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GENERALIZED SEMIGROUP KERNELS

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This paper is concerned with the problem of generalizing the notion of a kernel of a semigroup. Various kernels are introduced and their mutual relationships are investigated. Conditions are found on a semigroup which are necessary and sufficient in order that certain of its kernels be trivial.

The "generalized" kernels we introduce here have properties which are reminiscent of the notion of a radical. Our results, however, are quite different from certain of the investigations along these lines (see, for example, [3] and [13]). Our work is more closely related to that of Schwartz [10], [11], and [12]. We refer to [2] for definitions not explicitly given.

1. Mutually annihilating sums and kernels. The following definition seems to be due to Ljapin [6]. If S is a semigroup, then S is said to be a mutually annihilating sum of semigroups $\{S_\lambda\}_{\lambda \in A}$ if and only if S is (isomorphic to) a semigroup with zero such that if 0 is the zero of S , then

- (i) for λ in A , S_λ is a subsemigroup of S with contains 0 ,
- (ii) each member of S is in S_λ for some λ in A , and
- (iii) for λ and γ in A , $\lambda \neq \gamma$, $S_\lambda \cap S_\gamma = \{0\} = S_\lambda S_\gamma$.

We shall be concerned with semigroups S which are mutually annihilating sums of semigroups each of which has some one fixed semigroup property P (to say that P is a semigroup property means that P is a property such that if one of two isomorphic semigroups has property P , then so does the other). There is a rather obvious connection between mutually annihilating sums and subdirect sums which we make explicit in the lemma below.

We use the concept of a subdirect sum as in the theory of rings, *i.e.*, to say that S is a subdirect sum of semigroups $\{T_\mu\}_{\mu \in \Omega}$ means that S is (isomorphic to) a subsemigroup of the direct product $\prod_{\mu \in \Omega} T_\mu$ such that if for some $\nu \in \Omega$, π_ν is the projection of $\prod_{\mu \in \Omega} T_\mu$ onto T_ν , then the homomorphism $\pi_\nu|_S$ is onto T_ν . The following lemma is not difficult to prove.

LEMMA 1.1 *If S is a semigroup with zero, then S is a mutually annihilating sum of semigroups each having property P if and only if there is a collection $\{T_\mu\}_{\mu \in \Omega}$ of semigroups such that*

(1) *for each $\mu \in \Omega$, T_μ is a semigroup with zero which has property P , and*

(2) S is a subdirect sum of the collection $\{T_\mu\}_{\mu \in \Omega}$ such that each member of S , when viewed as a member of $\prod_{\mu \in \Omega} T_\mu$, has at most one nonzero component.

Let K_P denote the set $\{I \mid I \text{ is an ideal and } S/I \text{ has property } P\}$.

THEOREM 1.2. *Suppose P is a semigroup property, S is a semigroup, and J is an ideal of S . Then S/J is a mutually annihilating sum of semigroups each having property P if and only if there is a subset K of K_P such that (i) J is the K -kernel of S , and (ii) if I and I' are distinct members of K , then $S = I \cup I'$.*

Proof. Assume K is a subset of K_P such that (i) and (ii) of the theorem are true. It is clear from (i) that S/J is, in a natural way, a subdirect sum of the collection $\{S/I \mid I \in K\}$. Property (ii) implies that each member of the subdirect sum has no more than one nonzero component. It then follows from (i) and the lemma that S/J is a mutually annihilating sum of semigroups each having property P .

Now assume J is an ideal of S and S/J is a mutually annihilating sum of a collection $\{S_\lambda\}_{\lambda \in \Lambda}$, where, for each λ in Λ , S_λ is a semigroup having property P . Let φ denote the natural homomorphism from S onto S/J . For each $\lambda \in \Lambda$, let I_λ denote the set of all x in S such that either $\varphi(x)$ is zero or $\varphi(x)$ is not in S_λ . If $K = \{I_\lambda \mid \lambda \in \Lambda\}$, then K satisfies (i) and (ii) of the theorem.

REMARK 1.3. In case S has a zero and J is zero, the theorem asserts that S is a mutually annihilating sum of semigroups each of which has property P if and only if there is a subset K of K_P such that (i) $\bigcap K = 0$ and (ii) if I and I' are in K , $I \neq I'$, then $S = I \cup I'$.

