow \alpha(A) \in \operatorname{Ker}(\phi) \\ &\Longrightarrow \alpha(A)\cdot E_\phi = 0 . \end{aligned}$$

We have arrived at a contradiction, thus showing that $\bar{\alpha}(E_\theta) = E_\theta$ for $\theta \in \mathfrak{V}(I_\sigma)$, i.e. (2) holds for $\rho = \sigma$.

By transfinite induction, $\bar{\alpha}(E_\pi) = E_\pi$ for all $\pi \in \hat{\mathfrak{A}} (= \mathfrak{V}(I_\delta))$. Since the centre of $\mathfrak{A}^-$ is generated as a von Neumann algebra by the E_π and $\bar{\alpha}$ is ultra-weakly continuous (*cf.* Lemma 1), $\bar{\alpha}$ leaves the centre elementwise fixed. But $\mathfrak{A}^-$ is Type I, so by Kaplansky's theorem [7] $\bar{\alpha}$ is inner, which proves the theorem.

COROLLARY 1. *If $\mathfrak{A}$ is postliminal, then $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A})$.*

Proof. By Lemma 2 and Theorem 1 we have

$$\tau_0(\mathfrak{A}) \subseteqq \tau(\mathfrak{A}) = \iota_\psi(\mathfrak{A}) \subseteqq \tau_0(\mathfrak{A}) .$$

COROLLARY 2. *If $\mathfrak{A}$ is postliminal, $\alpha \in \tau(\mathfrak{A})$ and ϕ is an irreducible representation of $\mathfrak{A}$, then α induces a weakly-inner automorphism α_ϕ of $\phi(\mathfrak{A})$.*

Proof. Suppose that $\mathfrak{A}$ is given in its reduced atomic representation. ϕ is unitarily equivalent to the map $A \rightarrow AE_\pi$ (for some $\pi \in \hat{\mathfrak{A}}$). By Theorem 1, $\alpha(A) = U^* AU$ (for all $A \in \mathfrak{A}$) for some $U \in \mathfrak{A}^-$. The map $AE_\pi \rightarrow (UE_\pi)^* AE_\pi(UE_\pi)$ is then unitarily equivalent to the required automorphism of $\phi(\mathfrak{A})$.

Our results so far have mirrored those of Miles [8] on derivations. In the case of derivations, it is now known ([5] and [9]) that every derivation of a C^*-algebra is permanently weakly-inner. We shall now show that the analogous result holds for ideal-preserving automorphisms of (separable) postliminal C^*-algebras, by making use of the decomposition of a representation of such an algebra as a direct integral of irreducible representations. For an account of this decomposition, see [1; Chap. II] and [2; § 8].

LEMMA 3. *If $\mathfrak{A}$ is a C^*-algebra, $\alpha \in \tau_0(\mathfrak{A})$ and $\mathfrak{B}$ is any homomorphic image of $\mathfrak{A}$, then α induces an automorphism in $\tau_0(\mathfrak{B})$.*

Proof. Let ψ be a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$, with kernel I. Define a map $\tilde{\alpha}$ on $\mathfrak{B}$ by $\tilde{\alpha}(\psi(A)) = \psi(\alpha(A))$. $\tilde{\alpha}$ is well-defined since α preserves I. It is clearly a homomorphism, with range the whole of $\mathfrak{B}$, and since α preserves I carefully it is injective. Thus it is an automorphism.

If J is a closed two-sided ideal in $\mathfrak{B}$ then $\psi^{-1}(J)$ is a closed two-sided ideal in $\mathfrak{A}$ containing I and is carefully preserved by α, from which it follows that $\tilde{\alpha}$ carefully preserves J. Thus $\tilde{\alpha} \in \tau_0(\mathfrak{B})$.

