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# SEQUENCES OF HOMEOMORPHISMS WHICH CONVERGE TO HOMEOMORPHISMS

JEROME L. PAUL

**A technique often used in topology involves the inductive modification of a given mapping in order to achieve a limit mapping having certain prescribed properties. The following definition will facilitate the discussion. Suppose $X$ and $Y$ are topological spaces, and $\{W_i\}, i = 1, 2, \cdots,$ is a countable collection of subsets of $X$. Then a sequence $\{f_i\}, i \geqq 0,$ of mappings from $X$ into $Y$ is called stable relative to $\{W_i\}$ if $f_i \mid (X - W_i) = f_{i-1} \mid (X - W_i), i, = 1, 2, \cdots$. Note, in the above definition, that if $\{W_i\}$ is a locally finite collection, then $\lim_{i \to \infty} f_i$ is necessarily a well defined mapping from $X$ into $Y$, and is continuous if each $f_i$ is continuous. In a typical smoothing theorem, a $C^r$-mapping $f: M \to N$ between $C^\infty$ differentiable manifolds $M$ and $N$ is approximated by a $C^\infty$-mapping $g: M \to N$, where the mapping $g$ is constructed as the limit of a suitable sequence $\{f_i\}$(with $f_0 = f$) which is stable relative to a locally finite collection $\{C_i\}$ of compact subsets of $M$. On the other hand, instead of improving $f$, it is also of interest to approximate $f$ by a mapping $g$ which has bad behavior at, say, a dense set of points of $M$. In this paper, such a mapping $g$ is constructed as the limit of a sequence $\{f_i\}$ (with $f_0 = f$) which is stable relative to $\{C_i\}$, but where the $C_i$ are more "clustered" than a locally finite collection. The case of interest here is where a sequence of homeomorphisms $\{H_i\}$, which is stable relative to $\{U_i\}$, necessarily converges to a homeomorphism. Theorem 1 of this paper gives a sufficient condition that the latter be satisfied for homeomorphisms of a metric space. In Theorem 1, the collection $\{U_i\}$ is not, in general, locally finite (in fact, the $U_i$ satisfy a certain "nested" condition). Theorem 1 is used to establish a result concerning the distribution of homeomorphisms (of a differentiable manifold) which have a dense set of spiral points.**

Let $M$ be a metric space with metric $d$. We denote the (open) ball, of radius $r$, and centered at the point $x \in M$, by $B(x, r) = \{y \in M \mid d(x, y) < r\}$. The diameter of a nonempty subset $A$ of $M$ is $\delta(A) = \sup\{d(x, y) \mid x \in A, y \in A\}$. When $M$ is euclidean $n$-space $E^n$, we write the points of $E^n$ as $x = (x^1, \cdots, x^n)$, and provide $E^n$ with the usual euclidean norm and metric

$$|| x || = \left[\sum_{i=1}^{n} (x^i)^2\right]^{1/2}, \quad d(x, y) = || x - y || .$$

The boundary of $B(x, r)$ in $E^n$ is the $(n-1)$-sphere $S(x, r) =$

$\{y \in E^n \mid d(x, y) = r\}$. If $A$ is a subset of $M$, we denote the closure of $A$ in $M$ by $\bar{A}$. If $\bar{A}$ is compact, we say that $A$ is *relatively* compact. Let $Z^+$ denote the set of positive integers. The identity mapping will be denoted by $I$, without regard to domain.

THEOREM 1. *Let $\{U_i\}$, $i \in Z^+$, be a sequence of nonempty relatively compact open subsets of $M$ such that*

$$(1) \qquad U_i \cap U_j \neq \phi \Rightarrow U_i \supset \bar{U}_j \qquad [i < j] .$$

*Suppose $\{F_i\}$, $i \in Z^+$. is a sequence of homeomorphisms of $M$ onto itself such that*

$$(2) \qquad F_i \mid (M - U_i) = I ,$$

*and*

$$(3) \qquad d(F_i(x), F_i(y)) \geqq \zeta_i d(x, y) \qquad [x, y \in M] ,$$

*for some constant $\zeta_i$(depending on $F_i$). Set*

$$(4) \qquad H_i = F_i F_{i-1} \cdots F_1 ,$$

*(note that the sequence $\{H_i\}$ is stable relative to $\{U_i\}$). If, for $i \geqq 2$,*

$$(5) \qquad \delta(U_i) < \zeta_{i-1}\zeta_{i-2} \cdots \zeta_1/2^i ,$$

*then $H = \lim_{i\to\infty} H_i$ is a homeomorphism of $M$ on itself.*

*Proof*. We note from (2) that $\zeta_i \leqq 1$, $i \in Z^+$. Therefore, we see from (2) and (5) that $d(F_i(x), x) < 1/2^i$, and hence

$$(6) \qquad d(H_i(x), H_{i-1}(x)) < 1/2^i \qquad [i \in Z^+, x \in M] .$$

Given any fixed $x \in M$, we first show that $\lim_{i\to\infty} H_i(x)$ exists. We have two cases to consider.