For each semigroup S , let $\mathcal{M} = \mathcal{M}_S$ denote the set of all maximal ideals of S . The following corollaries are immediate applications of Remark 1.3.

COROLLARY 1.4. *Assume S is a semigroup with zero. Then $\bigcap \mathcal{M} = 0$ if and only if S is a mutually annihilating sum of semigroups each of which either is a null semigroup of order two or is a 0-simple semigroup.*

COROLLARY 1.5. *If S is a semigroup and J is the \mathcal{M} -kernel of S , then the \mathcal{M} -kernel of S/J is zero.*

The \mathcal{M} -kernel of a semigroup determines, to some extent, which maximal ideals are prime (an ideal J of a semigroup S is said to be prime if and only if either $J = S$ or the complement of J is a sub-semigroup of S).

THEOREM 1.6. *Suppose S is a semigroup which has a maximal ideal. If J denotes the \mathcal{M} -kernel of S , then each maximal ideal of S is prime if and only if there is a collection $\{S_\alpha\}_{\alpha \in A}$ of simple subsemigroups of S such that*

- (1) $S = J \cup \bigcup_{\alpha \in A} S_\alpha$,
- (2) for $\alpha \in A, \beta \in A, \alpha \neq \beta, S_\alpha \cap S_\beta$ is void and $S_\alpha S_\beta \subseteq J$, and
- (3) for each $\alpha \in A, J \cap S_\alpha$ is void.

Proof. First assume that each maximal ideal of S is prime and that J is the \mathcal{M} -kernel of S . From previous arguments, it is known that S/J is isomorphic to a mutually annihilating sum of semigroups each of which is isomorphic to S/M for some $M \in \mathcal{M}$. Since S/M is a simple semigroup with zero for each $M \in \mathcal{M}$ (recall that M is prime), it follows that there is a collection $\{S_\alpha\}_{\alpha \in A}$ of simple semigroups such that S/J is a mutually annihilating sum of $\{S_\alpha^0 \mid \alpha \in A\}$. For each $\alpha \in A$, we identify S_α with the subsemigroup T_α of S such that

$$(T_\alpha \cup J)/J = S_\alpha$$

and $T_\alpha \cap J$ is void. Then the collection $\{S_\alpha\}_{\alpha \in A}$ satisfies (1), (2), and (3) of the theorem.

Assume, on the other hand, that $\{S_\alpha\}_{\alpha \in A}$ is a collection of simple subsemigroups of S such that (1), (2), and (3) hold where J denotes the \mathcal{M} -kernel of S . Then each maximal ideal of S is of the form

$$J \cup \bigcup_{\alpha \in A \setminus \{\beta\}} S_\alpha$$

for some $\beta \in A$. Thus each maximal ideal of S is prime.

2. The \mathcal{P} -kernel of a semigroup. We now turn our attention to a different kind of kernel of a semigroup. Let \mathcal{P} denote the set of all prime ideals of S . We now characterize the \mathcal{P} -kernel of S . First we need some notation and definitions.

To say that S is a band means that S is an idempotent semigroup. S is said to be a rectangular band if and only if S is a band and $a b a = a$ for all a and b in S . Rectangular bands may be characterized as semigroups of the form $X \times Y$ where X and Y are arbitrary sets and where the operation on $X \times Y$ is defined by

$$(x, y)(x', y') = (x, y')$$

for x, x' in X and y, y' in Y (see, for example, [4] or [7]).

We assume, from this point on, that S is any semigroup, that E is the maximal semilattice homomorphic image of S , and that η is

the natural homomorphism from S onto E . Define a relation φ on S by $(a, b) \in \varphi$ if and only if there exists $x \in S^1, y \in S^1, c \in S, d \in S$, and positive integers m and n such that $a = xcy, b = xdy$, and $c^m = d^n$. Clifford has observed, [1], that if φ' is the transitive closure of φ , then S/φ' is the maximal band homomorphic image of S . He also noted that the maximal semilattice homomorphic image of S/φ' is the maximal semilattice homomorphic image of S . Each φ' -congruence class of S will be called an archimedean component of S . This definition, which agrees with the usual one in case S is commutative, has not been used before in case S is not commutative. Clifford's observation may be rephrased, "any semigroup is a semilattice union of semigroups each of which is a rectangular band of archimedean components of S ".

