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**SOME PROPERTIES OF SEQUENCES, WITH AN  
APPLICATION TO NONCONTINUABLE POWER SERIES**

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# SOME PROPERTIES OF SEQUENCES, WITH AN APPLICATION TO NONCONTINUABLE POWER SERIES

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**For a real sequence  $f = \{f(n)\}$  and positive integer  $N$ , let  $F^N$  denote the sequence of  $N$ -tuples  $\{(f(n+1), \dots, f(n+N))\}$ . A functional equation method due to Kemperman is used to obtain a sufficient condition on  $s$  in order that  $s^N$  have an independent  $N$ -tuple among its cluster points. If a bounded  $s$  has the latter property, and if  $g = rs$ , where  $r(n) \rightarrow \infty$  and  $r(n+1)/r(n) \rightarrow 1$  as  $n \rightarrow \infty$ , then there is a subsequence  $S$  of the sequence of positive integers such that, for almost all real  $\alpha$ , the restriction of  $\alpha g^N$  to  $S$  is uniformly distributed (mod 1) in the  $N$ -cube.**

Let  $F$  be an analytic function whose Maclaurin series has bounded coefficients  $\{a_n\}$  which satisfy the additional requirement

$$\lim_{M \rightarrow \infty} \inf_{0 \leq k < \infty} \sum_{n=k}^{k+M} |a_n| = \infty.$$

If  $a_n = |a_n| \exp \{2\pi i f(n)\}$ , then the density (mod 1) of  $f^N$  for each  $N$  is sufficient in order that  $F$  have the unit circle as a natural boundary. Hence, a metric result for noncontinuable series is obtained from the results for sequences.

**1. Notation.** For  $x$  real, let  $((x)) = x - [x]$ , and  $e(x) = \exp(2\pi ix)$ .  $h_1, \dots, h_N$  will denote an  $N$ -tuple of integers, not all of which are zero. The sequence of nonnegative integers will be denoted by  $Z$ , and subsequences of  $Z$  by  $S_1, S_2$ , etc. For a real sequence  $f$ , we denote by  $\Delta f$  the sequence  $\{f(n+1) - f(n)\}$  and

$$\Delta^{j+1}f = \Delta(\Delta^j f), \quad (j = 1, 2, \dots)$$

**2. The property (PN).**

**DEFINITION.** A bounded sequence  $s$  of real numbers will be said to have *property* (PN) if there is an independent  $N$ -tuple among the cluster points of  $s^N$ . In other words,  $s$  has property (PN) if there is a subsequence  $S$  of  $Z$  such that for every  $N$ -tuple  $h_1, \dots, h_N$  of integers not all zero, there holds

$$(2.1) \quad \lim_{n \rightarrow \infty} |h_1 s(n+1) + \dots + h_N s(n+N)| > 0, \quad (n \in S).$$

We shall be interested in sequences  $s$  of the following form:

$$(2.2) \quad s(n) = \varphi(\psi(n)) , \quad (n \in Z) ,$$

where  $\varphi$  is a function of period 1 with at most a nowhere dense set of points of discontinuity, and  $\psi$  has the property (QN).

(QN) There exists a subsequence  $S_1$  of  $Z$  such that

$$(2.3) \quad \begin{aligned} & \text{(i)} \quad \Delta^j \psi(n) \text{ converges (mod 1) for } n \rightarrow \infty , \\ & \quad \quad n \in S_1 \quad (j = 2, \dots, N) \\ & \text{(ii)} \quad \{((\psi(n)), ((\Delta \psi(n)))): n \in S_1\} \text{ is not nowhere dense.} \end{aligned}$$

**THEOREM 2.1.** *Let  $s$  be of the form (2.2), where  $\varphi$  and  $\psi$  have the properties listed above. Then either  $s$  has property (PN), or else  $\varphi$  agrees on some interval  $I \subset [0, 1]$  with a polynomial of degree  $N-2$  at most.*

*Proof.* Under the conditions on  $\varphi$  and  $\psi$ , it is possible to obtain a subsequence  $S_2$  of  $S_1$  and an open disk  $D$  in the plane such that

$$(2.4) \quad \begin{aligned} & \text{(i)} \quad \lim_{n \rightarrow \infty} \Delta^j \psi(n) = \tau_j \pmod{1}, \quad (n \in S_2), \quad (j = 2, \dots, N), \\ & \text{(ii)} \quad \{((\psi(n)), ((\Delta \psi(n)))): n \in S_2\} \text{ is dense in } D, \\ & \text{(iii)} \quad \text{for every } (\tau_0, \tau_1) \text{ in } D, \text{ and} \\ & \quad \text{every } p, 1 \leq p \leq N, \text{ the point} \end{aligned}$$

$$\tau_0 + p\tau_1 + \sum_{j=2}^p \binom{p}{j} \tau_j$$

is a point of continuity for  $\varphi$ .