THEOREM 2. *If $\mathfrak{A}$ is a separable postliminal C^*-algebra then $\pi(\mathfrak{A}) = \tau(\mathfrak{A})$.*

Proof. We have already noted that $\pi(\mathfrak{A}) \subseteqq \tau(\mathfrak{A})$. Suppose $\alpha \in \tau(\mathfrak{A})$, and let ϕ be any faithful representation of $\mathfrak{A}$. We have to show that α is weakly-inner in the representation ϕ. Since $\mathfrak{A}$ is postliminal, the weak closure $\overline{\phi(\mathfrak{A})}$ is a Type I von Neumann algebra, so is isomorphic to an algebra with abelian commutant, i.e. ϕ is quasi-equivalent to a multiplicity-free representation (*cf.* [2; 5.4.1]). Since the property of being weakly-inner is preserved by quasi-equivalence, we may suppose that ϕ is multiplicity-free and $\phi(\mathfrak{A})'$ is abelian (we use a prime to denote the commutant of a set of operators). Since we are assuming that $\mathfrak{A}$ is separable, $\overline{\phi(\mathfrak{A})}$ is generated (as a von Neumann algebra) by a countable set of operators.

Let E be a cyclic projection in $\phi(\mathfrak{A})'$ (which is the centre of $\overline{\phi(\mathfrak{A})}$). The restriction of $\phi(\mathfrak{A})$ to E is a homomorphic image of $\mathfrak{A}$, so by Lemma 3 α induces an ideal-preserving automorphism on it. If the automorphism so induced on each cyclic portion of the centre of $\overline{\phi(\mathfrak{A})}$ is weakly-inner, then (taking a maximal orthogonal family of cyclic central projections) it follows that α is weakly-inner. We may thus restrict to a cyclic central projection and we can therefore assume that ϕ acts on a separable Hilbert space H.

There exist [2; 8.3.2] a standard Borel space Z, a bounded positive measure μ on Z, a measurable field $\zeta \rightarrow H_\zeta$ of Hilbert spaces on Z, a measurable field of representations $\zeta \rightarrow \pi_\zeta$ of $\mathfrak{A}$ on the field (H_ζ) and an isometry from H onto $\int^\oplus H_\zeta d\mu(\zeta)$, which transforms $\phi(\mathfrak{A})'$ into the diagonal operators and ϕ into $\int^\oplus \pi_\zeta d\mu(\zeta)$. We shall equate H, $\phi(\mathfrak{A})$, &c. with their transforms under this equivalence. Since $\phi(\mathfrak{A})'$ consists of diagonal operators, almost every π_ζ is irreducible [2; 8.5.1]. For almost all $\zeta \in Z$, α induces an automorphism α_ζ of $\pi_\zeta(\mathfrak{A})$, which by Corollary 2 of Theorem 1 is weakly-inner, and so in particular extends to an automorphism (which we still call α_ζ) of $\mathfrak{L}(H_\zeta)$. Define $\alpha_\zeta = 0$ on the exceptional null set. α_ζ is ultra-weakly continuous, hence strongly continuous on bounded sets. Thus we have a field (which we do not yet know to be measurable) of automorphisms α_ζ, such that for each $A \in \mathfrak{A}$, $\phi(\alpha(A)) = \int^\oplus \alpha_\zeta(\pi_\zeta(A))d\mu(\zeta)$.