*Case* 1. There exists an integer $j(x)$ such that $H_k(x) = H_{j(x)}(x)$ for all $k \geqq j(x)$. Then, of course, $\lim_{i\to\infty} H_i(x) = H_{j(x)}(x)$.

*Case* 2. There exists a sequence $l_1 < l_2 < \cdots$ such that $H_{l_i}(x) \neq H_{l_{i+1}}(x)$, $i \in Z^+$. Then from (4) and (2), we see that there exists a sequence $m_1 < m_2 \cdots$ such that $H_{m_i-1}(x) \in U_{m_i}$, and $\bar{U}_{m_{i+1}} \subset U_{m_i}$, $i \in Z^+$. Note that $\bigcap_{i=1}^{\infty} U_{m_i} = \bigcap_{i=1}^{\infty} \bar{U}_{m_i} \neq \phi$, the last inequality holding since $\{\bar{U}_{m_i}\}$, $i \in Z^+$, is a decreasing sequence of nonempty compact subsets of the compact set $\bar{U}_{m_1}$. Since $\delta(U_{m_i}) \to 0$ as $i \to \infty$, we see that there is a unique point $z = \bigcap_{i=1}^{\infty} U_{m_i}$. Hence $\lim_{i\to\infty} H_{m_i}(x) = z$. Then, using (6), $\lim_{i\to\infty} H_i(x) = z$. This shows that $H$ is a well defined mapping of $M$ into itself. Moreover, using (6), $H$ is the limit of a uniformly convergent sequence of continuous mappings, and hence is itself continuous.

We now show that $H$ is one-to-one. Suppose, then, that $x, y$ are two distinct points of $M$. We have three cases.

*Case* 1. There exist integers $j(x), k(y)$ such that $H_l(x) = H_{j(x)}(x)$ for $l \geqq j(x)$, and $H_m(y) = H_{k(y)}(y)$ for $m \geqq k(y)$. Then, setting $q = \max\{j(x), k(y)\}$, we have $H(x) = H_q(x) \neq H_q(y) = H(y)$.

*Case* 2. Same as Case 2 above. Then, as above, there exists a sequence $m_1 < m_2 < \cdots$ such that $H_{m_i}(x) \in U_{m_i}$, $i \in Z^+$. Choose $m_i = p$ so large that $1/2^p < d(x, y)$. Then using (1) we have, in particular, $H_{p-1}(x) \cup H(x) \subset U_p$. Using (3) and (4),

$$d(H_{p-1}(x), H_{p-1}(y)) \geqq \zeta_{p-1} \cdots \zeta_1 \cdot d(x, y) .$$

On the other hand, using (5) and our choice of $p$, it follows that $\delta(U_p) < \zeta_{p-1} \cdots \zeta_1/2^p < \zeta_{p-1} \cdots \zeta_1 \cdot d(x, y)$. Hence $H_{p-1}(y) \notin U_p$, and, using (1), (2), and (4), $H(y) \notin U_p$. Therefore, $H(x) \neq H(y)$.

*Case* 3. There exists a sequence $n_1 < n_2 < \cdots$ such that $H_{n_i}(y) \neq H_{n_{i+1}}(y)$. The proof that $H(x) \neq H(y)$ in this case is entirely analogous to Case 2. This completes the proof that $H$ is one-to-one.

We now show that $H$ maps $M$ *onto* itself. Let $y$ be an arbitrary point of $M$. If there exists an integer $j(y)$ such that $z = H_k^{-1}(y) = H_{j(y)}^{-1}(y)$ for all $k \geqq j(y)$, then $H(z) = y$. Suppose, then, that there exists a sequence $k_1 < k_2 < \cdots$ such that $H_{k_i}^{-1}(y) \neq H_{k_{i+1}}^{-1}(y)$. Then, using (1), (2), and (4), there exists a sequence $l_1 < l_2 < \cdots$ such that $H_{l_i}^{-1}(y) \in U_{l_i}$ and $U_{l_i} \supset \bar{U}_{l_{i+1}}$, $i \in Z^+$. Letting $z$ be the unique point $z = \bigcap_{i=1}^{\infty} U_{l_i}$, we see that $\lim_{i\to\infty} H_{l_i}^{-1}(y) = z$. But then $H(z) = \lim_{i\to\infty} H_{l_i}(H_{l_i}^{-1}(y)) = y$, where the first equality follows from the fact that a uniformly convergent sequence of functions is *continuously* convergent. Hence, $H$ is an onto mapping.