For each  $(\tau_0, \tau_1)$  in  $D$ , a subsequence  $S_3 = S_3(\tau_0, \tau_1)$  of  $S_2$  can be chosen so that the corresponding subsequence of (2.4 (ii)) converges to  $(\tau_0, \tau_1)$ . In this case, as  $n \rightarrow \infty$ ,  $n \in S_3$ , one has for every  $h_1, \dots, h_N$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p s(n+p) &= \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p \varphi(\psi(n)) \\ &\quad + p \Delta \psi(n) + \sum_{j=2}^p \binom{p}{j} \Delta^j \psi(n) \end{aligned}$$

so that

$$(2.5) \quad \lim_{n \rightarrow \infty} \sum_{p=1}^N h_p s(n+p) = \sum_{p=1}^N h_p \varphi\left(\tau_0 + p\tau_1 + \sum_{j=2}^p \binom{p}{j} \tau_j\right), \quad (n \in S_3) .$$

Suppose now that  $s$  does not have property (PN). Then for each  $(\tau_0, \tau_1)$  in  $D$ , there is an  $N$ -tuple  $h_1, \dots, h_N$  such that the right hand member of (2.5) is zero. Hence  $D$  is a countable union of closed sets

$$F = F(h_1, \dots, h_N) = \{(\tau_0, \tau_1) \in D: (2.5) \text{ vanishes}\} .$$

Some  $F$ , then, must contain an open subdisk  $D_1$ , with center

$(\tau'_0, \tau'_1)$ . That is, there exists an  $N$ -tuple  $h_1, \dots, h_N$  of integers not all zero with the property that for all sufficiently small positive  $h$  and  $k$ ,

$$\sum_{p=1}^N h_p \varphi \left( h + pk + \tau'_0 + p\tau'_1 + \sum_{j=2}^p \binom{p}{j} \tau_j \right) = 0.$$

The assertion of the theorem follows upon taking

$$\varphi_p(x) = h_p \varphi \left( x + \tau'_0 + p\tau'_1 + \sum_{j=2}^p \binom{p}{j} \tau_j \right)$$

in the following lemma, a weak version of one due to Kemperman [4, p. 41]. The proof is included for completeness.

**LEMMA.** *Let  $a > 0$ , and let  $\varphi_1, \dots, \varphi_N$  be real functions, with  $\varphi_j$  defined and continuous on  $I_j = (- (j+1)a, (j+1)a)$ , ( $j = 1, \dots, N$ ). Suppose that for all  $x, y$  in  $(-a, a)$ , there holds*

$$(2.6) \quad \sum_{j=1}^N \varphi_j(x + jy) = 0.$$

*Then  $\varphi_j$  is equal on  $I_j$  to a polynomial of degree  $N-2$  at most.*

*Proof.* We may suppose that  $N \geq 2$  (the case  $N = 1$  is trivial), and that the lemma holds for  $N - 1$ . Let  $0 < b < a$ , and let  $I'_j = (-(j+1)b, (j+1)b)$ .

Next, we choose and keep fixed a number  $h$ ,  $0 < h < \min(b, a - b)$ . For this  $h$ , and  $j = 1, \dots, N$ , let

$$\tilde{\varphi}_j(x) = \varphi_j(x + (1 - j/N)h) - \varphi_j(x), \quad (x \in I'_j).$$

We note that each  $\tilde{\varphi}_j$  is continuous, and  $\tilde{\varphi}_N \equiv 0$ . Moreover, if  $x, y$  are in  $(-b, b)$ , then  $x, y, x + h$ , and  $y - h/N$  are in  $(-a, a)$ .