We now show that α is weakly continuous on the unit ball of $\mathfrak{A}$ (in the representation ϕ). To do this it suffices, by [4; Remark 2.2.3], to show that α is weakly continuous at zero on the set of positive operators in the unit ball of $\mathfrak{A}$. Since H is separable, the unit ball is metrizable in the weak topology, and we need only deal with sequences. Suppose that $I \geqq A_n \geqq 0$ and $\phi(A_n) \rightarrow 0$ weakly. Then $\phi(A_n^{1/2}) \rightarrow 0$ strongly and by [1; Chap. II § 2 Prop. 4 (i)] there is a subsequence (n_k) such that, locally almost everywhere, $\pi_\zeta(A_{n_k}^{1/2}) \rightarrow 0$ strongly. Since α_ζ is strongly continuous on bounded sets, we have locally almost everywhere, $\pi_\zeta(\alpha(A_{n_k}^{1/2})) = \alpha_\zeta(\pi_\zeta(A_{n_k}^{1/2})) \rightarrow 0$ strongly. Since the sequence (A_{n_k}) is bounded, it follows from [1; Chap. II § 2 Prop. 4 (ii)] that $\alpha(A_{n_k}^{1/2}) \to 0$ strongly and so $\alpha(A_{n_k}) \to 0$ weakly. Thus α (and similarly α^{-1}) is weakly continuous on bounded sets in the representation ϕ, hence ultra-weakly continuous, and so α is extendable to an automorphism $\bar{\alpha}$ of $\overline{\phi(\mathfrak{A})}$.

We shall next show that the field of automorphisms $\zeta \to \alpha_\zeta$ induces $\bar{\alpha}$ on $\overline{\phi(\mathfrak{A})}$ (and so is measurable). Let A be a fixed element of $\overline{\phi(\mathfrak{A})}$, and let $\zeta \to A_\zeta$ be a measurable operator field representing A. Let $\zeta \to B_\zeta$ be a measurable operator field representing $\bar{\alpha}(A)$. By metrizability of the strong topology [1; p. 33] and Kaplansky's Density Theorem [1; Chap. I § 3 Th. 3], we can choose a sequence (A_n) in $\mathfrak{A}$ such that $\|A_n\| \leqq \|A\|$ and $\phi(A_n) \to A$ strongly. By passing to a subsequence and using [1; Chap. II § 2 Prop. 4(i)] again, we can even suppose that $\pi_\zeta(A_n) \to A_\zeta$ strongly, locally almost everywhere. Since $\bar{\alpha}$ is strongly continuous on bounded sets, $\phi(\alpha(A_n)) \to \bar{\alpha}(A) = \int^{\oplus} B_\zeta d\mu(\zeta)$ strongly, and there is a subsequence (A_{n_k}) of (A_n) such that $\pi_\zeta(\alpha(A_{n_k})) \to B_\zeta$ strongly, locally almost everywhere. But since α_ζ is strongly continuous on bounded sets, we have $\pi_\zeta(\alpha(A_{n_k})) = \alpha_\zeta(\pi_\zeta(A_{n_k})) \to \alpha_\zeta(A_\zeta)$ strongly, locally a.e. Hence, locally almost everywhere, we have $B_\zeta = \alpha_\zeta(A_\zeta)$. Thus $\bar{\alpha}(A) = \int^{\oplus} \alpha_\zeta(A_\zeta) d\mu(\zeta)$, as required.

Now since $\bar{\alpha}$ is induced by the field $\zeta \to \alpha_\zeta$, it is clear that $\bar{\alpha}$ leaves each diagonal operator fixed, i.e. $\bar{\alpha}$ leaves the centre of $\overline{\phi(\mathfrak{A})}$ elementwise fixed. Hence by Kaplansky's Theorem $\bar{\alpha}$ is inner (since $\overline{\phi(\mathfrak{A})}$ is Type I), and the proof is complete.

It is possible for an automorphism of a postliminal algebra to be weakly-inner in some representation without being π-inner, as the following example shows. Let ν denote Lebesgue measure on the interval $[0, 1]$, and let $H = L_2([0, 1], \nu)$. Let $\mathfrak{K}$ denote the set of compact operators on H. For $f \in C([0, 1])$ let T_f denote the operator defined by

$$T_f x(t) = f(t)x(t) ,$$

and let $\mathfrak{T} = \{T_f : f \in C([0, 1])\} \subseteqq \mathfrak{L}(H)$. Then $\mathfrak{A} = \mathfrak{K} + \mathfrak{T}$ is a C^*-algebra [2; 1.8.4] and is postliminal since $\{(0), \mathfrak{K}, \mathfrak{A}\}$ is a composition series for which each difference algebra has Hausdorff structure space (because $\mathfrak{A} - \mathfrak{K} \cong \mathfrak{T}$). Let $U \in \mathfrak{L}(H)$ be the unitary operator defined by