To show that $H$ is a homeomorphism of $M$ on itself, it remains to verify the continuity of $H^{-1}$ (note that when $M$ is an open subset of $E^n$, Brouwer's theorem on invariance of domain implies that $H^{-1}$ is continuous). We do this by showing that the limit set of $H$ is empty, i.e., given any $y \in M$, and any sequence $\{x_n\}$ of points of $M$ having no convergent subsequence, we shall show that the sequence $\{H(x_n)\}$ does *not* converge to $y$. Since $H$ is onto, let $z \in M$ be such that $H(z) = y$. We have two cases to consider.

*Case* 1. There exists an integer $j(z)$ such that $H_k(z) = H_{j(z)}(z) = y$

for all $k \geqq j(z)$. Now since $\{x_n\}$ contains no convergent subsequence, we may assume $d(x_n, z) \geqq \xi > 0$ for some fixed $\xi$ and all $n \in Z^+$. Let $p > 0$ be a *fixed* integer so large that $1/2^p < \xi/2$. Now for an arbitrary $n \in Z^+$, we have, from (3) and (4), $d(H_{p-1}(x_n), y) \geqq \zeta_{p-1} \cdots \zeta_1 \cdot \xi$, whereas, from (5), $\delta(U_p) < \zeta_{p-1} \cdots \zeta_1/2^p < \zeta_{p-1} \cdots \zeta_1 \cdot \xi/2$. Hence the points $H_{p-1}(x_n)$ and $y$ are not *both* contained in $\bar{U}_p$. A similar analysis shows that

($\alpha$) the points $H_{k-1}(x_n)$ and $y$ are not *both* contained in $\bar{U}_k$, for all $k \geqq p$, and all $n \in Z^+$.

Now given any $k \geqq p$, let $N_k$ denote those points $x_j$ of the sequence $\{x_n\}$ such that $H_{k-1}(x_j) \in U_k$. If $N_k \neq \phi$, then from ($\alpha$) above we see that $y \notin \bar{U}_k$. Setting $W_k = M - \bar{U}_k$ if $N_k \neq \phi$, and $W_k = M$ if $N_k = \phi$, we see that $W_k$ is a neighborhood of $y$ such that

$$H(N_k) \cap W_k = \phi\ . \tag{7}$$

Setting $\eta = \zeta_p \cdots \zeta_1 \cdot \xi$, we see from (3) and (4) that $d(H_p(x_n), y) \geqq \eta$ for all $n \in Z^+$. Now choose an integer $q > p$ so large that $\sum_{i=q}^{\infty} 1/2^i < \eta/2$. Then for any $x_l \notin \{N_p \cup N_{p+1} \cup \cdots \cup N_q\}$, we have $H_{q-1}(x_l) = H_{q-2}(x_l) = \cdots = H_p(x_l)$. Then $d(H(x_l), H_p(x_l)) \leqq \sum_{i=q}^{\infty} 1/2^q < \eta/2$, whereas $d(H(x_l), y) \geqq d(H(x_l), y) - d(H_p(x_l), H(x_l)) \geqq \eta - \eta/2 = \eta/2$. Hence we see that

$$H(x_l) \notin B(y, \eta/4) \qquad [x_l \notin \{N_p \cup N_{p+1} \cup \cdots \cup N_q\}]\ . \tag{8}$$

Setting $V = B(y, \eta/4) \cap W_p \cap W_{p+1} \cap \cdots \cap W_q$, we see from (7) and (8) that $V$ is a neighborhood of $y$ in $M$ such that $H(x_n) \notin V$ for all $n \in Z^+$. Hence the sequence $H(x_n)$ does *not* converge to $y$.