The following theorem is due to Petrich (see [8] and [9]).

THEOREM 2.1. (*Petrich*) *In order that P be a prime ideal of the semigroup S it is necessary and sufficient that there exists a prime ideal Q of E such that $P = \bigcup_{e \in Q} \eta^{-1}(e)$.*

The following corollary is immediate.

COROLLARY 2.2. *The \mathcal{P} -kernel of the semigroup S is precisely the inverse image of the \mathcal{P} -kernel of E under η (even in case either is void).*

LEMMA 2.3. *If E is a semilattice, then the \mathcal{P} -kernel of E is void in case E contains no zero element and otherwise is the zero of E .*

Proof. Suppose z is in the \mathcal{P} -kernel of E . If z were not a zero of E , then $\{x \in E \mid x \not\leq z\}$ would be a prime ideal of E which does not contain z .

The next theorem follows immediately from previous results.

THEOREM 2.4. *The \mathcal{P} -kernel of the semigroup S is void in case E does not contain a zero and otherwise is the inverse image of the zero of E under η .*

COROLLARY 2.5. *If the semigroup S contains a zero, then the \mathcal{P} -kernel of S is zero if and only if the equations $acb = 0$ and $c^n = d^m$ imply $adb = 0$ for $a \in S^1, b \in S^1, c \in S, d \in S$, and positive integers m and n . Note that in case S is commutative the latter*

condition merely asserts that 0 is the only nilpotent member of S .

PROOF. By Theorem 2.4 the \mathcal{S} -kernel of S is $\gamma^{-1}(z)$ where z is the zero of E . Since $\gamma^{-1}(z)$ contains the zero of S , it must contain only one archimedean component of S . Thus the \mathcal{S} -kernel of any semigroup with zero is precisely the archimedean component containing the zero. The corollary now follows from the way φ was defined.

The following corollaries are evident.

COROLLARY 2.6. *The following statements are equivalent:*

- (1) *the maximal semilattice homomorphic image of S is trivial,*
- (2) *the \mathcal{S} -kernel of S is S , and*
- (3) *S is a rectangular band of its archimedean components.*

COROLLARY 2.7. *The maximal band image of a semigroup is a rectangular band if and only if the maximal semilattice image is trivial.*

Finally we consider an application to semilattice theory. To say that F is a face of a semilattice E means that F is a (nonvoid) sub-semigroup of E such that either F is E or the complement of F in E is a prime ideal of E . A prime ideal P of E is principal if and only if it is of the form $\{x \in E \mid e \not\leq x\}$ for some $e \in E$ (in this case e is called the generator of P).

THEOREM 2.7. *If E is a semilattice, then each proper face of E is finite if and only if*

- (1) *each proper prime ideal of E is principal,*
- (2) *each ascending chain in E is finite, and*
- (3) *each nonzero element of E is covered by at most a finite number of elements of E .*

Proof. First assume each proper face of E is finite. If P is a proper prime ideal of E , then P is principal and has as generator the product of all elements of E not in P . It is equally clear that (2) and (3) follow.

Now assume (1), (2), and (3) are true. Let F denote any proper face of E . Then $E \setminus F$ is a proper prime ideal and thus is principal. Let e denote the generator of $E \setminus F$. Then $x \in F$ if and only if $x \geq e$. Define a sequence A of subsets of E inductively by

(i) $x \in A_1$ if and only if $x \in E$ and x covers e , and (ii) if k is a positive integer, $x \in A_{k+1}$ if and only if $x \in E$ and x covers some member of A_k .

For each positive integer i , A_i is finite. One can show that there is a positive integer n such that A_n is void by assuming otherwise and by constructing an infinite ascending chain in E . Thus

$$F = \bigcup_{i=1}^n A_i$$

and F is finite. The theorem follows.