Thus, for all  $x, y$  in  $(-b, b)$ , we have

$$\sum_{j=1}^{N-1} \tilde{\varphi}_j(x + jy) = \sum_{j=1}^N \varphi_j(x + h + j(y - h/N)) - \sum_{j=1}^N \varphi_j(x + jy) = 0.$$

The induction hypothesis implies that, for  $j = 1, \dots, N - 1$ ,  $\tilde{\varphi}_j$  is a polynomial of degree  $N - 3$  at most on  $I'_j$ . Hence  $\varphi_j$  is, on  $I'_j$ , the sum of a polynomial of degree  $N - 2$  at most and a function of period  $(1 - j/N)h$ . But such a representation is given for every sufficiently small positive  $h$ , which, with the continuity of  $\varphi_j$ , implies that  $\varphi_j$  is a polynomial of degree  $N - 2$  at most on  $I'_j$ , ( $1 \leq j \leq N - 1$ ). From the arbitrariness of  $b$ ,  $\varphi_j$  is such a polynomial on  $I_j$ . Finally, (2.6) shows that  $\varphi_N$  is also such a polynomial on  $I_N$ .

In a previous paper [1], results of *v. d. Corput* were used to

obtain various sufficient conditions on a real sequence  $\psi$  in order that  $\psi$  satisfy condition (I):

(I) *There exists a sequence  $S$  such that  $\lim \Delta^j \psi(n)$  ( $n \in S$ ) exists for all  $j \geq r$ , while  $\{(\psi(n), \Delta \psi(n), \dots, \Delta^{r-1} \psi(n)); n \in S\}$  is uniformly distributed (mod 1) in the  $r$ -dimensional unit cube.*

(I) clearly implies that  $\psi$  has property (QN) for every  $N \geq 2$ . The reader is referred to the paper for details and proofs.

### 3. A metric result for uniform distribution in the $N$ -cube.

**THEOREM 3.1** *Let  $g = \{g(n); n \in Z\}$  be a sequence of real numbers. Let there exist a subsequence  $S_0$  of  $Z$  such that, for every  $N$ -tuple  $h_1, \dots, h_N$  of integers not all zero there holds*

$$(3.1) \quad \lim \left| \sum_{p=1}^N h_p g(n+p) \right| = \infty, \quad \text{as } n \rightarrow \infty, \quad n \in S_0.$$

*Then there exists a subsequence  $S$  of  $S_0$  such that, for almost all real  $\alpha$ , the sequence  $(\alpha g^N) | S$  is uniformly distributed (mod 1) in the  $N$ -cube.*

*Proof.* Let the set of all such  $N$ -tuples be ordered, with, say,  $h'_1, \dots, h'_N$  as the first. Let a subsequence  $S_1 \subset S_0$  be taken such that

$$\sum_{p=1}^N h'_p \{g(n+p) - g(m+p)\}$$

is either greater than 1 for every  $n, m$  in  $S_1$ , with  $n > m$ , or else is less than  $-1$  for every such  $n$  and  $m$ . Successively extracting subsequences  $S_1 \supset S_2 \supset \dots$  in this way, and then using a diagonal procedure, one finally obtains a sequence  $S$  such that, for every  $N$ -tuple  $h_1, \dots, h_N$ , there is an  $m_0 = m_0(h_1, \dots, h_N)$  such that one has either

$$(3.2) \quad \sum_{p=1}^N h_p \{g(n+p) - g(m+p)\} \geq 1$$

for all  $n$  and  $m$  in  $S$  with  $n > m \geq m_0$   
or else

$$(3.3) \quad \sum_{p=1}^N h_p \{g(n+p) - g(m+p)\} \leq -1$$

for all such  $n$  and  $m$ .

By a well-known result of Weyl [6, p. 348], either condition (3.2) or (3.3) implies that, for almost all real  $\alpha$ , the sequence

$$(3.4) \quad \alpha \sum_{p=1}^N h_p g(n+p) \quad (n \in S)$$

is uniformly distributed (mod 1). There being only countably many  $N$ -tuples, it follows that, for almost all  $\alpha$ , (3.4) is uniformly distributed (mod 1) for every  $N$ -tuple  $h_1, \dots, h_N$ . But this shows [2, p. 66] that for almost all  $\alpha$  the sequence  $(\alpha g^N) \mid S$  is uniformly distributed (mod 1) in the  $N$ -cube.