*Case* 2. There exists a sequence $m_1 < m_2 < \cdots$ such that $H_{m_i}(z) \neq H_{m_{i+1}}(z)$, $i \in Z^+$. Then, as seen before, there exists a sequence $k_1 < k_2 < \cdots$ such that $y = H(z) = \bigcap_{i=1}^{\infty} U_{k_i}$. As before, letting $\xi > 0$ be such that $d(x_n, y) \geqq \xi$ for all $n \in Z^+$, we take $p = k_j$ so large that $1/2^p < \xi$. Then, using (3), (4), and (5), we see that $y \in U_p$, whereas $H_{p-1}(x_n) \notin U_p$ for all $n \in Z^+$. Then by (4) and (2), $H(x_n) \notin U_p$ for all $n \in Z^+$. Since $U_p$ is a neighborhood of $y$ in $M$, it follows that $\{H(x_n)\}$ does not converge to $y$. This completes the proof that $H^{-1}$ is continuous, and hence Theorem 1 is completely proven.

REMARKS AND EXAMPLES. One verifies that the *biuniqueness* of the limit mapping $H$ is still valid if condition (5) is weakened to requiring only that $\delta(U_i) < \zeta_{i-1} \cdots \zeta_1/a_i$, where the positive constants $a_i$ are subject to the condition $\lim_{i\to\infty} a_i = +\infty$. The necessity for this latter condition is illustrated by the following example. Let $M = E^n$, and for any $i \in Z^+$, set $U_i = B(0, 1/2^{i+1})$. Let $F_i$ be a diffeomorphism of $E^n$ on itself defined by $F_i(x) = \alpha_i(\|x\|)x$, where $\alpha_i$ is a smooth monotonic real-valued function of the real variable $t$ such that $\alpha_i(t) = 1/2$ for $t \leqq 1/2^{i+2}$, and $\alpha_i(t) = 1$ for $t \geqq 1/2^{i+1}$. Then $d(F_i(x), F_i(y)) \geqq (1/2)d(x, y)$ for all $x, y \in E^n$ and $i \in Z^+$. Hence, setting $\zeta_i = 1/2$, $i \in Z^+$, we see that conditions (1), (2), and (3) of Theorem 1 are satisfied by $U_i$ and $F_i$. Condition (5) is violated, but we have, nevertheless,

$$\delta(U_i) = 2(1/2^{i+1}) = 1/2^i < 1/2^{i-1} = \zeta_{i-1} \cdots \zeta_1 .$$

It is easily seen that the mapping $H = \lim_{i\to\infty} F_i F_{i-1} \cdots F_1$ is a continuous mapping of $E^n$ on itself, but $H$ is *not* one-to-one since $H(B(0, 1/8)) = 0$.

The diffeomorphisms $F_i$ in the above example are members of an important class of homeomorphisms of $E^n$ which satisfy a condition such as (3): namely, the class of diffeomorphisms of $E^n$ which are the identity outside some compact subset of $E^n$. Condition (3) is not, in general, satisfied for *homeomorphisms* of $E^n$ which are the identity outside some compact subset of $E^n$, even for those which are, in addition, diffeomorphisms on the complement of a single point. For consider the following example. Let $F$ be a $C_0^\infty$-diffeomorphism of $E^2$ on itself (i.e., $F$ is a homeomorphism of $E^2$ on itself such that $F|(E^2 - 0)$ is a $C^\infty$-diffeomorphism) such that $F$ is the identity on the subset $\{E^2 - B(0, 1)\} \cap \{\bigcup_{n=1}^\infty S(0, 1/n)\} \cup \{0\}$, and such that the spheres $S(0, 1/n - 10^{-n})$, $n \in Z^+$, are rotated by $F$ through 180 degrees. Such a homeomorphism is readily constructed. One verifies that

$$\begin{aligned} &d(F((0, 1/n)), F((0, -1/n + 10^{-n})) \\ &\qquad = (10^n \cdot 2/n - 1)^{-1} \cdot d((0, 1/n), (0, -1/n + 10^{-n})) , \end{aligned}$$

and hence there can *not* exist a number $\zeta$ such that $d(F(x), F(y)) \geqq \zeta d(x, y)$ for all $x, y \in E^2$.