If \mathcal{I} is any collection of ideals of a semigroup S , $S/(\cap \mathcal{I})$ is always a subdirect sum of the collection $\{S/T \mid T \in \mathcal{I}\}$. In case $S = E$ is a semilattice the intersection of the collection of all prime ideals of E is void or is a zero of E . Thus one obtains the following corollary of Theorem 2.7.

COROLLARY 2.8. *Assume E is a semilattice such that*

- (1) *each proper prime ideal of E is principal,*
- (2) *each ascending chain in E is finite, and*
- (3) *each nonzero element of E is covered by at most a finite number of elements of E .*

Then E is a subdirect sum of the collection $\{F^\circ \mid F \text{ is a finite face of } E\}$.

3. Relationships among various kernels. As in the previous section S denotes any semigroup, E its maximal semilattice homomorphic image, and η the natural homomorphism from S onto E . Throughout this section K_T will denote the intersection of all ideals of the semigroup T and will be called the kernel of T . If N denotes the void set, we define $K_N = N$. Likewise P_T and M_T will denote the \mathcal{P} and \mathcal{M} kernels of T respectively.

THEOREM 3.1. *If A is an ideal of the semigroup S , then $K_A = K_S$. Thus we have*

$$K_S = K_{P_S} = K_{M_S} .$$

Proof. Let A denote any ideal of S . If K_S is not void, then for each ideal J of A

$$K_S = K_S J K_S \subseteq K_S J K_S .$$

Thus $K_S J K_S$ is an ideal of K_S . Since K_S is simple,

$$K_S = K_S J K_S \subseteq A J A \subseteq J .$$

Thus $K_S \subseteq K_A$.

Conversely, if K_A is not void, then K_S is equal to the intersection

of the collection \mathcal{H} where $J \in \mathcal{H}$ if and only if $J = I \cap A$ for some ideal J of S . But each such J is an ideal of A , thus $K_S \subseteq K_A$.

It follows that $K_S = K_A$ for each ideal A of S . Clearly if P_S or M_S is void so is K_S . The theorem follows.

COROLLARY 3.2. *If S is a semigroup, then the kernel of S is the same as the kernel of P_S and thus is the kernel of a rectangular band of archimedean components of S .*

In order to obtain the relationship between the \mathcal{M} -kernel, M_S , and the \mathcal{P} -kernel, P_S , we need more information about the maximal ideals of S . The next theorem provides such information and has some interest in its own right. First we need another definition. An ideal I of $\eta^{-1}(e)$, for $e \in E$, is said to be induced by S if and only if $I \cup (S \setminus \eta^{-1}(e))$ is an ideal of S . It is easy to see that an ideal I of $\eta^{-1}(e)$ is induced by S if and only if

- (1) $f_1 \in E \setminus \{e\}, f_2 \in E \setminus \{e\}$, and $f_1 f_2 = e$ imply $\eta^{-1}(f_1) \eta^{-1}(f_2) \subseteq I$ and
- (2) $f \in E$ and $f > e$ imply $\eta^{-1}(f) I \subseteq I$ and $I \eta^{-1}(f) \subseteq I$.

THEOREM 3.3. *If M is a subset of the semigroup S , then M is a maximal ideal of S if and only if there exists $e \in E$ such that either*

- (1) *e is a maximal element of E such that $\eta^{-1}(e)$ is simple and $M = \bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$, or*

- (1) *there is a maximal ideal M_e of $\eta^{-1}(e)$ such that M_e is induced by S and $M = M_e \cup \bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$.*

Proof. Suppose M is a maximal ideal of S and that $a \in S \setminus M$. Let e denote $\eta(a)$. First we show that $\eta^{-1}(f) \subseteq M$ for all $f \in E \setminus \{e\}$. Consider $F = \{x \in E \mid x \not\geq e\}$. The set $M \cup \eta^{-1}(F)$ is a proper ideal of S . Since M is maximal, $\eta^{-1}(F) \subseteq M$. Thus if $f \in E$ such that $\eta^{-1}(f) \not\subseteq M$, then $e \leq f$. Suppose there exists $f_0 \in E$ such that

$$\eta^{-1}(f_0) \not\subseteq M$$

such that $f_0 > e$. If $F_0 = \{x \in E \mid x \not\geq f_0\}$, then $M \cup \eta^{-1}(F_0)$ is a proper ideal of S . Thus $\eta^{-1}(F_0) \subseteq M$. But $f_0 > e$ implies that $e \in F_0$ and that $\eta^{-1}(e) \not\subseteq M$, contrary to the choice of $e \in E$. It follows that $\eta^{-1}(f) \subseteq M$ for each $f \neq e$.