It is easy to see that if  $\theta > 1$  is a transcendental number and  $g(n) = \theta^n$ , then Theorem 3.1 is applicable. The next result shows the less obvious fact that Theorem 3.1 also applies if, for instance,  $g(n) = n^3 \log n \sin n^2$ .

**THEOREM 3.2.** *Let  $g = \{g(n): n \in Z\}$  be of the form*

$$(3.5) \quad g(n) = r(n)s(n), \quad n \in Z,$$

*where  $s$  has property (PN), while*

$$(3.6) \quad \lim r(n) = \infty, \quad \lim (r(n+1)/r(n)) = 1.$$

*Then there is a subsequence  $S_0$  of  $Z$  such that (3.1) holds for every  $N$ -tuple  $h_1, \dots, h_N$  of integers not all zero.*

*Proof.* For  $p = 1, 2, \dots, N$ , it follows from (3.6) that

$$r(n+p) = r(n)(1 + o(1)), \quad \text{as } n \rightarrow \infty.$$

Therefore we have

$$(3.7) \quad g(n+p) = r(n)s(n+p)(1 + o(1)), \quad \text{as } n \rightarrow \infty, \quad p = 1, \dots, N.$$

Since  $s$  has property (PN), there exists a subsequence  $S_0$  of  $Z$  such that

$$(2.1) \quad \lim_{n \rightarrow \infty} |h_1 s(n+1) + \dots + h_N s(n+N)| > 0, \quad (n \in S_0)$$

for all  $N$ -tuples  $h_1, \dots, h_N$  of integers not all zero. But (3.6), (3.7), and (2.1) imply (3.1).

**4. An application to noncontinuable power series.** Perry [5] has proved that, for every real sequence  $f = \{f(n): n \in Z\}$ , there exists a sequence of moduli  $\{|a_n|: n \in Z\}$  such that the power series

$$(4.1) \quad \sum_{n=0}^{\infty} |a_n| e(f(n)) z^n$$

has radius of convergence 1 and the analytic function it represents can be continued analytically across a semicircle of the unit circle. However, if the additional requirements

$$(4.2) \quad |a_n| = o(1) \quad \text{as } n \rightarrow \infty$$

and

$$(4.3) \quad \lim_{N \rightarrow \infty} \inf_{0 \leq k < \infty} \sum_{n=k+1}^{k+N} |a_n| = \infty$$

are imposed, then there are conditions on  $f$  sufficient that (4.1) represent a function with  $|z| = 1$  as its natural boundary. Some such conditions were given in [1]. Theorem 4 gives a metric result in this direction.

**THEOREM 4.** *Let  $\{|a_n|: n \in Z\}$  satisfy (4.2) and (4.3). Let  $g$  be a real sequence which, for each  $N$ , satisfies the hypothesis of Theorem 3.1. For each real  $\alpha$ , let*

$$(4.4) \quad F_\alpha(z) = \sum_{n=0}^{\infty} |a_n| e(\alpha g(n)) z^n, \quad |z| < 1.$$

*Then the set of  $\alpha$  for which  $F_\alpha$  can be continued across an arc of the unit circle has measure zero.*

*Example.*  $\sum e(\alpha n \sin n^2) z^n$  has  $|z| = 1$  as its natural boundary for almost all  $\alpha$ .

For  $N = 2, 3, \dots$ , let  $A_N$  be the set of those real  $\alpha$  for which  $\alpha g^N$  is dense (mod 1) in the unit  $N$ -cube.

By Theorem 3.1,  $A_N$  contains almost all  $\alpha$ , and it follows that almost all  $\alpha$  are in  $A_N$  for every  $N$ . For each such  $\alpha$ , and each  $z_0 = e(\theta_0)$ , there holds

$$(4.5) \quad \limsup_{k \rightarrow \infty} \left| \sum_{k+1}^{k+N} a_n e(\alpha g(n) + n\theta_0) \right| \geq \liminf_{k \rightarrow \infty} \sum_{k+1}^{k+N} |a_n|.$$

In view of (4.3), (4.5) shows that the partial sums of the series in (4.4) are unbounded at  $z_0$ . By (4.2) and a well-known theorem of Fatou [3, p. 391], it follows that  $z_0$  is a singularity for  $F$ .

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