We now use Theorem 1 to establish a result concerning spiral points of homeomorphisms of nonbounded differentiable manifolds. The reader is referred to [1] for the relevant definitions and results. We recall that if $f: U \to E^n$, $n \geqq 2$, is a homeomorphism, where $U$ is an open set in $E^n$, then a point $x \in U$ is a *spiral point of* $f$ if, and only if, the following is satisfied: given *any* $C^p$-imbedding $(p > 0)\sigma: [0, 1] \to U$ such that $\sigma(1) = x$, *any* diffeomorphism $H$ of $E^n$ on itself, and *any* $(n-1)$-hyperplane $P$ in $E^n$ through $Hf(x)$, then there exists a sequence of points $t_i \in [0, 1]$ converging to 1 and such that $Hf\sigma(t_i) \in P$. The notion is extended to differentiable manifolds in the natural way. It is readily verified (cf. Proposition 2 of [1]) that if $f: M^n \to N^n$ is a homeomorphism, where $M^n$, $N^n$ are nonbounded differentiable $n$-manifolds, then the set of nonspiral points of $f$ is (uncountably) dense in $M^n$. Nevertheless, there always exist (Theorem 1 of [1]) homeomorphisms of $M^n$ on itself (or into $N^n$) having a dense set of spiral points. We generalize this result result and show that the homeomorphisms of $M^n$ into $N^n$ which have a dense set of spiral points form a dense subset, in the fine $C^0$ topology, of the set $H(M^n, N^n)$ of homeomorphisms of $M^n$ into $N^n$.

THEOREM 2. *Let $f: U \to E^n$, $n \geqq 2$, be a homeomorphism, where $U$ is an open subset of $E^n$, and let $\varepsilon: U \to E^1$ be a real-valued positive continuous function. Then there exists a homeomorphism $g: U \to E^n$ such that $g$ has a dense set (in $U$) of spiral points, and $d(f(x), g(x)) < \varepsilon(x)$, $[x \in U]$.*

REMARKS. It can be seen from the constructions in § 8 of [1] that Theorem 2 above is valid for *diffeomorphisms* $f$. Indeed, using the techniques in § 8 of [1], one can construct a homeomorphism $h$ of $U$ on itself having a dense set of spiral points, and, moreover, such that $d(fh(x), f(x)) < \varepsilon(x)$, for all $x \in U$. Then $g = fh$ satisfies the requirements of Theorem 2 relative to $f$. The difficulty that arises when $f$ is not a diffeomorphism is that a point $x \in U$ can be a spiral point of the homeomorphism $g$, and the point $g(x)$ can be a *piercing point* (cf. Definition 1 of [1]) of the homeomorphism $f$, and yet $x$ can be a piercing point of $fg$ (and hence, in particular, $x$ is *not* a spiral point of $fg$). However, the generality afforded by condition 3 of Theorem 1 (i.e., the constants $\zeta_i$ vary with $F_i$), as opposed to the *uniform* constant $\delta$ appearing in property ($\beta$) of [1], will allow us to overcome the above difficulty.

*Proof of Theorem* 2. Let $X$ be a countable dense subset in $U$ of distinct points $x_i$, $i \in Z^+$. We will construct a sequence of homeomorphisms $H_i$ of $U$ on itself, of the type described in Theorem 1 above, and such that if $H = \lim_{i\to\infty} H_i$, then $X$ consists entirely of spiral points of $fH$, and $d(fH(x), f(x)) < \varepsilon(x)$ for all $x \in U$. This latter condition will be satisfied if $d(H(x), x) < \tau(x)$, provided $\tau: U \to E^1$ is a suitably chosen real-valued positive continuous function. We assume below that a fixed choice for such a function $\tau$ has been made. Note from (2) and (4) that if $\delta(U_i) < \min\{\tau(x) \mid x \in U_i\}$, for all $i \in Z^+$, then the above approximation conditions are necessarily satisfied.

Before defining the homeomorphisms $H_i$, we need some definitions. For $c = (c^1, c^2, \cdots, c^n)$, $x = (x^1, x^2, \cdots, x^n)$, $0 < r < d(c, E^n - U)$, $i \in \{1, 2, \cdots, n-1\}$, $i < j \leqq n$, and $m \in Z^+$, we define the homeomorphism $F_{c,r,i,j,m}$ of $U$ on itself as follows:

$$F_{c,r,i,j,m}(x) = x \qquad [x \in \{U - B(c, r)\} \cup \bar{B}(c, r/2)] , \tag{9}$$

while for $x \in \bar{B}(c, r) - B(c, r/2)$, the components of $F_{c,r,i,j,m}$ are:

$$(9)' \qquad F^k_{c,r,i,j,m}(x) = x^k, \; k \neq i, j \; ,$$

$$(9)'' \qquad F^i_{c,r,i,j,m}(x) = (x^i - c^i)\cos\alpha_m(x) - (x^j - c^j)\sin\alpha_m(x) + c^i \; ,$$