We now show that if $M \cap \eta^{-1}(e)$ is not void, then $M_e = M \cap \eta^{-1}(e)$ is a maximal ideal of $\eta^{-1}(e)$ which is induced by S . Suppose there is an ideal J of $\eta^{-1}(e)$ such that $M_e \subset J \subset \eta^{-1}(e)$. Then $M \cup J$ is an ideal of S such that $M \subset M \cup J \subset S$. Thus no such J exists and M_e is a maximal ideal of $\eta^{-1}(e)$. Clearly M_e is induced by S .

Similar reasoning shows that if $M \cap \eta^{-1}(e)$ is void, then e is maximal in E and that $\eta^{-1}(e)$ is simple.

The proof of the other half of the theorem is easy and is omitted.

LEMMA 3.4. The \mathcal{M} -kernel of a semigroup is never void.

Proof. Assume the \mathcal{M} -kernel of some semigroup S is void. Then the \mathcal{M} -kernel of S° is zero and thus S° is a mutually annihilating sum of semigroups each of which either is a null semigroup of order two or is a simple semigroup with zero. Since 0 is a prime ideal of S° , S° must be a simple semigroup with zero. Thus S is simple and the \mathcal{M} -kernel of S is S contrary to the assumption that the \mathcal{M} -kernel of S is void.

THEOREM 3.5. *In order that the \mathcal{M} -kernel of S be a subset of the \mathcal{P} -kernel of S it is necessary and sufficient that E contain a zero z and that for each $e \in E \setminus \{z\}$, e is maximal in E and $\eta^{-1}(e)$ is simple.*

REMARK. We do not require in the previous theorem that E contain elements other than z .

Proof. Assume $M_S \subseteq P_S$. Since M_S is not void, neither is P_S ; thus there is a zero z in E and $P_S = \eta^{-1}(z)$. Assume $e \in E \setminus \{z\}$. We show that e is maximal in E and that $\eta^{-1}(e)$ is simple. To do this it suffices, by Theorem 3.3, to show that $\bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$ is a maximal ideal of S . Assume $\bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$ is not a maximal ideal of S . Since $M_S \subseteq P_S$, $x \in \eta^{-1}(e)$ implies that there exists a maximal ideal M_x of S such that $x \in M_x$. Since $\bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$ is not a maximal ideal of S , Theorem 3.3 implies that there exists a maximal ideal N_x of $\eta^{-1}(e)$ such that $M_x = N_x \cup \bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$. Since for each $x \in \eta^{-1}(e)$, $x \in N_x$, we have that $\bigcap_{x \in \eta^{-1}(e)} N_x$ is void. But $\bigcap_{x \in \eta^{-1}(e)} N_x$ contains the \mathcal{M} -kernel of $\eta^{-1}(e)$ which, by the lemma, is not void. We have established the necessity of the condition.

Now assume E has a zero and that if $e \in E \setminus \{z\}$, then e is maximal in E and $\eta^{-1}(e)$ is simple. If E contains no element other than z , then $M_S \subseteq S = \eta^{-1}(z) = P_S$. Assume E contains elements other than z . For each $e \in E \setminus \{z\}$ it is easy to see that $\bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)$ is a maximal ideal of S . Thus $\bigcap_{e \in E \setminus \{z\}} [\bigcup_{f \in E \setminus \{e\}} \eta^{-1}(f)] = \eta^{-1}(z)$ contains $\bigcap \mathcal{M}$ and $M_S \subseteq \eta^{-1}(z) = P_S$. The theorem follows.

COROLLARY 3.6. *If E has a zero z and $\eta^{-1}(z)$ is simple, then the following statements are equivalent:*

- (1) $M_S = P_S$, and
- (2) whenever $e \in E \setminus \{z\}$, e is maximal in E and $\eta^{-1}(e)$ is simple.

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