$$U x(t) = x(1 - t) ,$$

then U induces an automorphism of $\mathfrak{A}$: for if $K \in \mathfrak{K}$, $T_f \in \mathfrak{T}$ then $U^*(K + T_f) U = U^* K U + T_g$ (where $g(t) = f(1 - t)$). Let

$$I_0 = \{T_f \in \mathfrak{T}: f(t) = 0 \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2}\}$$

and let $I_1 = \mathfrak{K} + I_0$, then it is easy to see that $U^* \cdot U$ does not preserve I_1, so by Theorem 2, $U^* \cdot U$ is not π-inner. (In fact, it is not weakly-inner in the representation of $\mathfrak{A}$ on $H \oplus H$ defined by $K + T \rightarrow (K + T) \oplus T$.) But it is clearly weakly-inner in the given representation, since this is irreducible.

This example also shows that an automorphism of a postliminal C^*-algebra can leave the centre elementwise fixed and yet not be π-inner: for the centre of $\mathfrak{K} + \mathfrak{T}$ consists just of scalar multiples of the identity.

We conclude with a few remarks about the antiliminal case. Let $\mathfrak{A}$ be a factor of Type II_1. Then $\mathfrak{A}$ has no nonzero proper closed two-sided ideals, so that $\tau_0(\mathfrak{A}) = \tau(\mathfrak{A}) = \alpha(\mathfrak{A})$ in this case. On the other hand, there are many outer automorphisms of $\mathfrak{A}$. Thus the sets $\tau_0(\mathfrak{A})$ and $\tau(\mathfrak{A})$ are probably not of great interest when $\mathfrak{A}$ is antiliminal.

Let $\mathfrak{A}$ be an antiliminal algebra with a faithful irreducible representation. Then $\mathfrak{A}$ has uncountably many such representations, all inequivalent [2; 4.7.2]. Intuitively, it seems unlikely that an automorphism would be weakly-inner in all these representations without actually being inner. In [6; Ex. a] an example is given of such an algebra (the Fermion algebra $\mathfrak{F}$) together with an automorphism of $\mathfrak{F}$ which is weakly-inner in one representation, but not π-inner. It would be interesting to have an example of an automorphism of $\mathfrak{F}$ which is π-inner but not inner.

REFERENCES

1. J. Dixmier, *Les algèbres d'opérateurs dans l'espace hilbertien*, Gauthier-Villars, Paris, 1957.
2. ———, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris, 1964.
3. J. Glimm and R. V. Kadison, *Unitary operators in C^*-algebras*, Pacific J. Math. **10** (1960), 547-556.
4. R. V. Kadison, *Unitary invariants for representations of operator algebras*, Ann. of Math. **66** (1957), 304-379.
5. ———, *Derivations of operator algebras*, Ann. of Math. **83** (1966), 280-293.
6. R. V. Kadison and J. R. Ringrose, *Derivations and automorphisms of operator algebras*, Commun. Math. Phys. **4** (1967), 32-63.
7. I. Kaplansky, *Algebras of type I*, Ann. of Math. **56** (1952), 460-472.
8. P. Miles, *Derivations on B^*-algebras*, Pacific J. Math. **14** (1964), 1359-1366.
9. S. Sakai, *Derivations of W^*-algebras*, Ann. of Math. **83** (1966), 273-279.

Received January 17, 1967.

UNIVERSITY OF NEWCASTLE UPON TYNE

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be typewritten (double spaced). The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts may be sent to any one of the four editors. All other communications to the editors should be addressed to the managing editor, Richard Arens at the University of California, Los Angeles, California 90024.

50 reprints per author of each article are furnished free of charge; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley 8, California.

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Pacific Journal of Mathematics

Vol. 23, No. 3 May, 1967