$$(9)''' \qquad F^j_{c,r,i,j,m}(x) = (x^i - c^i)\sin\alpha_m(x) + (x^j - c^j)\cos\alpha_m(x) + c^j \; ,$$

where $\alpha_m(x) = 4m\pi((r - \| x - c \|)/r)$. We then define the homeomorphism $F_{c,r,m}$ of $U$ on itself by setting

$$F_{c,r,m} = F_{c,r,1,2,m}F_{c,r,1,3,m} \cdots F_{c,r,n-1,n,m} \; .$$

It is readily seen that there exists a positive constant $\zeta(m)$ such that $d(F_{c,r,m}(x), F_{c,r,m}(y)) \geqq \zeta(m)d(x, y)$, $[x, y \in U]$.

A homeomorphic image $\Omega$ of $E^{n-1}$ in $E^n$ will be called a *sufficiently planar topological* $(n-1)$-*hyperplane relative to* $y \in E^n$ if the following conditions are satisfied: (i) $y \in \Omega$, (ii) $E^n - \Omega$ is not connected, and (iii) there exists a (true)$(n-1)$-hyperplane $P$ in $E^n$ through $y$ such that for all $x \in \Omega$, the secant line joining $x$ to $y$ makes an angle of less than one degree with $P$. Given a homeomorphism $g: U \to E^n$, we will say that $F_{c,r,m}$ is of *spiral type relative to* $g$ if the following condition holds: if $\sigma$ is any arc joining a point of $S(c, r/2)$ to a point of $S(c, r)$, and such that $\sigma$ lies in one component of $E^n - P^*$, where $P^*$ is some $(n-1)$-hyperplane in $E^n$ through $c$, and if $\Omega$ is any sufficiently planar topological $(n-1)$-hyperplane relative to $g(c)$, then $gF_{c,r,m}(\sigma) \cap \Omega \neq \phi$.

We now can construct, inductively, the required homeomorphisms $H_i$. The inductive description is most conveniently carried out by stages, i.e., setting $\sigma(k) = 1 + 2 + \cdots + k - 1 = k(k-1)/2$, at stage $k$, the homeomorphisms $H_{\sigma(k)+1}, H_{\sigma(k)+2}, \cdots, H_{\sigma(k+1)}$ are constructed. To further orient our discussion, we remark that the point $H_{\sigma(k)}(x_k)$ is added to our discussion at stage $k$, and relative to the constants $r_{js}, m_{js}$ chosen below, the subscript $j$ refers to $x_j$, while the subscript $s$ denotes stage $s$, $j \leqq s$.

*Stage* 1. Select a positive constant $r_{11}$ such that

$$r_{11} < \min\{1/2, 1/2 \min\{\tau(x) \mid x \in B(x_1, r_{11})\}, d(x_1, E^n - U)\} \; ,$$

and $S(x_1, r_{11}) \cap X = \phi$. Then choose the positive integer $m_{11}$ so large that the homeomorphism $F_{x_1, r_{11}, m_{11}}$ is of spiral type relative to $f$. We set $H_1 = F_1 = F_{x_1, r_{11}, m_{11}} \zeta(m_{11}) = \zeta_1$, and $U_1 = B(x_1, r_{11})$.

*Stage* 2. Select a positive constant $r_{12}$ such that

$$(10) \qquad r_{12} < \min\{r_{11}/2, \zeta_1/2^2\} \; ,$$

(11) $$S(H_1(x_1), r_{12}) \cap H_1(X) = \phi ,$$

(12) $$H_1(x_2) \notin \bar{B}(H_1(x_1), r_{12}) .$$

In each step, a condition such as (11) is crucial in the construction of the $U_i$ satisfying (1), and can be achieved since $X$ is countable. Then choose the positive integer $m_{12}$ so large that $F_{H_1(x_1), r_{12}, m_{12}}$ is of spiral type relative to $fH_1$. We set $F_2 = F_{H_1(x_1), r_{12}, m_{12}}$, $\zeta(m_{12}) = \zeta_2$, $U_2 = B(H_1(x_1), r_{12})$, and $H_2 = F_2F_1$. Now consider the point $H_2(x_2)$. Using (9), (12), and our choice of $r_{11}$, we have

$$H_2(x_2) = H_1(x_2) \notin S(x_1, r_{11}) \cup S(H_1(x_1), r_{12}) .$$

We then have two cases to consider.

*Case* 1. $H_2(x_2) \in B(x_1, r_{11})$. Then select $r_{22}$ such that

(13) $$r_{22} < \zeta_2\zeta_1/2^3 ,$$

(14) $$S(H_2(x_2), r_{22}) \cap H_2(X) = \phi ,$$

(15) $$\bar{B}(H_2(x_2), r_{22}) \cap \bar{B}(H_1(x_1), r_{12}) = \phi ,$$

(16) $$\bar{B}(H_2(x_2), r_{22}) \subset B(x_1, r_{11}) .$$

Then choose $m_{22}$ so large that $F_{H_2(x_2), r_{22}, m_{22}}$ is of spiral type relative to $fH_2$. Set $F_3 = F_{H_2(x_2), r_{22}, m_{22}}$, $\zeta(m_{22}) = \zeta_3$, $U_3 = B(H_2(x_2), r_{22})$, and $H_3 = F_3F_2F_1$.

*Case* 2. $H_2(x_2) \in U - \bar{B}(x_1, r_{11})$. Then select $r_{22}$ such that

$$r_{22} < \min \{1/2 \min \{\tau(x) \mid x \in B(H_2(x_2), r_{22})\}, d(H_2(x_2), E^n - U)\} ,$$

and, moreover, relations (13), (14), and (15) are satisfied, together with the following relation analogous to (16):

(17) $$\bar{B}(H_2(x_2), r_{22}) \subset U - \bar{B}(x_1, r_{11}) .$$

Then let $F_3$, $\zeta_3$, $U_3$, and $H_3$ be determined as in Case 1. One verifies that $F_i$ and $U_i$ satisfy all the conditions of Theorem 1. It also is readily verified, in particular, that

(18) $$H_3(X) \cap \{S(x_1, r_{11}) \cup S(H_1(x_1), r_{12}) \cup S(H_2(x_2), r_{22})\} = \phi .$$

Suppose, inductively, that stages 1 through $k-1$ have been constructed, i.e., that positive constants $r_{js}$, and positive integers $m_{js}$, $j = 1, 2, \cdots, k-1, j \leqq s \leqq k-1$, have been chosen, together with homeomorphisms $H_0 = I, H_1, \cdots, H_{\sigma(k)}$ of $U$ on itself such that the following conditions are satisfied. First, for $1 \leqq m \leqq \sigma(k)$, $H_m = F_mF_{m-1} \cdots F_1$, where $F_{\sigma(s)+j} = F_{H_{\sigma(s)+j-1}(x_j), r_{js}, m_{js}}$, and $m_{js}$ is so large that $F_{\sigma(s)+j}$ is of spiral type relative to $fH_{\sigma(s)+j-1}$. Before stating further conditions, we simplify our notation by setting $F_{js} = F_{\sigma(s)+j}$, $U_{js} = U_{\sigma(s)+j}$, $\zeta_{js} = \zeta_{\sigma(s)+j-1} = \zeta(m_{js})$, $H_{js} = H_{\sigma(s)+j-1}$, $B_{js} = B(H_{js}(x_j), r_{js})$, and $S_{js} = S(H_j(x_j), r_{js})$. Continuing, now, the enumeration of the conditions satisfied in stages 1 through $k-1$, we have:

$$r_{js} < r_{js-1}/2 < \cdots < r_{jj}/2 \ , \tag{19}$$

(20)

$$r_{js} < \min\{\zeta_{js} \cdots \zeta_1/2^{\sigma(s)+j}, d(H_{js}(x_j), E^n - U), 1/2 \min\{\tau(x) \mid x \in B_{js}\}\} \ ,$$

$$S_{js} \cap H_{js}(X) = \phi \ , \tag{21}$$

$$\bar{B}_{js} \cap \bar{B}_{ls} = \phi \qquad [j \neq l,\ s \text{ fixed}] \ , \tag{22}$$

$$H_{js}(x_j) \in B_{lt} \Rightarrow \bar{B}_{js} \subset B_{lt} \qquad [t < s] \ , \tag{23}$$

$$H_{js}(x_j) \in U - \bar{B}_{lt} \Rightarrow \bar{B}_{js} \subset U - \bar{B}_{lt} \qquad [t < s] \ . \tag{24}$$

It is readily verified, using (9), (21), and (22), that

$$\left\{\bigcup_{t \leqq s,\, l<j} S_{lt}\right\} \cap H_{j+1s}(X) = \phi \ , \tag{25}$$

and that (23) and (24) cover the possible locations of $H_{js}(x_j)$. Setting $U_{js} = B_{js}$, one verifies, using (19)-(24), that $F_m$ and $U_m$, $1 \leqq m \leqq \sigma(k)$, satisfy the conditions of Theorem 1, as well as the condition $\delta(U_m) < \min\{\tau(x) \mid x \in U_m\}$. Clearly, we may choose positive constants $r_{jk}, m_{jk}, j = 1, \cdots, k$, and define $U_{k1}, \cdots, U_{kk}, F_{k1}, \cdots, F_{kk}$ as above so that relations (19)-(24) remain valid for $j = 1, \cdots, k, j \leqq s \leqq k$, and $H_m = F_m F_{m-1} \cdots F_1, 1 \leqq m \leqq \sigma(k+1)$, and, moreover, $F_{js}$ is of spiral type relative to $fH_{js}$. This completes the induction, and we set $H = \lim_{i\to\infty} H_i$. Using Theorem 1, $H$ is a homeomorphism of $U$ on itself, and from (9) and (20), $d(H(x), x) < \tau(x)$, for all $x \in U$. It is readily seen (compare with § 8 of [1]) that $X$ consists entirely of spiral points of $fH$. Since, by our choice of $\tau$, $d(f(x), fH(x)) < \varepsilon(x)$ for all $x \in U$, the proof of Theorem 2 is complete.

COROLLARY 1. *In Theorem 2, the homeomorphism $g = fH$ can be taken as $g = fK_1$, where $K_t, t \in [0, 1]$, is a continuous family of homeomorphisms of $U$ on itself such that $K_0 = I$.*

*Proof.* In the proof and notation of Theorem 2, we replace $H_k = F_k \cdots F_1$ by $(H_k)_t = (F_k)_t \cdots (F_1)_t$, where if $F_k = F_{c,r,m}$, then $(F_k)_t$ is defined as follows. First, $(F_k)_t(x) = F_k(x)$ for $x \in U - B(c, r)$. Now for $x \in \bar{B}(c, r) - B(c, r/2)$, the formulas for the components of $(F_k)_t$ are obtained from the corresponding formulas (cf. (9)-(9)''') for the components of $F_k$ by replacing $\alpha_m(x)$ by $t\alpha_m(x)$. Finally, we set

$$(F_k)_t(x) = \frac{2\,||\,x-c\,||}{r}\Big[(F_k)_t\Big(\frac{r(x-c)}{2\,||\,x-c\,||}+c\Big)-c\Big]+c$$

for $x \in \bar{B}(c, r/2) - 0$, and set $(F_k)_t(0) = 0$. These are overdefinitions, but are consistant, and define a homeomorphism of $U$ on itself, for each $t \in [0, 1]$. Note that $(H_k)_0 = I$, and $(H_k)_1 = H_k$, for each $k \geqq 1$. It is clear that $d((F_k)_t(x), (F_k)_t(y)) \geqq \zeta(m)d(x, y)$, $[x, y \in U]$, where $\zeta(m)$ is the constant verifying the corresponding inequality for $F_k$. Hence, setting $K_t = \lim_{k\to\infty} (H_k)_t$, we see by Theorem 1 that $K_t$ is a homeomorphism of $U$ on itself, for each $t \in [0, 1]$. Note also that $K_0 = I$ and $K_1 = H$. To complete the proof, one verifies that $K_t$ is a continuous family by noting that

$$d(K_t(x), H_k(x)) \leqq \sum_{i=k}^{\infty} 1/2^i,\ [x \in U, t \in [0, 1], k \in Z^+]$$

and

$$d((H_k)_s(x), (H_k)_t(y)) \leqq d(x, y) + 1/2^{k-1},\ [x, y \in U, s, t \in [0, 1], k \in Z^+]\ .$$

COROLLARY 2. *Let $f\colon M^n \to N^n$ be a homeomorphism of $M^n$ into $N^n$, where $M^n$ and $N^n$ are nonbounded differentiable n-manifolds. Suppose $\varepsilon\colon M^n \to E^1$ is an arbitrary real-valued positive continuous function. Then there exists a continuous family $K_t, t \in [0, 1]$, of homeomorphisms of $M^n$ on itself such that $K_0 = I$, and the homeomorphism $g = fK_1$ has a dense set (in $M^n$) of spiral points, and $d(f(x), g(x)) < \varepsilon(x)$, for all $x \in M^n$.*

*Proof.* With the aid of Corollary 1, a proof of Corollary 2 can be patterned after the proof of Theorem 1 of [1].

## REFERENCE

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PURDUE UNIVERSITY
LAFAYETTE, INDIANA

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